Nonlinear heat structures and arc-discharge electrode spots

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Domainlike temperature distributions (solitary structures) are studied, which may arise in the half space with the surface heated by the external heat flux depending in a nonlinear way on the local surface temperature. Solutions describing stationary axisymmetric structures for some model dependencies of the external heat flux density on the surface temperature are found analytically or numerically. The stability of stationary axisymmetric structures is analyzed. When applied to an analysis of spots on electrodes in arc discharges, for given characteristics of the near-electrode plasma layer such an approach results in the complete description of the stationary spots on smooth surfaces, including the spot radius and the integral current. An evaluation for cathode spots in vacuum arcs is given; the results are inside the usual range of characteristics of macrospots.

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I. INTRODUCTION

We consider the problem of heat conduction in the half space heated from outside, the density q of the incoming heat flux being a nonlinear function of the local value of the surface temperature. At large distances from the origin of coordinates the temperature distribution is uniform: $T=T_\infty$, where T_∞ is a given constant (obviously, this implies that the value of q, corresponding to T_∞, is zero).

The solution of the considered problem is in a general case nonunique. There is the trivial solution $T=T_\infty$ corresponding to the situation when the temperature in the whole half space is unperturbed, and nontrivial solutions describing localized temperature perturbations (solitary structures). The length scale of these structures is of order of $\kappa T/q$, where κ is the thermal conductivity of the material. Just these structures are studied in the present paper.

This problem is of substantial theoretical interest as one of the simplest models giving rise to solitary dissipative structures. In addition, it is of interest due to its relation to spots on the electrodes in arc discharges. Consider for definiteness the cathode spots of vacuum arcs. It is well known (e.g., [1]) that the thickness of the near-cathode plasma layer in which the flux of ions to the cathode is formed and which gives the main contribution to the total near-cathode voltage drop is considerably less than the radius of the spots. Hence, the current transfer across this layer is locally one dimensional and for the given voltage drop in the layer the density of the heat flux from the plasma to the cathode surface may be considered as a function of the local surface temperature. Thus, one arrives at the above-described statement of the problem, the structure being nothing else than the cathode spot.

An important specific feature that distinguishes the spots from other possible nonlinear heat structures is their channel-like character. The electric current and, correspondingly, the heat flux coming to the electrode surface from the plasma are localized in an area with a more or less distinct boundary (in the current spot). In the simplest—axially symmetric—case this is a circle of a certain radius. This feature is well known [2] and results from the fact that some of the processes involved are of the Arrhenius type with a high activation energy, such as electron emission, ionization of neutral particles, and evaporation. A correct account of this feature is important for any theory of the spots.

In view of the above, we may conclude that a proper way of constructing the theory of the electrode arc spots is to consider the spot as a nonlinear heat structure with account for the Arrhenius nature of the processes involved. A very large number of papers has been published on various aspects of the theory of cathode spots of vacuum arcs; see, e.g., [1–6] and references therein. However, there is no theory that can without using empirical parameters (such as the value of the integral current to the spot) or arbitrary suppositions (such as some or other implementation of the "principle" of minimum voltage) predict the radius of the spot on a smooth surface. In contrast to this, the above-described approach provides for given characteristics of the near-electrode plasma layer the complete description of the spot. When the above-described problem is solved, the temperature distribution in the cathode bulk and on the surface, the current density distribution in the spot, the radius of the spot, the integral current to the spot, the integral power dissipated in the spot, etc., corresponding to the considered value of the voltage drop in the near-electrode layer, will be determined. By solving the problem for different values of the voltage drop, one can calculate the current-voltage characteristic of the spot. Close to such an approach we mention investigations [7,8] in which the idea of treating the spot as a nonlinear heat structure had been put forward. Unfortunately the way to combine it with the idea of a channel-like spot has not been found; in fact, in [8] consideration of the spot as a nonlinear structure was replaced by the channel-like model with constant parameters within the current spot, the spot radius being determined from the certain implementation of the minimum principle.

The formulation of the problem is considered in Sec. II. Stationary axisymmetric structures are treated in Sec. III. Stability of these structures is analyzed in Sec. IV. Finally, in Sec. V, among other concluding remarks, the application to cathode spots in vacuum arcs is given.

II. GOVERNING EQUATION AND BOUNDARY CONDITIONS

Consider the half space (electrode bulk) with the surface heated by the energy flux from the adjacent medium (plasma); see Fig. 1. We denote by q the density of the heat flux from the surface into the bulk; for example, if the theory refers to the cathode spots of a vacuum arc, q may be taken equal to the density of the energy influx caused by ion impact and neutralization minus losses caused by electron emission cooling and evaporation cooling. q will be treated as a prescribed nonlinear function of the local surface temperature and of a control parameter U (the near-electrode voltage drop which is assumed constant along the surface): $q=q(T,U)$. It is supposed that q tends to zero faster than $(T-T_\infty)^2$ as T decreases down to the unperturbed temperature T_∞, which implies that the derivative $\partial q/\partial T$ in some range of the temperature values is positive. Note that from the point of view of the conventional heat-exchange theory the latter is not quite usual: In heat-exchange problems without the electric current increase of the surface temperature results in decrease of the external heat flux to this surface, i.e., dq/dT is negative.

As the thermal conductivity κ of the bulk material is in a general case variable (it depends on the temperature), it is convenient to introduce in place of T the new variable [9]

$$\psi(T)=\int_{T_\infty}^{T}\kappa(T)dT \ . \tag{1}$$

In cases when it cannot cause confusion ψ will be for brevity referred to as the temperature.

Neglecting ohmic heating in the bulk, we have [the cylindrical coordinates (r,φ,z) are introduced as shown in Fig. 1]

$$\frac{1}{\chi}\frac{\partial\psi}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial\psi}{\partial r}\right]+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\varphi^2}+\frac{\partial^2\psi}{\partial z^2} \ , \tag{2}$$

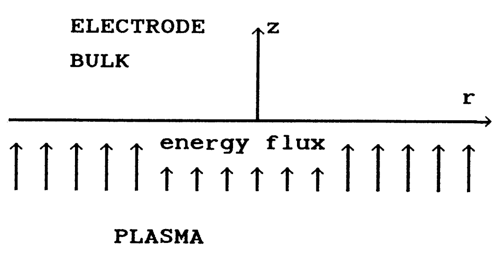

FIG. 1. Geometry of the problem.

$$\frac{\partial\psi}{\partial z}=-q(\psi,U) \ , \quad z=0 \ ,$$
$$\psi\to 0 \ , \quad r+z\to\infty \ . \tag{3}$$

Here t designates time and χ is the thermal diffusivity of the material (the thermal conductivity divided by the density and by the specific heat).

III. STATIONARY AXISYMMETRIC SOLUTIONS

The problem governing the stationary axisymmetric solution $\psi=\psi^{(1)}(r,z)$ is obtained from Eqs. (2) and (3) by dropping in Eq. (2) the term on the left-hand side and the second term on the right-hand side. For some purposes (for example, for the numerical solution [7]) it is convenient to employ also an (equivalent) statement of the problem in terms of an integral equation. To derive this equation, we represent the solution as the integral transform [9]

$$\psi^{(1)}(r,z)=\int_0^\infty A(k)J_0(kr)e^{-kz}dk \ , \tag{4}$$

where J_0 is the zero-order Bessel function of the first kind, and $A(k)$ is a function to be specified. Differentiating (4) with respect to z, setting $z=0$, and inverting the obtained relationship, one finds

$$A(k)=\int_0^\infty q(\Theta(\xi),U)J_0(k\xi)\xi\,d\xi \ . \tag{5}$$

Here $\Theta=\Theta(r)=\psi^{(1)}(r,0)$.

Substituting (5) in (4), setting $z=0$, changing order of integration, and expressing with the help of [10] the integral of the product $J_0(k\xi)J_0(kr)$ in terms of the complete elliptic integral of the first kind $K=K(m)$ [11], one obtains the desired integral equation for the distribution of the surface temperature $\Theta(r)$

$$\Theta(r)=\frac{2}{\pi r}\int_0^r q(\Theta(\xi),U)\xi K\left[\frac{\xi^2}{r^2}\right]d\xi + \frac{2}{\pi}\int_r^\infty q(\Theta(\xi),U)K\left[\frac{r^2}{\xi^2}\right]d\xi \ . \tag{6}$$

Due to the properties of the function $K(m)$ [11], this equation may be written also in the form

$$\Theta(r)=\frac{2}{\pi}\int_0^\infty q(\Theta(\xi),U)\frac{\xi}{r+\xi}K\left[\frac{4r\xi}{(r+\xi)^2}\right]d\xi \ . \tag{7}$$

Note that $K(0)=\pi/2$, so Eq. (7) for large r in the first approximation assumes the form

$$\Theta(r)=\frac{Q}{2\pi r} \ , \tag{8}$$

where Q is the integral heat flux removed by heat conduction,

$$Q=2\pi\int_0^\infty q(\Theta(\xi),U)\xi\,d\xi \ . \tag{9}$$

The physical sense of Eq. (8) is quite clear: At large distances from the structure the temperature distribution in the first approximation coincides with that created by a point heat source with the intensity Q. According to the supposition mentioned in Sec. II concerning the function q, $q(\Theta(\xi),U)$ decreases for $\xi\to\infty$ faster than ξ^{-2}, and the above integrals are convergent.

The considered problem has a nonunique solution. There is the trivial solution $\Theta=0$ corresponding to the situation when the structure is absent and the temperature field is unperturbed, and a nontrivial solution which we are interested in. The general question of what are the conditions satisfied by the function q for a nontrivial solution to exist, and of uniqueness of this solution, is beyond the scope of the present paper. Instead, solutions are studied below for three particular cases.

The most convincing example of the existence of a nontrivial solution is that in which an exact solution is found analytically. Therefore, we first consider the case when the dependence of the heat flux density on the temperature is described by the step function

$$q(\psi,U)=\begin{cases} q_* & \text{for } \psi>\psi_* \\ 0 & \text{for } \psi<\psi_* \end{cases} . \tag{10}$$

Here and further q_* and ψ_* designate some prescribed quantities which are independent of ψ but may depend on U.

Substituting (10) in (6) and evaluating integrals with the help of [10] under the supposition that the equation $\Theta(r)=\psi_*$ has only one root, one obtains the solution that may be written as follows:

$$\alpha(\rho)=\begin{cases} E(\rho^2) & \text{for } \rho\le 1 \\ \rho E\left[\dfrac{1}{\rho^2}\right]-\left[\rho-\dfrac{1}{\rho}\right]K\left[\dfrac{1}{\rho^2}\right] & \text{for } \rho\ge 1 \end{cases} . \tag{11}$$

Here and further $\alpha=\Theta/\psi_*$ and $\rho=2rq_*/\pi\psi_*$ are the normalized temperature of the surface and the normalized distance from the center of the structure, $E=E(m)$ is the complete elliptic integral of the second kind [11].

The local heat flux density equals to q_* for $\rho<1$ and to zero for $\rho>1$, so heat production is localized in the circle (current spot) of the radius $\pi\psi_*/2q_*$. The temperature decreases monotonically, inside the circle from the value $\Theta(0)=(\pi/2)\psi_*$ to the "switching" temperature ψ_*, and outside the circle from ψ_* to zero. The integral heat flux Q to the surface is $\pi^3\psi_*^2/4q_*$.

A graph of the function $\alpha(\rho)$ described by Eq. (11) is shown in Fig. 2 by the solid line. The dashed line represents the main term of the asymptotic expansion of $\alpha(\rho)$ for $\rho\to\infty$, which equals to $\pi/4\rho$.

The above solution is unique in the class of functions such that the equation $\Theta(r)=\psi_*$ has one root, i.e., in the class of solutions describing a circular current spot. One could think also of solutions with two roots (describing a ring spot), three roots (a circular spot and a ring spot concentric with it), etc. However, it is not evident that such solutions exist. Leaving this question beyond the scope of the present paper, we mention only that the solution with two roots does not exist, which can be easily proven.

For the case when the dependence of the heat flux density on the temperature may be approximated by the Arrhenius-type function

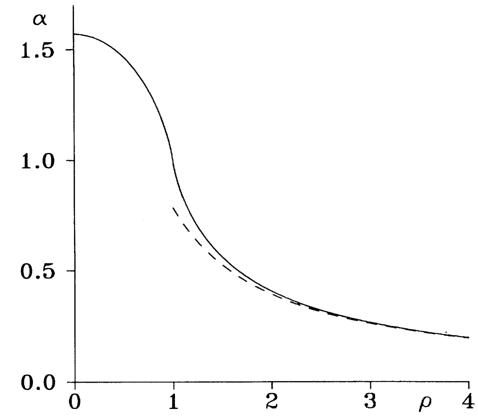

FIG. 2. Distribution of the normalized surface temperature for the step-function model (10).

$$q(\psi,U)=q_*\exp(-\psi_*/\psi) , \tag{12}$$

Eq. (7) has been solved numerically. A usual technique of solving nonlinear integral equations [12,13] that includes linearization of the right-hand side by means of the Newton's method was employed; note that straightforward iterations which are suggested by the form of Eq. (7) proved to be unstable. The convergence of iterations turned out to be insensitive to the choice of the initial approximation; the obtained solution is shown in Fig. 3 by the solid lines (here and further $q_u=q/q_*$).

Again, $\alpha(\rho)$ decreases monotonically. In contrast to the preceding case, the current spot does not have a distinct boundary. $\alpha(0)=0.5638$, i.e., is of order unity; in other words, the temperature $\psi^{(1)}(0,0)$ in the center of the spot is of the order of the activation temperature ψ_*. The value of the normalized integral heat flux $Q/(\pi^3\psi_*^2/2q_*)$ is 0.4746. The main term of the asymptotic expansion of $\alpha(\rho)$ for $\rho\to\infty$, calculated by means of (8) using this value, is shown in Fig. 3 by the dashed line.

Consider now the case when the heat flux density is nonzero only in a narrow range of the surface tempera-

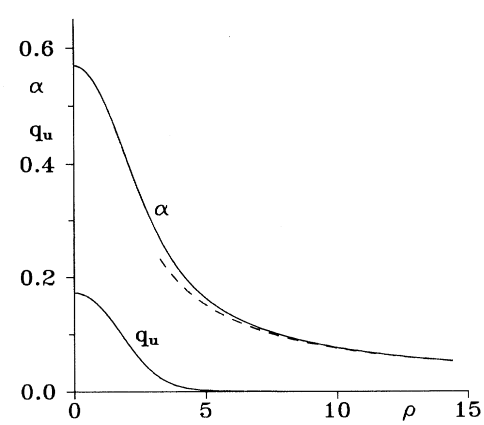

FIG. 3. Distributions of the normalized surface temperature and of the normalized heat flux for the Arrhenius model (12).

tures:

$$q(\psi,U)=q_* q_u\left[\frac{\psi-\psi_*}{\epsilon\psi}\right], \tag{13}$$

where ϵ is a prescribed small parameter and q_u is a prescribed function which tends to zero fast enough for large positive or negative values of the argument. (Note that one could consider a little simpler expresson obtained by replacing ψ in the denominator of the argument of q_u by ψ_*, but this expression would not satisfy for finite ϵ the condition of vanishing at $\psi=0$.) For definiteness, it will be supposed that q_u monotonically grows for negative values of the argument and monotonically decreases for positive values, $q_u(0)$ being equal to unity. The simple example of such a function is

$$q_u=\left[\cosh\frac{\beta-1}{\epsilon\beta}\right]^{-1}, \tag{14}$$

where $\beta=\psi/\psi_*$.

To find the general appearance of the solution for such-type functions, the problem has been solved numerically for the function (14). The solutions obtained for $\epsilon=0.1$, 0.05 are shown in Fig. 4 by the solid lines. The surface temperature decreases monotonically, as in both preceding cases. Similar to the case of the step function and in contrast to the Arrhenius case, the boundary of the current spot is well pronounced. The surface temperature in the spot varies only slightly and is a little higher than ψ_*, which seems quite understandable: If the temperature inside the spot exceeded ψ_* appreciably, the local heat flux would be zero and such a temperature could not be maintained.

To describe the main features of the solution analytically, the technique of the matched asymptotic expansions in the small parameter ϵ may be employed. The straightforward asymptotic expansion is

$$\beta=\beta_1(\rho_1,\eta_1)+\cdots, \tag{15}$$

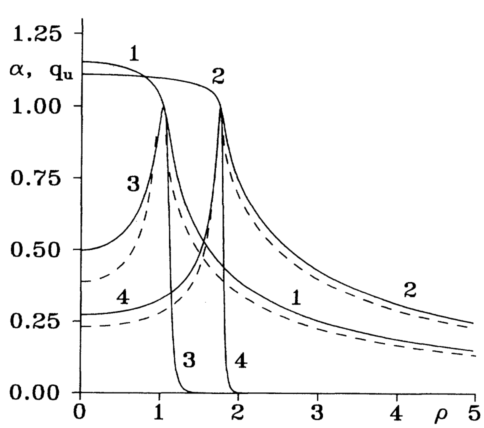

FIG. 4. Distributions of the normalized surface temperature α (lines 1,2) and of the normalized heat flux q_u (3,4) for the model (14). 1,3: $\epsilon=0.1$; 2,4: $\epsilon=0.05$.

where $\rho_1=\rho/\rho_*$, $\eta_1=\eta/\rho_*$, $\eta=2zq_*/\pi\psi_*$, ρ_* being an unknown parameter (the normalized radius of the spot). The function β_1 satisfies the Laplace equation, is equal to unity on the surface in the unit circle $\rho_1\le 1$, satisfies the homogeneous Neumann condition on the surface outside this circle, and vanishes at infinity. This function coincides with the distribution of the electrostatic potential, created in vacuum by a biased disk, and may be written as (e.g., [9,14])

$$\beta_1=\frac{2}{\pi}\arcsin\frac{2}{[(\rho_1-1)^2+\eta_1^2]^{1/2}+[(\rho_1+1)^2+\eta_1^2]^{1/2}}. \tag{16}$$

The distribution of the normalized heat flux density inside the spot, derived from (16), is

$$q_u=\frac{4}{\pi^2\rho_*}\frac{1}{(1-\rho_1^2)^{1/2}}. \tag{17}$$

The corresponding integral heat flux is

$$Q=\frac{2\pi\rho_*\psi_*^2}{q_*}. \tag{18}$$

In principle, (17) allows one to find the variation of the surface temperature inside the spot [or, in other words, to evaluate the second term of the expansion (15) inside the unit circle on the surface]. This variation should correspond to the heat flux distribution (17).

Distribution (17) and the surface temperature outside the spot, described by (16), both calculated using the value of ρ_* from the numerical solution, are shown in Fig. 4 by the dashed lines. Agreement with the exact results is reasonable. The difference between the value (18) and the exact value of Q also is reasonably small: 11% for $\epsilon=0.1$ and 7% for $\epsilon=0.05$.

Expansion (15) should be applicable in the whole half space $\eta_1\ge 0$ with the exception of the vicinity of the ring $\rho_1=1$ in the plane $\eta_1=0$. It follows, in particular, that the heat flux density described by (17) should not exceed 1 for any fixed $\rho_1<1$. Hence, the order of ρ_* exceeds unity. To obtain a more definite estimate, the abovementioned vicinity, i.e., the transition region between the spot and the surrounding current-free area, should be considered.

Supposing that the variation of the temperature in this vicinity is of order $\epsilon\psi_*$ while the heat flux density is of order q_*, one finds that the local length scale is $\epsilon\psi_*/q_*$. Accordingly, the corresponding two-term asymptotic expansion is

$$\beta=1+\epsilon\beta_2(\rho_2,\eta_2)+\cdots, \tag{19}$$

where $\rho_2=(\rho-\rho_*)/\epsilon$ and $\eta_2=\eta/\varepsilon$. β_2 is governed by the problem

$$\frac{\partial^2\beta_2}{\partial\rho_2^2}+\frac{\partial^2\beta_2}{\partial\eta_2^2}=0, \tag{20}$$

$$\frac{\partial\beta_2}{\partial\eta_2}=-\frac{\pi}{2}q_u(\beta_2), \quad \eta_2=0, \tag{21}$$

$$\beta_2 = -C\sqrt{\rho_3}\cos\frac{\varphi_3}{2} + \cdots \ , \quad \rho_3 \to \infty \ . \tag{22}$$

Here $C=(8/\pi^2\epsilon\rho_*)^{1/2}$, ρ_3 and φ_3 are the polar coordinates in the plane (ρ_2,η_2): $\rho_2=\rho_3\cos\varphi_3$, $\eta_2=\rho_3\sin\varphi_3$. Boundary condition (22) is derived from matching of the first and the second terms of the expansion (19) with the first term of the outer expansion (15).

Note that the first item on the right-hand side of (22) vanishes at $\varphi_3=\pi$. Hence, the order of magnitude of β_2 in the limit $\rho_2\to-\infty$, η_2 fixed is smaller than $\sqrt{-\rho_2}$. To find asymptotic behavior in this limit, the second term of the outer expansion should be included in matching. It follows from (17) that β_2 tends to plus infinity in such a way that

$$q_u(\beta_2) = \frac{C}{\pi\sqrt{-\rho_2}} + \cdots \ . \tag{23}$$

It is expected that the constant C cannot be chosen arbitrarily; this is similar to the case of the constant Q, which governs asymptotic behavior of the function $\Theta(r)$ for large r [Eq. (8)] and which also cannot be chosen arbitrarily but rather is to be determined as a part of the solution of the problem. Hence, C in (22) and (23) should be replaced by $O(1)$, and when the resulting problem is solved and β_2 is found, one will be able to determine C. ρ_* also can be determined, so the asymptotic solution of the problem on the whole will be complete. Evidently, $\rho_*=O(1/\epsilon)$.

The value of C should be determined separately for each function q_u. For example, for the function (14) this value, if estimated in terms of ρ_* taken from the numerical solution shown in Fig. 4, is approximately 3.

The contribution of the considered transition region to the integral heat flux to the surface is of the order of ψ_*^2/q_*. This is asymptotically small as compared to the contribution of the inner part of the spot, governed by Eq. (18), which is of the order of $\psi_*^2/\epsilon q_*$. Hence, the integral heat flux in the first approximation is determined by the inner part of the spot and may be evaluated by means of Eq. (18).

IV. STABILITY OF STATIONARY AXISYMMETRIC SOLUTIONS

We assume for simplicity $\chi=$const and represent perturbations as $\psi^{(2)}(r,z)e^{\lambda\tau+i\mu\varphi}$, where $\tau=\chi t$ is normalized time and $\mu=0,\pm1,\pm2,\ldots$. The amplitude $\psi^{(2)}$ and the increment λ are the eigenfunction and the eigenvalue of the linear problem

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial\psi^{(2)}}{\partial r}\right] + \frac{\partial^2\psi^{(2)}}{\partial z^2} - \left[\lambda+\frac{\mu^2}{r^2}\right]\psi^{(2)} = 0 \ , \tag{24}$$

$$\frac{\partial\psi^{(2)}}{\partial z} = -\frac{\partial q}{\partial\psi}[\psi^{(1)},U]\psi^{(2)}, \quad z=0 \ , \tag{25}$$

$$\psi^{(2)}\to 0, \quad r+z\to\infty \ . \tag{26}$$

It follows trivially that

$$\lambda\int_0^\infty\int_0^\infty|\psi^{(2)}|^2 r\,dr\,dz = \int_0^\infty|\psi^{(2)}(r,0)|^2\frac{\partial q}{\partial\psi}(\Theta(r),U)r\,dr - \int_0^\infty\int_0^\infty\left[\left|\frac{\partial\psi^{(2)}}{\partial r}\right|^2+\left|\frac{\partial\psi^{(2)}}{\partial z}\right|^2 + \frac{\mu^2}{r^2}|\psi^{(2)}|^2\right]r\,dr\,dz \ . \tag{27}$$

Evidently, λ is real; positive values of λ are not excluded since the first term on the right-hand side of (27) may be positive (note once again that we consider the situation when, in contrast to the conditions of the conventional heat exchange, the derivative $\partial q/\partial\psi$ in some temperature range is positive).

Equations (24)–(26) may be reduced to the eigenvalue problem for the homogeneous Fredholm equation of the second kind for the function $\omega=\psi^{(2)}(r,0)$ describing the perturbation of the surface temperature

$$\omega(r) = \int_0^\infty \omega(\xi)\frac{\partial q}{\partial\psi}(\Theta(\xi),U)W\xi\,d\xi \ , \tag{28}$$

where

$$W = W(\xi,r;\lambda,\nu) = \int_a^\infty \frac{k}{(k^2+\lambda)^{1/2}}J_\nu(k\xi)J_\nu(kr)dk \ . \tag{29}$$

Here $a=0$ for $\lambda\geq0$ and $a=\sqrt{-\lambda}$ for $\lambda<0$, $\nu=|\mu|$.

Note that

$$W(\xi,r;0,\nu) = (-1)^\nu\frac{\sqrt{\pi}}{\Gamma(\nu+1/2)}\frac{1}{|r^2-\xi^2|^{1/2}} \times P^{\nu}_{-1/2}\left[\frac{r^2+\xi^2}{|r^2-\xi^2|}\right] , \tag{30}$$

$$W(\xi,r;0,0,) = \frac{2}{\pi}\frac{1}{r+\xi}K\left[\frac{4r\xi}{(r+\xi)^2}\right] . \tag{31}$$

Here Γ is the gamma function, $P^{\nu}_{-1/2}$ is the accoatiated Legendre function of the first kind [11].

Leaving the full treatment of Eq. (28) beyond the scope of this paper, we present some considerations on stability against the zeroth and first harmonics $\nu=0$ and $\nu=1$ supposing that for each harmonic the biggest eigenvalue λ corresponds to the eigenfunction ω which does not vanish in the open interval $(0,\infty)$. We mention in this connection the investigation of the stability of cylindrical current filaments in semiconductors [15].

For $\nu=1$, Eq. (28) is satisfied by

$$\lambda=0 \ , \quad \omega=\frac{d\Theta}{dr} \ , \tag{32}$$

which is verified explicitly in the Appendix. This is the Goldstone mode [15], corresponding to a quasistationary infinitesimal translation of the structure in the direction $\varphi=0$. It is expected that the surface temperature distribution is monotonic, so $d\Theta/dr$ is negative in the open interval $(0,\infty)$. Then the eigenvalue $\lambda=0$ is the biggest and the structure is stable against the first harmonic.

Proceed to the stability against cylindrically symmetric perturbations $\nu=0$. We are interested in the variation of the sign of the biggest eigenvalue λ, produced by the variation of U. In other words, we shall study the change of stability along the (continuous) family of stationary states corresponding to the given function $q(\psi,U)$.

Generally speaking, the variation of U along the above-mentioned family is not monotonic, so the function $\lambda(U)$ must not be single valued. To study this function, it is convenient to introduce another parameter γ which varies monotonically, and to analyze the functions $U(\gamma)$ and $\lambda(\gamma)$. The choice of this parameter depends on the function $q(\psi,U)$; a natural candidate is the value of the maximum surface temperature $\Theta(0)$.

Suppose that at a certain point γ_m the biggest eigenvalue vanishes: $\lambda(\gamma_m)=0$. We differentiate (7) with respect to γ and set $\gamma=\gamma_m$

$$\left[\frac{\partial\Theta(r)}{\partial\gamma}\right]_m-\int_0^\infty\left[\frac{\partial\Theta(\xi)}{\partial\gamma}\right]_m\frac{\partial q}{\partial\psi}(\Theta_m(\xi),U_m)\times W(\xi,r;0,0)\xi\,d\xi=\left[\frac{dU}{d\gamma}\right]_m\int_0^\infty\frac{\partial q}{\partial U}(\Theta_m(\xi),U_m)W(\xi,r;0,0)\xi\,d\xi\ , \tag{33}$$

where the index m denotes quantities at $\gamma=\gamma_m$.

Equation (33) may be considered as a linear inhomogeneous equation governing the function $(\partial\Theta/\partial\gamma)_m$. The corresponding homogeneous equation coincides with Eq. (28) and is satisfied by the (nontrivial) function ω_m, so for Eq. (33) to be solvable its right-hand side should satisfy a certain orthogonality condition. To obtain this condition, we multiply (33) by $r\omega_m(r)\partial q/\partial\psi(\Theta_m(r),U_m)$ and integrate in r. The left-hand side vanishes and the result is

$$\left[\frac{dU}{d\gamma}\right]_m\int_0^\infty\omega_m(\xi)\frac{\partial q}{\partial U}(\Theta_m(\xi),U_m)\xi\,d\xi=0\ . \tag{34}$$

Further consideration is restricted with functions q with positive derivative $\partial q/\partial U$ (the heat flux density grows when U is increased at a fixed temperature). The eigenfunction $\omega_m(\xi)$ (corresponding to the biggest eigenvalue) does not change sign. Hence, the integral in (34) is not zero, so $(dU/d\gamma)_m=0$. Thus, the neutral stability may occur only for states corresponding to extreme points of the function $U(\gamma)$ and one may set

$$\omega_m(r)=\left[\frac{\partial\Theta(r)}{\partial\gamma}\right]_m\ . \tag{35}$$

Consider now stability in the vicinity of the point $\gamma=\gamma_m$. We set in (29) $\nu=0$ and rewrite it as

$$W(\xi,r;\lambda,0)=W(\xi,r;0,0)+\sqrt{|\lambda|}\int_b^\infty\frac{\zeta-(\zeta^2+\mathrm{sgn}\lambda)^{1/2}}{(\zeta^2+\mathrm{sgn}\lambda)^{1/2}}\times J_0(\xi\sqrt{|\lambda|}\zeta)J_0(r\sqrt{|\lambda|}\zeta)d\zeta\ , \tag{36}$$

where $b=0$ and $\mathrm{sgn}\lambda=1$ for $\lambda>0$, $b=1$ and $\mathrm{sgn}\lambda=-1$ for $\lambda<0$. If follows that the asymptotic expansion of $W(\xi,r;\lambda,0)$ for small $|\lambda|$ is

$$W(\xi,r;\lambda,0)=W(\xi,r;0,0)-\mathrm{sgn}(\lambda)\sqrt{|\lambda|}+\cdots\ . \tag{37}$$

Differentiating (28) with respect to γ, setting $\gamma=\gamma_m$, and making use of (37), one finds

$$\left[\frac{\partial\omega(r)}{\partial\gamma}\right]_m-\int_0^\infty\left[\frac{\partial\omega(\xi)}{\partial\gamma}\right]_m\frac{\partial q}{\partial\psi}(\Theta_m(\xi),U_m)W(\xi,r;0,0)\xi\,d\xi=\int_0^\infty[\omega_m(\xi)]^2\frac{\partial^2q}{\partial\psi^2}(\Theta_m(\xi),U_m)W(\xi,r;0,0)\xi\,d\xi-\left[\frac{d}{d\gamma}(\mathrm{sgn}(\lambda)\sqrt{|\lambda|})\right]_m\int_0^\infty\omega_m(\xi)\frac{\partial q}{\partial\psi}(\Theta_m(\xi),U_m)\xi\,d\xi\ . \tag{38}$$

The condition of solvability of this linear inhomogeneous equation reads

$$\left[\frac{d}{d\gamma}(\mathrm{sgn}(\lambda)\sqrt{|\lambda|})\right]_m\left\{\int_0^\infty\omega_m(\xi)\frac{\partial q}{\partial\psi}(\Theta_m(\xi),U_m)\xi\,d\xi\right\}^2=\int_0^\infty[\omega_m(\xi)]^3\frac{\partial^2q}{\partial\psi^2}(\Theta_m(\xi),U_m)\xi\,d\xi\ . \tag{39}$$

It follows that the function $\mathrm{sgn}(\lambda)d\sqrt{|\lambda|}/d\gamma$ is continuous in the point $\gamma=\gamma_m$. As $d\sqrt{|\lambda|}/d\gamma$ changes its sign in this point (it is negative for $\gamma<\gamma_m$ and positive for $\gamma>\gamma_m$), λ also changes its sign. For example, if the right-hand side of Eq. (39) is negative, λ is as depicted in Fig. 5 and states $\gamma>\gamma_m$ are stable while states $\gamma<\gamma_m$ are unstable.

To evaluate the right-hand side of Eq. (39), we twice differentiate Eq. (7) with respect to γ, set $\gamma=\gamma_m$, multiply by $r\omega_m(r)\partial q/\partial\psi(\Theta_m(r),U_m)$, and integrate in r. It follows that the right-hand side of (39) equals to

$$-\left[\frac{d^2U}{d\gamma^2}\right]_m\int_0^\infty\omega_m(\xi)\frac{\partial q}{\partial U}(\Theta_m(\xi),U_m)\xi\,d\xi\ . \tag{40}$$

As has already been mentioned, the function $\omega_m(\xi)$

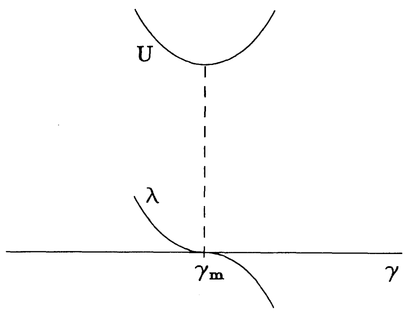

FIG. 5. Increment of the zeroth harmonic and the voltage drop (schematic).

does not change sign. According to (35), its sign is determined by the choice of the parameter γ. We assume for definiteness that $\omega_m(\xi)$ is positive. Then the right-hand side of (39) is negative in the case $[d^2U/d\gamma^2]_m>0$ [$\gamma=\gamma_m$ is the point of minimum of the function $U(\gamma)$; see Fig. 5], and positive in the case $[d^2U/d\gamma^2]_m<0$ (the maximum point). It follows that in both cases states with positive values of $dU/d\gamma$ are stable and states with negative values are unstable.

According to the above, information on stability against the cylindrically symmetric perturbations (uniform expansion or shrinking) may be obtained from analysis of properties of the family of the stationary states corresponding to the given function $q(\psi,U)$. For example, if the dependence $U(\gamma)$ is nonmonotonic and the function $\partial\Theta/\partial\gamma$ evaluated for any extreme point of $U(\gamma)$ is positive in the open interval $(0,\infty)$, the states on sections of growth of $U(\gamma)$ are stable while states on sections of decrease are unstable. As another example, we mention the situation when the function $\partial\Theta/\partial\gamma$ for an extreme point of $U(\gamma)$ vanishes at finite nonzero r. In such a situation the eigenvalue $\lambda=0$ is not the biggest and all the states in the vicinity of the extreme point are unstable. Also, we mention the case when the dependence $U(\gamma)$ is monotonic while $\partial\Theta/\partial\gamma$ evaluated for any stationary state is positive (i.e., the temperature at any fixed point of the surface increases monotonically for the whole family of steady states). It is natural to expect that in such a case all the states are stable if the dependence $U(\gamma)$ is growing, and are unstable if $U(\gamma)$ is decreasing.

If the role of the parameter γ may be attributed to the integral electric current to the spot and the increase of the integral current is accompanied by an increase of the surface temperature, the change of stability occurs in the extreme points of the current-voltage characteristic of the spot: The growing section is stable and the falling section is unstable. It should be emphasized that while previously the relationship between the stability of current systems under constant voltage and the sign of their differential resistance was considered for spatially homogeneous or one-dimensional steady states (e.g., [15,16]), present analysis refers to the two-dimensional states.

V. DISCUSSION OF RESULTS AND APPLICATION TO THE CATHODE SPOTS OF VACUUM ARCS

There are some features in common between the heat structures studied in the present paper and the cylindrical arc column in a thermal plasma. In both cases there is a curent structure (current spot or current channel, respectively; discussion of the latter from the point of view of the asymptotic theory of the arc column is given in [17]) positioned inside the heat structure; the boundary of the current structure may or may not be distinct; the current structure heats the adjacent current-free region by means of heat conduction. However, there is also a very substantial difference: An external length scale which is present in the problem of the arc column (the radius of the discharge vessel) is absent in the theory considered in the present paper. Note that due to just this difference the temperature in the center of the spot for the Arrhenius case (12) is of the order of the activation temperature, while the temperature at the axis of the arc column depends on the radius of the vessel and is usually much smaller than the activation temperature (the ionization potential), this inequality allowing one to obtain an approximate analytic solution [17].

For the step function (10), the radius of the current spot is determined by the condition that the temperature value at the spot edge be equal to the switching temperature ψ_*. Again, this condition is similar to that determining the radius of the current channel of the cylindrical arc in a thermal plasma with the stepwise dependence of the conductivity on the temperature. For the model (13), the situation is more complicated and the spot radius is determined from an analysis of the transition region separating the spot and the surrounding current-free area. It should be emphasized once again that in all the cases, both in the theory of the present paper and in the theory of the cylindrical arc column, the radius of the current spot or of the current channel is uniquely determined by the equations; there is no need of the principle of minimum voltage. Note that in the theory of the normal current density effect in the near-cathode region of the glow discharge, which is another traditional area of application of this principle, there is also no need of this principle; in fact, the normal voltage does not coincide with the minimum voltage [18,19].

In all the cases the longitudinal length scale of the spot is inversely proportional to the characteristic heat flux density q_*, in agreement with the estimate given in the Introduction from dimensionality considerations. The reason is that the temperature gradients are inversely proportional to the spot dimension. The integral heat flux is proportional to the spot area times q_* and therefore is inversely proportional to q_*.

To apply the above theory to analysis of the cathode spots of vacuum arcs, the heat flux from the surface into the cathode bulk should be specified. In the present paper for this purpose the simple model is employed, which is in the spirit of the conventional macroscopic approach [1–3] and close to the models used for similar purposes in [20,21].

The function q is written as

$$q = j_i(U + U_i - \varphi_1) - j_e\varphi_1 + \varphi_2(j_i - eC_v) , \quad (41)$$

where j_i is the density of the ion current from the plasma to the cathode surface, j_e is the electron emission current density, U_i is the ionization potential, φ_1 is the effective work function, φ_2 is the vaporization energy per atom divided by the electron charge e, C_v is the flux of the evaporated atoms. j_e is estimated by means of the Richardson-Schottky formula (the cathode surface is hot enough in the considered conditions). C_v is evaluated in terms of the vapor pressure. To obtain an estimate for the ion current from the plasma, it is supposed that the backflow coefficient is unity, i.e., all the atoms evaporated from the surface get ionized and go back to the cathode, then $j_i = eC_v$. The electric-field strength at the cathode surface, which is involved in the Richardson-Schottky formula, is evaluated via the Poisson equation.

The heat flux density described by this model for the copper cathode is shown in Fig. 6. If T is less than some critical value which lies in the range 4000–4500 K and is weakly dependent on U (it slightly decreases as U increases), the first term on the right-hand side of (41), describing the flux of energy delivered by the ions, dominates and q is close to the Arrhenius function, i.e., approximately proportional to $\exp(-\mathrm{const}/T)$. For T near the critical value the second term on the right-hand side of (41), describing the electron emission cooling, becomes important and the dependence of q on T reaches the maximum and then becomes falling. Note that the dependence of q on ψ is of the same character.

If the temperature in the center of the spot were less than the above-mentioned critical value, Eq. (12) would be applicable in the whole temperature range of interest. It follows from Sec. III that in such a case the temperature in the center would be of the order of the activation temperature. However, the latter for the present conditions is around 3.5×10^4 K and far above the critical temperature. Hence, this situation cannot be realized and the temperature in the center should be close to the critical temperature.

To illustrate appearance of the function q in the vicinity of the critical temperature, the data shown in Fig. 6 are transformed to the independent variable ψ, normalized by the values ψ_* and q_* which correspond to the point of maximum of the dependence of q on ψ (or on T), and plotted on the linear scale in Fig. 7. One can see that q_u is comparable to unity only in the narrow range of the values of β and is in the first approximation independent of U. Hence, the function q may be considered as being of the type (13) with ϵ and the form of the function q_u independent of U.

The asymptotic analysis given in Sec. III for the function (13) should be applicable. Furthermore, as the width of the graph in Fig. 7 at the half of the maximum is close to that for the function (14) with $\epsilon = 0.05$, it is expected that the results of the numerical calculations for the latter case are representative also for the function considered here. Then distributions of the temperature and of the heat flux density over the cathode surface are illustrated by the lines 2 and 4 in Fig. 4. In particular, the temperature in the spot is a little higher than the temperature T_* corresponding to the maximum of the dependence of q on T. The radius r_* of the spot may be estimated as $\pi\rho_*\psi_*/2q_*$, where the value of ρ_* again is taken from the numerical solution for the function (14) with $\epsilon = 0.05$ and equals to 1.76. The integral heat flux Q removed by heat conduction is determined by Eq. (18). To calculate the integral current to the spot, we note that the current density $j = j_i + j_e$ may be written for the considered model as

$$j = \frac{j_i(U + U_i) - q}{\varphi_1} . \quad (42)$$

Integrating over the spot surface and supposing for simplicity that the ion current density in the spot is constant and equal to the value j_{i*} corresponding to the maximum of the dependence of q on T, one finds for the integral current

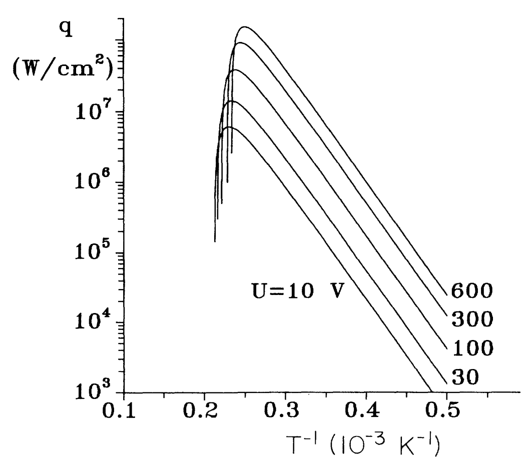

FIG. 6. Heat flux density in the cathode spot of a vacuum arc vs the surface temperature.

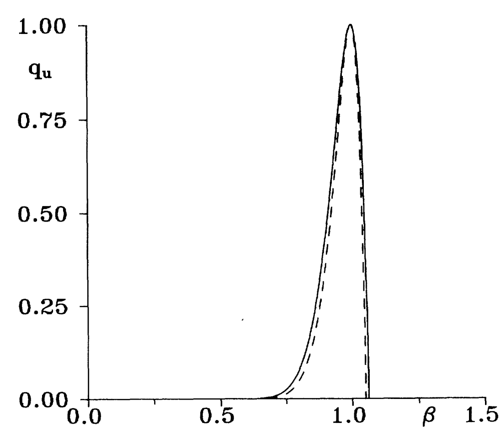

FIG. 7. Normalized heat flux density in the cathode spot of a vacuum arc. $U = 10$ V (solid line), $U = 600$ V (dashed line).

TABLE I. Evaluated parameters of the cathode spots of vacuum arcs.

U (V)	T_* (K)	q_* (MW/cm^2)	ψ_* (kW/cm)	r_* (μm)	I (A)	$\langle j \rangle$ (MA/cm^2)	Q (W)	j_i/j_e	Q_J/Q
10	4340	6	8.05	37	126	3	119	0.59	0.28
15	4320	8.1	8.02	27	87	3.7	88	0.43	0.24
20	4310	10	8.01	22	68	4.5	70	0.33	0.23
30	4285	14	7.97	16	47	6	50	0.23	0.21
60	4245	25	7.92	8.8	26	11	28	0.12	0.21
100	4200	38	7.86	5.7	17	16	18	0.07	0.20
300	4090	91	7.71	2.4	7.5	43	7.2	0.02	0.24
600	4000	151	7.58	1.4	4.7	78	4.2	0.01	0.27

$$I=\pi r_*^2 j_{i*}\frac{U+U_i}{\varphi_1}-\frac{4\psi_* r_*}{\varphi_1} . \tag{43}$$

The results of evaluation for several values of U are given in Table I. Besides the above mentioned, the following parameters are presented: the averaged current density in the spot $\langle j \rangle = I/\pi r_*^2$; the ratio of the ion current to the electron emission current, evaluated for given U and $T=T_*$; the ratio Q_J/Q, where $Q_J=\langle j \rangle^2 r_*^3/\sigma$ characterizes the joule heat production in the cathode bulk [20] (σ is the conductivity of the cathode material evaluated at the temperature in the spot). One can see that the role of the joule heat production is not decisive.

Detailed comparison of the results with some or other specific experimental data is hardly advisable because values of such parameters as the current density reported by experimentalists differ from work to work by orders of magnitude (e.g., [4,5,22,23] and references therein); on the other hand, the above-used simple model for the function q is rather crude. However, we note that the results for the experimentally observed values of U on the level 15–20 V are inside the usual range of characteristics of macrospots.

The current-voltage characteristic of the spot is falling. The increase of the voltage drop is accompanied by the decrease of the integral current. The temperature in the spot and the radius of the spot also decrease. If the analysis of Sec. IV were applicable, one could expect in such a situation that the structure is unstable. However, $\partial q/\partial U$ for some T is negative, which is clearly seen from Fig. 6, and this analysis cannot be applied directly. Thus, the question of stability of the spots requires further investigation.

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APPENDIX

After the substitution $x=\xi/r$ Eq. (7) may be rewritten as

$$\Theta(r)=r\int_0^\infty q(\Theta(rx),U)\frac{dB}{dx}dx , \tag{A1}$$

where

$$B=B(x)=\frac{2}{\pi}\int_0^x \frac{x}{x+1}K\left[\frac{4x}{(x+1)^2}\right]dx . \tag{A2}$$

We differentiate (A1) with respect to r, integrate by parts, and rewrite the result as

$$\frac{d\Theta(r)}{dr}=\int_0^\infty \frac{d\Theta(\xi)}{dr}\frac{\partial q}{\partial\psi}(\Theta(\xi),U)\left\{x\frac{dB}{dx}-B\right\}d\xi . \tag{A3}$$

It may be shown that

$$x\frac{d}{dx}\left\{\frac{2}{\pi}\frac{x}{x+1}K\left[\frac{4x}{(x+1)^2}\right]\right\} = -2\frac{d}{dx}\left[\frac{x}{|x^2-1|^{1/2}}P^1_{-1/2}(\eta)\right] , \tag{A4}$$

where $\eta=(x^2+1)/|x^2-1|$. (To prove this, it is sufficient to express x in terms of η, K and $P^1_{-1/2}$ in terms of $P_{-1/2}$, and to use the differential equation governing the associated Legendre functions.) Integrating (A4) from 0 to x, one can see that the quantity in the curly brackets in (A3) coincides with $\xi W(\xi,r;0,1)$. Hence, $d\Theta/dr$ is the eigenfunction of Eq. (28), corresponding to $\lambda=0$, $\nu=1$.

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