BRIEF REPORTS

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One-parameter family of soliton solutions with compact support in a class of generalized Korteweg–de Vries equations

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We study the generalized Korteweg–de Vries (KdV) equations derivable from the Lagrangian $L(l,p)=\int[\frac{1}{2}\varphi_x\varphi_t-\frac{(\varphi_x)^l}{l(l-1)}+\alpha(\varphi_x)^p(\varphi_{xx})^2]dx$, where the usual fields $u(x,t)$ of the generalized KdV equation are defined by $u(x,t)=\varphi_x(x,t)$. For p an arbitrary continuous parameter $0<p\leq 2$, $l=p+2$ we find soliton solutions with compact support (compactons) to these equations which have the feature that their width is independent of the amplitude. This generalizes previous results which considered $p=1,2$. For the exact compactons we find a relation between the energy, mass, and velocity of the solitons. We show that this relationship can also be obtained using a variational method based on the principle of least action.

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Recently, Cooper *et al.* [1] obtained compacton solutions to a generalized sequence of Korteweg–de Vries (KdV) equations of the form

$$u_t=u_xu^{l-2}+\alpha[2u_{xxx}u^p+4pu^{p-1}u_xu_{xx}+p(p-1)u^{p-2}(u_x)^3] \quad (1)$$

for the case where $l=p+2$ and $0<p\leq 2$. In particular, they obtained compacton solutions (i.e., solitary waves with compact support) for the cases $p=1,2$. In [1], hereafter referred to as I, it was shown that the widths of the compactons were independent of the amplitude and that all the solutions to these equations obeyed the same first three conservation laws as found in the KdV case; namely, area, mass, and energy. In I it was also found that the energy, mass, and velocity of the compactons could be simply related for the integer values of p studied. The purpose of this Brief Report is to extend the work done in I to noninteger values of p. That is, we show that for arbitrary continuous values of p, $0<p\leq 2$, there exist compacton solutions to the above equation. We obtain the explicit expressions for the solitons and determine their mass (M) and energy (E) and show that

$$E=cM/(p+2) \quad (2)$$

where c is the velocity of the soliton. We also consider the class of variational wave functions of the form

$$u_v(x,t)=A(t)\exp\left[-\beta(t)|x-q(t)|^{2n}\right], \quad (3)$$

where n is an arbitrary continuous, real parameter. We find that this class of trial wave functions yields the exact relationship (2) as well as giving an excellent global approximation to the compacton, except at the end points.

Following I, we consider the traveling wave solution to Eq. (1) of the form

$$u(x,t)=f(\xi)=f(x+ct)\ . \quad (4)$$

As shown in I, the function f satisfies for $l=p+2$

$$\alpha f'^2=\frac{c}{2}f^{2-p}-\frac{f^2}{(p+1)(p+2)}. \quad (5)$$

On using the ansatz

$$f(\xi)=\beta\cos^\delta(\gamma\xi) \quad (6)$$

in Eq. (5), we then obtain the following one-continuous-parameter family of compacton solutions:

$$u=\left(\frac{c(p+1)(p+2)}{2}\right)^{1/p}\cos^{2/p}\left(\frac{p\xi}{\sqrt{4\alpha(p+1)(p+2)}}\right), \quad (7)$$

where

$$\left|\frac{p\xi}{\sqrt{4\alpha(p+1)(p+2)}}\right|\leq\frac{\pi}{2}\ . \quad (8)$$

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Note that for all these solutions, the width is independent of the velocity c of the compacton. For $p = 1, 2$ we immediately recover the solutions obtained in I by choosing $\alpha = 1/2, 3$, respectively. The mass M and energy E of these solutions are easily calculated. We find

$$M = \frac{2}{p}(2c)^{2/p}\alpha^{1/2}[(p+1)(p+2)]^{2/p+1/2}\frac{\Gamma^2(2/p+1/2)}{\Gamma(4/p+1)} ,$$

$$E = \frac{1}{p(p+2)}(2c)^{2/p+1}\alpha^{1/2}[(p+1)(p+2)]^{2/p+1/2} \times\frac{\Gamma^2(2/p+1/2)}{\Gamma(4/p+1)} , \quad (9)$$

and hence the relation (2) between E, M, and c follows immediately.

We can also study these solitary waves in the variational approximation defined by Eq. (3) as discussed in I. Following the arguments in I and using $l = p + 2$ we again discover that the soliton width β is independent of the mass M for arbitrary p. We also find that for arbitrary p we exactly satisfy the relationship (2) for any value of the variational parameter n. Minimizing the action gives us the optimum value of n for each p and again we find for the optimal n excellent numerical agreement for the conserved quantities and good agreement for the global compacton profile except near the places where the true compacton goes to zero. As a particular example for the case $p = 4/3$ we find that the exact energy (in units of M) is 0.024 527 7, whereas the variational result is 0.024 417 3, which is obtained at $n = 1.199$. In Fig. 1 we compare the exact and variational expressions for the soliton solutions for the case $M = 1$, and $\alpha = 1$.

The generalized sequence of KdV equations of Rosenau and Hyman [2] are very similar to those of Eq. (1); thus we expect (and also find) that a similar one-continuous-parameter family of compacton solutions also exists in their case when in their notation $m = n$ and $1 < m \leq 3$. The compacton equation of Rosenau and Hyman [2],

$$K(n,m): \; u_t + (u^m)_x + (u^n)_{xxx} = 0, \quad (10)$$

has the following solutions for arbitrary noninteger $m = n$:

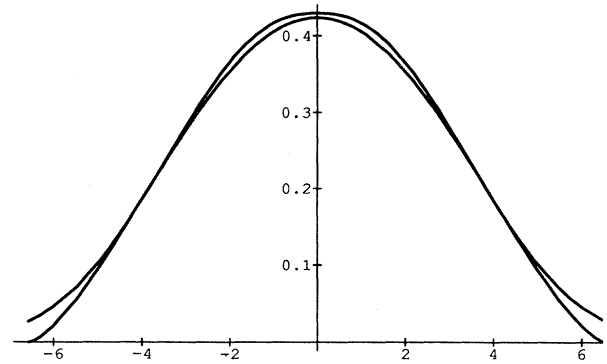

FIG. 1. u_{var} with $n = 1.199$ and u_{exact} for $p = 4/3$, $M = 1$, and $\alpha = 1$.

$$u = \left(\frac{2mc}{m+1}\right)^{1/(m-1)} \cos^{2/(m-1)}\left(\frac{m-1}{2m}\xi\right), \qquad \xi = x - ct. \quad (11)$$

Although the Rosenau-Hyman equation does not have a conserved Hamiltonian associated with a Lagrangian, it admits the two conservation laws:

$$Q_1 = \int u\,dx, \qquad Q_2 = \int u^{n+1}\,dx . \quad (12)$$

It is straightforward to show that these two conserved quantities are related by

$$Q_2 = c\,Q_1^2\, f(m),$$

$$f(m) = 4\left(\frac{m-1}{m+1}\right)\frac{\Gamma^2(\frac{m+1}{m-1})\Gamma^2(\frac{m+1}{m-1}+\frac{1}{2})}{\Gamma^4(\frac{m+1}{2(m-1)})\Gamma(\frac{3m+1}{m-1})} . \quad (13)$$

The Rosenau-Hyman equation also admits two further conservation laws which are discussed in [2].

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[1] F. Cooper, H. Shepard, and P. Sodano, Phys. Rev. E **48**, 4027 (1993).

[2] P. Rosenau and J.M. Hyman, Phys. Rev. Lett. **70**, 564 (1993).