Transport on the percolation backbone

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We investigated numerically the number of sites visited S_N by random walks on the backbone structure of the percolation cluster at the critical threshold. This quantity can be predicted by the scaling conjecture in terms of the fractal and the random-walk dimensions (d_f and d_w). Our results confirm this scaling with time, similar to the critical cluster. The scaling exponent (spectral dimension) is numerically calculated, and it is found to be $d_s^{BB} = 1.23$, while the scaling conjecture predicts a value of 1.19, suggesting that there are uncertainties in the d_f^{BB} and d_w^{BB} values. This value is also smaller (by about 5%) than d_s, the spectral dimension on the full percolation cluster, suggesting that the walk is less efficient on the backbone. Previous estimates of the d_w^{BB} suggested that the walk should be more efficient on the backbone. We investigate this apparent contradiction by calculating and comparing the full distributions of S_N for the backbone and the full percolating cluster. We investigated a few higher moments of this quantity and we found that they exhibit constant-gap scaling, similar to the percolation cluster. The backbone considerations help our understanding of the diffusion on the percolation cluster, especially the contribution of the dangling ends and the ramified parts of the structure, which are so characteristic of percolation at criticality.

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I. INTRODUCTION

The compactness and ramification of lattice clusters at the critical percolation threshold have become well known, especially in the last fifteen years with the renewed interest on the fractal character of the structure and dynamics of such entities [1,2]. A remarkable result is that diffusion slows down at the critical point, as the usual properties scale with exponents distinctly different than the usual lattices [1,2]. This is, of course, due to the distinct geometrical character of a ramified structure, and has been convincingly explained. Additional insight and information can be acquired by examining the diffusion properties on the backbone of the percolation cluster [3–12]. As the name suggests, the backbone is a subset of the percolation cluster formed by removing all dangling ends, branches, etc. that are not necessary for maintaining the connectivity of the structure throughout the lattice. But it spans the whole lattice similarly to the percolation cluster. It has considerably fewer fluctuations, exactly because fluctuations are mainly due to those parts of the percolation cluster that are now removed. If the ends of the lattice are placed on a potential difference, the backbone is the set of bonds that carry current continuously. For this reason, it has been suggested that the backbone model can be used for fluid flow in porous media [12]. Additional applications may include the conductivity of random systems [13,14].

The backbone is a fractal structure. Its fractal dimension has been calculated in the past for two-dimensional lattices as $d_f^{BB} = 1.60$, and it is considerably different than the fractal dimension of the percolation cluster, which is $d_f = 1.895$ [1,2]. This suggests that for increasingly larger lattices the backbone makes up only a small percentage of the original critical cluster, since most of it has been removed. Nevertheless, because of its long-range connectivity it is important to fully explain its dynamics. From random-walk calculations the walk dimension has been estimated to be about $d_w^{BB} = 2.69$ [1]. This is also different than the well-known value of d_w on the percolation cluster, $d_w = 2.87$, and it implies longer end-to-end distances on the backbone. We can predict the value of the spectral dimension on the backbone, d_s^{BB}, by employing the equation connecting the spectral dimension with the fractal and walk dimensions, i.e., $d_s = 2d_f/d_w$, which now becomes $d_s^{BB} = 2d_f^{BB}/d_w^{BB}$. Using the values of $d_f^{BB} = 1.60$ and $d_w^{BB} = 2.69$ we get $d_s^{BB} = 1.19$. This raises the question of the efficiency of the random walk on the backbone. By the term efficiency we mean the ability of the moving particle to get further away from the point of origin, and thus visit more sites. A strict definition of the efficiency ϵ is $\epsilon = S_N/N$, and, thus, one expects that when the end-to-end distance is longer, this would imply a larger ϵ value. The d_w^{BB} value suggests that the walk is more efficient on the backbone than on the percolation cluster. The d_s^{BB} value suggests the opposite, i.e., the walk may be less efficient. In the present work we try to clarify this apparent contradiction by calculating the complete distributions of the monitored quantities, and thus rationalize on this effect. We first calculate numerically the number of sites visited, S_N, in order to verify the scaling conjecture through the relation $S_N \sim N^{d_s/2}$. We also compare our calculated value to the predicted value for d_s^{BB}. Our calculations give a value of $d_s^{BB} = 1.23$, which is slightly higher than predicted.

In the second section we explain the algorithm used

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to construct the backbone, and in the third section we present the results and conclusions.

II. METHOD OF CALCULATION

The construction of the backbone structure is performed following techniques reported earlier in the literature [7]. First, the percolation cluster is made at the critical threshold, $p_c = 0.593$ for the two-dimensional square lattice. Then the largest cluster is isolated using the cluster multiple-labeling technique [15]. Eventually, the backbone is isolated from the percolation cluster by use of the "burning algorithm" [7]. In this method two endpoints P_1 and P_2 are chosen, which must be as far apart as possible in the percolation cluster, on diagonally opposite corners. Then three steps are followed. (a) Starting at point P_1, the cluster "burns" by examining all its neighbor sites and assigning specific index values, which increase by +1 at every step, and which denote the distance of the site from the point of origin P_1. This technique is well known, and it is used as a model for describing forest fires, or a model for finding the nearest-neighbor distance of points in a disordered medium. This step ends once the opposite point P_2 is found and assigned its index value. (b) In the second step, burning starts at P_2 and only those sites can be burned which have a smaller index value than the value of the site burned in the previous time step. This burning ends once point P_1 is found. (c) In the final step, one burns all the loops that have been formed in the previous two steps (loops which have been recorded). Again, as in the second step, a site can only be burned if its index value is smaller than the value of the site burned in the previous time step. For a given loop, it becomes part of the backbone only if the backbone can be reached in more than one way. If it is reached only via one path this means that the loop leads to a dangling end, and it is not included. When reached in more than one way, all the sites burned are added to the growing backbone. This last step must be repeated several times until no more parts can be added to the backbone. Once all three steps are finished the backbone is isolated and complete. Cyclic boundary conditions are used at the ends of the lattice so that if two sites at opposite ends both belong to the backbone they can communicate.

Random walks are performed on the backbone in the usual fashion [1], with the restriction now that the particle remains on the backbone structure. The "blind ant" model is used for time-keeping purposes. The end-to-end distance is monitored as a function of time by keeping track of the particle displacements in the four directions. The number of sites visited is also monitored as in previous studies [1]. It is worth noting that for the values of parameters used in this study most of the computer time is used to produce the backbone structure, while production of the percolation cluster and performing the random walks take up only a small fraction of computer time.

III. RESULTS AND DISCUSSION

The fractal dimension of the backbone is first calculated by use of lattices of several different sizes. We use lattices of sizes from 10×10 up to 250×250. By the standard method of plotting the mass vs linear size in log-log form we find a fractal dimension of $d_f^{\mathrm{BB}} = 1.62$, which is in very good agreement with the reported literature value of $d_f^{\mathrm{BB}} = 1.60 \pm 0.05$ [1]. We do not include details or plots of this calculation, since it was performed only to verify the validity of the algorithm for the construction of the backbone.

We first investigate the behavior of the end-to-end distance as given by the mean-square displacement $\langle R^2 \rangle$ as a function of time, in order to recover the proper scaling exponent values. In Fig. 1 we plot in log-log form the $\langle R^2 \rangle$ vs time, as well as two higher moments. The line for the average value ($q = 1$) produces a slope of 0.74, leading to a random-walk dimension of $d_w^{\mathrm{BB}} = 2.70$, in reasonable agreement with the most recently reported value of 2.69 ± 0.02 [1]. We also estimate our uncertainty in this value to be of the same order of 0.02. This value of 2.70 is about 6% smaller than the $d_w = 2.87$ value for walks on the regular percolation cluster at the critical point. This reduction signifies the trend that the particle "goes further" on the backbone than on the percolation cluster. This is because on the percolation cluster the particle spends time on the dangling bonds, on branches, the dead ends, etc., i.e., on isolated parts of the structure that consume time and restrict the span of the walk. On the other hand, on the backbone, by definition these particular structural patterns do not exist, so that the particle is free to move on the underbedding structure.

The same calculation is given by the S_N property. Figure 2 is the counterpart of Fig. 1. The slope of the first moment, $q = 1$, gives a value of 0.615 leading to a spectral dimension of $d_s^{\mathrm{BB}} = 1.23 \pm 0.02$. This value is about 5% smaller than the spectral dimension for walks on the percolation cluster, which is $d_s = 1.31 \pm 0.02$. It again signifies the different character of the motion; since in the backbone, there are no new sites to visit on the isolated branches, the particle spends more time revisiting the same sites again and again, eventually leading to the reduction of the spectral dimension. We see here that we have two opposite trends, and the natural question to ask is which one predominates. The results from the end-to-end distance show that on the backbone the particle moves further away from the point of origin. This would allow for more new sites to be visited. But the spectral dimension data show that this does not happen. The new

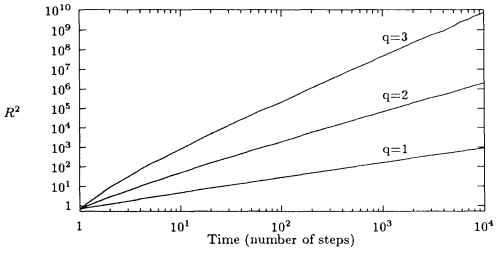

FIG. 1. Mean-square displacement, $\langle R^2 \rangle$, ($q = 1$), and two higher moments vs time for random walks on the backbone structure at the critical percolation threshold $p_c = 0.593$ for two-dimensional lattices in a log-log. The size of the lattice is 250×250. 10 000 realizations were carred out on ten different structures.

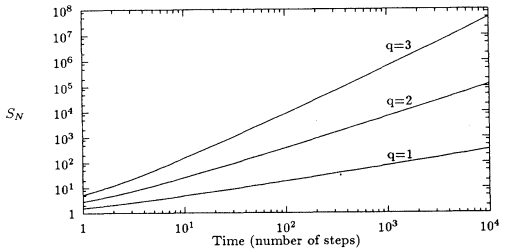

FIG. 2. A plot of the mean number of sites visited, $\langle S_N \rangle$, ($q = 1$) and two higher moments vs time, as in Fig. 1.

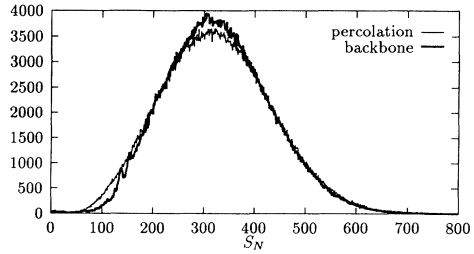

FIG. 4. The probability distribution of the S_N, similarly performed as in Fig. 3.

sites that are visited by the larger span of the walk are compensated by the sites that are on the branches that are now lost, and eventually fewer sites are visited on the backbone than on the percolation cluster.

A point to make is that the values of the scaling exponents are good asymptotically at very long times, as is usually the case. This can be easily seen in Figs. 1 and 2, for the curves $q = 1$. The above-mentioned values correspond to the slope at the interval 1000–10 000 steps. One can see that at earlier times the slope is higher in Fig. 1 and smaller in Fig. 2, as expected. The same trend holds for the higher moments.

To probe this question in more detail, we form the complete distribution of R and S_N and compare them to the equivalent distributions of the percolation cluster. These are given in Figs. 3 and 4. Figure 3 for the R distribution clearly shows that for long R, here $R > 30$, the backbone curve is considerably higher than the percolation curve, leading to a smaller walk dimension. Of course, the maximum is lower in order to normalize to the same value. The spikes close to the maximum value are of no special significance, as they are due to the fact that the R values are computed from the square root of R^2, and in order to form the distribution (binning), with integer R only, we round them off to the nearest integer. This loses some accuracy, especially at low R value, i.e. $R < 30$, causing this spiking.

Figure 4 is the complete distribution of the S_N quantity for the backbone and the full percolation cluster. The results are equivalent to those of Fig. 3. For small S_N values the distribution is higher for the percolation cluster, meaning that for specific small S_N values more walks are confined at the dangling ends than on the backbone. This is compensated by the lower maximum value, so that again the two curves normalize to the same quantity.

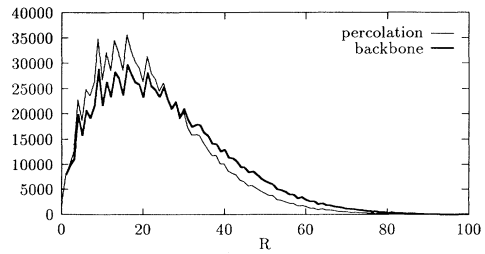

FIG. 3. The complete probability distribution of the distance R, the displacement from the origin, for walks on the percolation cluster at criticality and on the backbone of the percolation cluster. Calculations were done for 10 000 steps, and one million realizations were used for each case (using ten different structures).

Figures 3 and 4 unequivocally suggest that the exponents d_w and d_s are different on the backbone than on the percolation cluster. The difference is small, of the order of 5–6 %, but this is also clearly shown from the distribution of the measured quantities. The interesting observation is that one exponent (slope) is higher while the other one is lower (the R quantity is enhanced while the S_N quantity is reduced). As stated earlier, the predicted d_s^{BB} value is 1.19, while our calculated value is $d_s^{\mathrm{BB}} = 1.23$. The difference is barely within the nominal uncertainties that these quantities carry, which is estimated at ± 0.02 for each quantity. This suggests that there may be a slightly higher uncertainty than suggested by the reported literature values. Some preliminary data [16] from an independent calculation of d_s^{BB}, using density-of-states considerations and solution of the energy equation, give also a value of $d_s^{\mathrm{BB}} = 1.23 \pm 0.02$, in excellent agreement with the present study.

Also of interest is the question of the higher moments of the R^2 and S_N quantities, which are given in Figs. 1 and 2, $q = 2$ and $q = 3$, for the second and third moment, respectively. From Fig. 1 we get a slope of 1.48 ($q = 2$) and 2.21 ($q = 3$), which are obviously 2 and 3 times higher, respectively, than the $q = 1$ slope. From Fig. 2 we get a slope 1.22 ($q = 2$) and 1.83 ($q = 3$), again with the same conclusion. We remark that both these quantities exhibit constant gap scaling for the backbone structure rather than multifractal character. The situation is the same for percolation clusters, and one would not expect drastically different behavior at this point.

Summarizing, we calculated a critical exponent for the S_N quantity on the backbone structure, and found a value of $d_s^{\mathrm{BB}} = 1.23$. This suggests a lower visitation rate as compared to the percolation cluster at criticality, and this is due to the absence of the dangling bonds in the backbone cluster. The comparison with the end-to-end exponents help once again to elucidate the contribution of these characteristic structural parts in the percolation problem.

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