Center-manifold theory for low-frequency excitations in magnetized plasmas

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(Received 14 April 1993)

For the dissipative trapped-ion mode, a simple one-dimensional nonlinear model equation, including the effects of instability, dissipation, and dispersion, is investigated. The center-manifold theory is applied to the situation of more than one marginally stable mode, and the dynamics in the neighborhood of the onset of instability is elucidated. Depending on the (three) relevant parameters, stable solitary waves, mixed modes, heteroclinic orbits, etc., can exist; a scenario for the nonlinear dynamical behavior is developed. The bifurcation diagrams are drawn with quantitative predictions in parameter space. An important conclusion is that the codimension-two analysis utilized can predict successive bifurcations which cannot be captured by simple analysis of one unstable mode. The analytical calculations are checked by numerical simulations.

PACS number(s): 52.35.Qz, 52.35.Mw, 47.20.Ky

I. INTRODUCTION

Many years ago [1,2] a nonlinear model of the collisional trapped-ion mode was published. At that time, nonlinear physics was not yet in a flourishing state, and the mathematics needed for complicated nonlinear dynamics was not as developed as it is now. On this background, the simple-looking one-dimensional model [1,2] was a significant contribution and it is one of the first unstable, dissipative, and dispersive nonlinear systems being proposed for wave-saturation processes in plasma physics. Meanwhile it turned out that a great deal of fundamental physics and mathematics can be learned from this model, which goes far beyond its initial intention.

The model was taken up by Kawahara and Toh [3,4]. They noticed that for vanishing dispersion the Kuramoto-Sivashinski equation [5–8] is obtained. It was shown [9] that chaotic solutions can appear and that the chaos consists of spatially localized structures. This has some interesting consequences since it supports the idea [10] that a few active modes on a low-dimensional chaotic attractor of a partial differential equation are closely related to localized coherent structures and chaos may be described with these structures in a dynamical sense [11]. In the strongly dispersive case, computer simulations showed [3] that the growth of an initial perturbation is followed by formation of a row of solitons, such that the dispersive effect brings a kind of organization into the system that exhibits a turbulentlike behavior if the effect of dispersion is completely neglected. This aspect was studied, mainly numerically, in a series of interesting papers by Kawahara and Toh [12–14], also when damping is included. Two cases are of general interest: (i) solutions near marginal stability and (ii) fully developed nonlinear stages far from the onset of instability. In the latter case (ii), also now, mainly numerical methods are available to find out the dynamical behavior. In addition, techniques related to (approximate) inertial manifolds are getting more important for describing the global attractor (see, for instance, the monograph of Temam [15]). On the other hand, in the first case (i) during the past years very efficient mathematical tools have been developed, one being the center-manifold theory [16]. Applications to partial differential equations are discussed in Lanford [17], Marsden and McCracken [18] and Carr [19]. Here we apply this theory to the problem of saturation of the dissipative trapped-ion mode.

To elucidate the plan and organization of this paper in more detail let us start with the model equation

$$\frac{\partial\phi}{\partial t}+\phi\frac{\partial\phi}{\partial y}+\alpha\frac{\partial^2\phi}{\partial y^2}+\beta\frac{\partial^3\phi}{\partial y^3}+\frac{\partial^4\phi}{\partial y^4}+\nu\phi=0\ . \qquad (1)$$

Here, all quantities are nondimensional and normalized. The variable ϕ measures the drift-wave potential, y is the space coordinate in a slab model, and t is the time. The coefficient α measures the unstable tendency due to electron collisions in comparison to Landau damping $(\partial^4\phi/\partial y^4)$, β is the dispersion coefficient, and ν is a damping coefficient due to ion collisions. The coefficient of the Landau term is normalized to one by appropriate scaling of time t and potential ϕ. All coefficients α, β, and ν are non-negative. Note that the dissipative trapped-ion mode is a drift wave driven unstable by electron collisions. The wave is stabilized by ion collisional damping and Landau damping due to both circulating and trapped ions. In a series of papers [20–23] the physics of this mode and its relevance for fusion devices was discussed. Therefore we concentrate here more on its mathematical properties which follow from Eq. (1). Note also that the model (1) has more practical applications in other disciplines [24,25].

An insight into the interesting behavior of (1) can be obtained from the linear dispersion relation, i.e., after Fourier transforming the linearized equation (1) with $\phi\equiv 0$ as the equilibrium solution:

$$\omega=-\beta k_y^3+i\left(-k_y^4+\alpha k_y^2-\nu\right)\ . \qquad (2)$$

Here and in the following we take a unit cell of length

2π with periodic boundary conditions. Linear growth can occur for small wave numbers k_y. A typical dependence of the linear growth (or damping) rate $\gamma := \mathrm{Im}\,\omega$ is shown in Fig. 1 for $\alpha = 5.25$ and $\nu = 3.8$. Small-amplitude harmonic waves are growing for long wavelengths and damped for short wavelengths. The energy input due to self-excitation at small k_y values is transferred through mode coupling to small wavelengths and is expected to be balanced by damping because of fourth-derivative dissipation [3]. The existence of both, instability and dissipation, together with nonlinearity and dispersion, indicates the possibility of existence of rows of stable pulses.

Here we are less interested in the variety of stationary solutions far from the onset of instability. Kawahara and Toh [3,4,14] have already shown in the strongly dispersive case that numerically a row of pulses appears. Approximately, these pulses can be found by perturbation theory when the Korteweg–de Vries (KdV) terms are considered to be dominant. A stability analysis further shows that rows of pulses are stable when the distances between the pulses are not too large. Maxima and minima of the pulse amplitudes were discussed and their dependences on ν have been reported [12]. In the dispersionless case (we have already mentioned its relation to the Kuramoto-Sivashinski model) two-modes equilibria and chaotic behavior have been demonstrated.

All the numerical investigations mentioned above cannot, of course, claim completeness. Thus, for the latter a systematic analytical procedure is necessary, which on the other hand, when mathematical rigor is demanded, will be practicable only in a small parameter regime. But nevertheless we can address (and answer) the following questions: (i) How do we understand the appearance of chaos or coherence in the case $\beta \equiv 0$ (dispersionless)? (ii) Do we expect also chaos for large β? (iii) Are low-dimensional models (ODE's) available which mimic correctly the dynamical behavior of the infinite-dimensional (PDE) system (1)? In contrast to previous investigations we shall attack these problems close to the onset of instability for the cases when more (than one) unstable modes are excited.

The paper is organized as follows. In Sec. II we develop the general method and derive the amplitude equations. The dispersionless case ($\beta \equiv 0$) is treated in Sec. III whereas in Sec. IV specific aspects of the dispersion ($\beta \neq 0$) are discussed. The paper is concluded by a short summary and discussion in Sec. V.

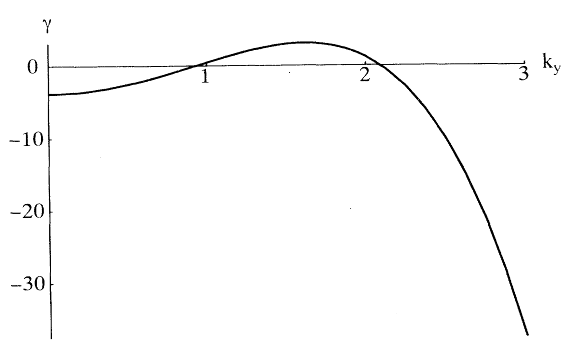

FIG. 1. The linear growth rate $\mathrm{Im}\,\omega$ vs k_y which follows from (2) when $\alpha = 5.25$ and $\nu = 3.8$.

II. CENTER MANIFOLD THEORY FOR TWO MARGINALLY STABLE WAVE NUMBERS

From the dispersion relation (2) [and Fig. 1] we can recognize that many interesting cases are possible when the stability of the $\phi \equiv 0$ solution is considered: No, one, a few, or many modes can be unstable. The first case is trivial, the second one has either been done [14] (for some special cases), or will follow from our calculations; the case with two unstable k_y values is already highly nontrivial and will be investigated now. As a by-product we shall gain also some *new insight into the often treated case of only one unstable mode. Whereas an analysis of one unstable mode can capture correctly only the first bifurcation, the analysis of more than one marginally stable mode allows the calculation of successive bifurcations when one distinct parameter is varied and the others are kept fixed.* This is an important conclusion which follows from the rather sophisticated mathematical treatment of the more general case. We come back to this point at the ends of Secs. III and IV.

The k_y values in the dispersion relation (2) can take the values $k_y = 0, 1, 2, \ldots$ because of periodic boundary conditions with unit cells of lengths 2π. Let us now choose $\alpha = \alpha_c \equiv 5$ and $\nu = \nu_c \equiv 4$. Then the modes

$$\phi^{(1)} = \sin y\,, \quad \phi^{(2)} = \cos y\,,$$
$$\phi^{(3)} = \sin 2y\,, \quad \phi^{(4)} = \cos 2y\,, \tag{3}$$

belonging to $k_y = 1$ and $k_y = 2$, respectively, are marginally stable. Damping occurs for $k_y = 0$ and $k_y \geq 3$.

We can introduce the four (real) amplitudes a_1, a_2, a_3, and a_4 for the four modes (3). In addition, we define

$$a_5 := \alpha - \alpha_c\,, \quad a_6 := \nu - \nu_c\,. \tag{4}$$

In the following we treat β as a fixed parameter. The aim is to derive a closed set of nonlinear amplitude equations

$$\dot{a}_n = f_n(a_1, \ldots, a_6)\,, \quad n = 1, \ldots, 6\,, \tag{5}$$

which are valid in the neighborhood of the critical point α_c, ν_c. [Note that we shall derive ordinary differential equations (ODE's) and we abbreviate in the following $\partial a_n/\partial t \equiv da_n/dt \equiv \dot{a}_n$.] We have $f_5 \equiv f_6 \equiv 0$. The other functions f_n are written as power series in a_n,

$$f_n = \sum_{1 \leq m \leq 6} A_n^m a_m + \sum_{1 \leq m \leq p \leq 6} A_n^{mp} a_m a_p + \cdots\,. \tag{6}$$

The justification for this procedure is the center-manifold theorem for flows [18] which proves that close to the critical point all trajectories converge to curves on the center manifold. A short summary of the ideas used here is presented in the Appendix.

Let us now proceed with the more or less technical steps. The dynamics on the center manifold is characterized by $a_1, ..., a_6$. Thus we can make the ansatz [26], which is a direct generalization of the finite dimensional case (see, e.g., Carr [19]),

$$\phi(y,t) = \sum_{1\le n\le 4} a_n(t)\phi^{(n)}(y) + \sum_{1\le n\le m\le 6} a_n(t)a_m(t)\phi^{(nm)}(y) + \cdots , \tag{7}$$

where the $\left[\binom{7}{2} = 21\right]$ new functions $\phi^{(nm)}$ and, of course, the next $\left[\binom{8}{3} = 56\right]$ functions $\phi^{(nmp)}$, and so on, can be chosen orthogonal to $\phi^{(n)}$, $n = 1, ..., 4$.

Introducing (7) into the PDE (1), we obtain to lowest order (a_n)

$$\sum_{1\le n\le 4}\left(\sum_{1\le m\le 6} A_n^m a_m \phi^{(n)} + \beta a_n \frac{d^3\phi^{(n)}}{dy^3}\right) = 0 . \tag{8}$$

Here we have used Eqs. (5) and (6) to eliminate the time derivatives. The 24 coefficients A_n^m are zero except

$$A_1^2 = -\beta , \quad A_2^1 = \beta , \quad A_3^4 = -8\beta , \quad A_4^3 = 8\beta . \tag{9}$$

Introducing

$$\mathfrak{L} := -\alpha_c \frac{d^2}{dy^2} - \beta\frac{d^3}{dy^3} - \frac{d^4}{dy^4} - \nu_c , \tag{10}$$

the next order $(a_n a_m)$ yields in a similar way

$$\begin{aligned}\sum_{1\le n\le m\le 6}&\left[a_n a_m \mathfrak{L}\phi^{(nm)} - \sum_{1\le p\le 6} a_p\left(A_n^p a_m + A_m^p a_n\right)\phi^{(nm)}\right] \\ &= \sum_{1\le m\le 4}\left[\sum_{1\le n\le p\le 6} A_m^{np} a_n a_p \phi^{(m)} + \sum_{1\le n\le 4} a_m a_n \phi^{(m)}\frac{d\phi^{(n)}}{dy} + a_m\left(a_5\frac{d^2\phi^{(m)}}{dy^2} + a_6\phi^{(m)}\right)\right] .\end{aligned} \tag{11}$$

In this equation we have to collect equal powers $a_r a_s$; the "coefficients" (being equated to zero) will determine the unknown functions $\phi^{(nm)}$ via ODE's. The latter contain the differential operator $\mathfrak{L}$. Taking into account the (periodic) boundary conditions, we have to satisfy the solvability conditions following from

$$\int_{-\pi}^{\pi}\phi^{(m)}\mathfrak{L}\phi^{(np)}dy = \int_{-\pi}^{\pi}\left(\mathfrak{L}^+\phi^{(m)}\right)\phi^{(np)}dy = 0 \tag{12}$$

for $m = 1, ..., 4$. Because of the orthogonality of the functions $\phi^{(m)}$ to all the functions $\phi^{(np)}$, the left-hand side of Eq. (11) will not contribute and the right-hand side of Eq. (11) leads to

$$\begin{aligned}0 = &\sum_{1\le n\le p\le 6}\pi A_r^{np} a_n a_p \\ &+ \sum_{1\le m\le 4}\sum_{1\le n\le 4} a_m a_n \int_{-\pi}^{\pi}\phi^{(r)}\phi^{(m)}\frac{d\phi^{(n)}}{dy}dy \\ &+ \sum_{1\le m\le 4} a_m \int_{-\pi}^{\pi}\phi^{(r)}\left(a_5\frac{d^2\phi^{(m)}}{dy^2} + a_6\phi^{(m)}\right)dy\end{aligned} \tag{13}$$

for $1 \le r \le 4$. Collecting equal products of the amplitudes a_n we find the solutions for the coefficients A_r^{np}. With these values we can solve for $\phi^{(mn)}$. This procedure can be continued, e.g., in the next order we obtain

$$\begin{aligned}0 = &\sum_{1\le n\le p\le q\le 6}\pi A_r^{npq} a_n a_p a_q \\ &+ \sum_{1\le m\le 4}\sum_{1\le n\le p\le 6} a_m a_n a_p \\ &\times\int_{-\pi}^{\pi}\phi^{(r)}\frac{d}{dy}\left(\phi^{(m)}\phi^{(np)}\right)dy ,\end{aligned} \tag{14}$$

and so on. Actually, when written explicitly we face a huge amount of work [in second order we have to solve for 84, in third order for 224 coefficients $A_r^{\cdots}$, and so on]. Thus one should try to simplify by making use of symmetries.

Translational invariance of Eq. (1) allows us to shorten the procedure considerably. If $\phi(y)$ is a solution of Eq. (1), also

$$\mathfrak{T}_{y_0}\phi(y) := \phi(y + y_0) \tag{15}$$

will fulfill (1), where y_0 is a real shift parameter. For $\phi(y)$ we made the ansatz (7), where the modes $\phi^{(nm)}$, $\phi^{(nmp)}$, and so on have to be determined from inhomogeneous differential equations, e.g., Eq. (11) for $\phi^{(nm)}$. The inhomogeneities contain (in nonlinear forms) the marginal modes $\phi^{(r)}$, $r = 1, ..., 4$. Thus the so-called slaved modes can be written in symbolic forms as

$$\phi^{(m\ldots)} = \mathfrak{h}^{(m\ldots)}\left[\{\phi^{(r)}\}\right] , \tag{16}$$

e.g., $\phi^{(mn)}$ as solutions of (11). Because of translational invariance the following symmetry should hold:

$$\mathfrak{T}_{y_0}\mathfrak{h}^{(m\ldots)}\left[\{\phi^{(r)}\}\right] = \mathfrak{h}^{(m\ldots)}\left[\{\mathfrak{T}_{y_0}\phi^{(r)}\}\right] . \tag{17}$$

To determine the consequences of this symmetry let us combine the marginal modes to

$$\varphi := \sum_{r=1}^{4} a_r\phi^{(r)} \equiv \mathrm{Im}\left(c_1 e^{iy} + c_2 e^{i2y}\right) \tag{18}$$

with the complex amplitudes $c_1 := a_1 + ia_2$, $c_2 := a_3 + ia_4$.

The (complex) amplitude equations [compare with (5)] are

$$\dot{c}_n = g_n\left(c_1, c_2, a_5, a_6\right) \;,\;\; n \,=\, 1,2 \;,$$
$$\dot{a}_m = 0 \;,\;\; m \,=\, 5,6 \;, \tag{19}$$

and again, close to the marginal point ($a_5 = a_6 = 0$) the functions g_1 and g_2 can be written as power series. Without making use of the symmetry we have

$$g_n = \sum_{p,q,r,s} B_n^{pqrs}\; c_1^p \bar{c}_1^q c_2^r \bar{c}_2^s \;,\; n = 1,2 \;, \tag{20}$$

where the bar denotes the complex conjugate and each $p, q, r,$ or s can be 0,1,2,.... Now using the fact

$$\mathfrak{T}_{y_0}\varphi = \operatorname{Im}\;\left(e^{iy_o}c_1 e^{iy} + e^{i2y_o}c_2 e^{i2y}\right) \;, \tag{21}$$

we obtain from (17) [in its equivalent form for the complex formulation]

$$e^{iny_0} g_n\left(c_1, c_2, a_5, a_6\right) = g_n\left(e^{iy_0}c_1, e^{i2y_0}c_2, a_5, a_6\right) \tag{22}$$

for $n = 1, 2$. The vector field (g_1, g_2) is thus equivariant with respect to the operation

$$(c_1, c_2) \longrightarrow \left(e^{iy_o}c_1, e^{i2y_o}c_2\right) \;. \tag{23}$$

Now, the most general vector fields being equivariant under the operation (23) can be written in the form [26]

$$(g_1, g_2) = \left(c_1 P_1 + \bar{c}_1 c_2 Q_1, c_2 P_2 + c_1^2 Q_2\right) \;, \tag{24}$$

where $P_1, P_2, Q_1,$ and Q_2 are polynomials in $|c_1|^2$, $|c_2|^2$, and Re $\left(c_1^2\bar{c}_2\right)$; of course they can also depend on a_5 and a_6.

Let us now proceed up to cubic order in the amplitudes. The general form (19) then reduces to

$$\dot{c}_1 = \lambda c_1 + \mathcal{A}\bar{c}_1 c_2 + \mathcal{C}c_1\left|c_1\right|^2 + \mathcal{E}c_1\left|c_2\right|^2 + O\left(|c|^4\right) ,$$
$$\dot{c}_2 = \mu c_2 + \mathcal{B}c_1^2 + \mathcal{D}c_2\left|c_1\right|^2 + \mathcal{F}c_2\left|c_2\right|^2 + O\left(|c|^4\right) ,$$
$$\dot{a}_5 = \dot{a}_6 \;=\; 0 \;. \tag{25}$$

As mentioned already, the coefficients $\lambda, \mu, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E},$ and $\mathcal{F}$, in general, depend on a_5 and a_6. However, it turns out that it is sufficient to take into account only the following behaviors: (i) linear dependences of λ and μ on a_5 and a_6 and (ii) evaluation of the other coefficients at $a_5 = a_6 = 0$. These statements imply a scaling which becomes obvious from the final equations.

Now comparing the system (25) with the set of equations (5) and (6), and making use of the relation (18) we arrive at the following important simplifying expressions:

$$\begin{aligned}
\operatorname{Re}\lambda &= A_1^1 + A_1^{15}a_5 + A_1^{16}a_6 \;=\; A_2^2 + A_2^{25}a_5 + A_2^{26}a_6 \;,\\
\operatorname{Im}\lambda &= A_2^1 + A_2^{15}a_5 + A_2^{16}a_6 \;=\; -A_1^2 - A_1^{25}a_5 - A_1^{26}a_6 \;,\\
\operatorname{Re}\mu &= A_3^3 + A_3^{35}a_5 + A_3^{36}a_6 \;=\; A_4^4 + A_4^{45}a_5 + A_4^{46}a_6 \;,\\
\operatorname{Im}\mu &= A_4^3 + A_4^{35}a_5 + A_4^{36}a_6 \;=\; -A_3^4 - A_3^{45}a_5 - A_3^{46}a_6 \;,\\
\operatorname{Re}\mathcal{A} &= A_1^{13} = A_1^{24} = A_2^{14} = -A_2^{23} \;,\\
\operatorname{Im}\mathcal{A} &= A_2^{13} = A_2^{24} = A_1^{23} = -A_1^{14} \;,\\
\operatorname{Re}\mathcal{B} &= A_3^{11} = \tfrac{1}{2}A_4^{12} = -A_3^{22} \;,\\
\operatorname{Im}\mathcal{B} &= A_4^{11} = -\tfrac{1}{2}A_3^{12} = -A_4^{22} \;,\\
\operatorname{Re}\mathcal{C} &= A_1^{111} = A_1^{122} = A_2^{112} = A_2^{222} \;,\\
\operatorname{Im}\mathcal{C} &= A_2^{111} = A_2^{122} = -A_1^{112} = -A_1^{222} \;,\\
\operatorname{Re}\mathcal{D} &= A_3^{113} = A_3^{223} = A_4^{114} = A_4^{224} \;,\\
\operatorname{Im}\mathcal{D} &= A_4^{113} = A_4^{223} = -A_3^{114} = -A_3^{224} \;,\\
\operatorname{Re}\mathcal{E} &= A_1^{133} = A_1^{144} = A_2^{233} = A_2^{244} \;,\\
\operatorname{Im}\mathcal{E} &= A_2^{133} = A_2^{144} = -A_1^{233} = -A_1^{244} \;,\\
\operatorname{Re}\mathcal{F} &= A_3^{333} = A_3^{344} = A_4^{334} = A_4^{444} \;,\\
\operatorname{Im}\mathcal{F} &= A_4^{333} = A_4^{344} = -A_3^{334} = -A_3^{444} \;.
\end{aligned} \tag{26}$$

Note that the coefficients A_n^m have already been evaluated; see Eq. (9). Previously we had to determine ($4 \times 21=$) 84 coefficients A_m^{rs} (since $A_m^{rs} = A_m^{sr}$) for $m = 1, ..., 4$ and $r, s = 1, ..., 6$. Now it is sufficient to determine 12 independent ones (in second order). With respect to the coefficients A_m^{rsp} we have reduced the amount from 224 to 8. This is a significant simplification.

A straightforward analysis leads to

$$\begin{aligned}
&\lambda = a_5 - a_6 + i\beta \;,\;\; &&\mu = 4a_5 - a_6 + i8\beta \;,\\
&\mathcal{A} = \tfrac{1}{2} \;, &&\mathcal{B} = -\tfrac{1}{2} \;,\\
&\mathcal{C} = 0 \;, &&\mathcal{D} = -\tfrac{3}{4(20 - i9\beta)} \;,\\
&\mathcal{E} = \tfrac{1}{2}\mathcal{D} \;, &&\mathcal{F} = -\tfrac{1}{12(15 - i4\beta)} \;.
\end{aligned} \tag{27}$$

Combining all these results we obtain the relevant amplitude equations (correct up to the cubic order $|a|^3$),

$$\begin{aligned}
\dot{a}_1 &= (\operatorname{Re}\lambda)\, a_1 - (\operatorname{Im}\lambda)\, a_2 + (\operatorname{Re}\mathcal{A})\left[a_1 a_3 + a_2 a_4\right]\\
&\quad + (\operatorname{Re}\mathcal{E})\, a_1\left[a_3^2 + a_4^2\right] - (\operatorname{Im}\mathcal{E})\, a_2\left[a_3^2 + a_4^2\right] \;,\\
\dot{a}_2 &= (\operatorname{Re}\lambda)\, a_2 + (\operatorname{Im}\lambda)\, a_1 + (\operatorname{Re}\mathcal{A})\left[a_1 a_4 - a_2 a_3\right]\\
&\quad + (\operatorname{Re}\mathcal{E})\, a_2\left[a_3^2 + a_4^2\right] + (\operatorname{Im}\mathcal{E})\, a_1\left[a_3^2 + a_4^2\right] \;,\\
\dot{a}_3 &= (\operatorname{Re}\mu)\, a_3 - (\operatorname{Im}\mu)\, a_4 + (\operatorname{Re}\mathcal{B})\left[a_1^2 - a_2^2\right]\\
&\quad + (\operatorname{Re}\mathcal{D})\, a_3\left[a_1^2 + a_2^2\right] - (\operatorname{Im}\mathcal{D})\, a_4\left[a_1^2 + a_2^2\right]\\
&\quad + (\operatorname{Re}\mathcal{F})\, a_3\left[a_3^2 + a_4^2\right] - (\operatorname{Im}\mathcal{F})\, a_4\left[a_3^2 + a_4^2\right] \;,\\
\dot{a}_4 &= (\operatorname{Re}\mu)\, a_4 + (\operatorname{Im}\mu)\, a_3 + 2\,(\operatorname{Re}\mathcal{B})\, a_1 a_2\\
&\quad + (\operatorname{Re}\mathcal{D})\, a_4\left[a_1^2 + a_2^2\right] + (\operatorname{Im}\mathcal{D})\, a_3\left[a_1^2 + a_2^2\right]\\
&\quad + (\operatorname{Re}\mathcal{F})\, a_4\left[a_3^2 + a_4^2\right] + (\operatorname{Im}\mathcal{F})\, a_3\left[a_3^2 + a_4^2\right] \;.
\end{aligned} \tag{28}$$

These are the basic equations for the marginally stable modes. They show that all are coupled, and we have to discuss in detail the consequences.

III. DISPERSIONLESS CASE ($\beta \equiv 0$)

For $\beta \equiv 0$ the problem simplifies considerably. Due to the mirror symmetry $\phi(y) = \phi(-y)$, all coefficients (27) become real, and the amplitude equations are

$$\begin{aligned}
\dot{a}_1 &= \lambda a_1 + \mathcal{A}\left[a_1 a_3 + a_2 a_4\right] + \mathcal{E} a_1 \left[a_3^2 + a_4^2\right] \ , \\
\dot{a}_2 &= \lambda a_2 + \mathcal{A}\left[a_1 a_4 - a_2 a_3\right] + \mathcal{E} a_2 \left[a_3^2 + a_4^2\right] \ , \\
\dot{a}_3 &= \mu a_3 + \mathcal{B}\left[a_1^2 - a_2^2\right] + \mathcal{D} a_3 \left[a_1^2 + a_2^2\right] + \mathcal{F} a_3 \left[a_3^2 + a_4^2\right], \\
\dot{a}_4 &= \mu a_4 + 2\mathcal{B} a_1 a_2 + \mathcal{D} a_4 \left[a_1^2 + a_2^2\right] + \mathcal{F} a_4 \left[a_3^2 + a_4^2\right] \ .
\end{aligned} \tag{29}$$

Similar equations have been investigated for different physical situations [26,27]. At $(a_1, a_2, a_3, a_4, \lambda, \mu) = (0,0,0,0,0,0)$ we can have a codimension-two bifurcation, i.e., the bifurcation diagram has to be drawn in two-dimensional parameter space. In Fig. 2, we have shown the eight possible solutions in their relevant parameter regions. First, the region $\lambda, \mu < 0$ is trivial. Here, all the modes are damped and we have the simple solution (I) $a_1 = a_2 = a_3 = a_4 = 0$. Already the next case (II) is interesting. For $\mu > 0$ but $\lambda < 0$, the mode with $k_y = 2$ is linearly unstable, whereas the mode $k_y = 1$ is linearly damped. Saturation occurs for

$$\begin{aligned}
a_1 = 0 \ , &\quad a_2 = 0 \ , \\
a_3 = \cos(\vartheta) \left(-\frac{\mu}{\mathcal{F}}\right)^{1/2} \ , &\quad a_4 = \sin(\vartheta) \left(-\frac{\mu}{\mathcal{F}}\right)^{1/2} \ .
\end{aligned} \tag{30}$$

This is what would be expected already from an analysis of one unstable mode. The solution (30) is linearly stable in the region II. This can be demonstrated by linearizing the equations (29) with respect to the stationary solution (30). The condition for stability (i.e., the region II) is

$$\lambda < \frac{\mathcal{E}}{\mathcal{F}}\mu - \mathcal{A}\left(-\frac{\mu}{\mathcal{F}}\right)^{1/2} \ . \tag{31}$$

If, for example, we would have reduced (29) by the *ad hoc* ansatz $a_1 \equiv a_2 \equiv 0$ for $\lambda < 0$, as it is often done

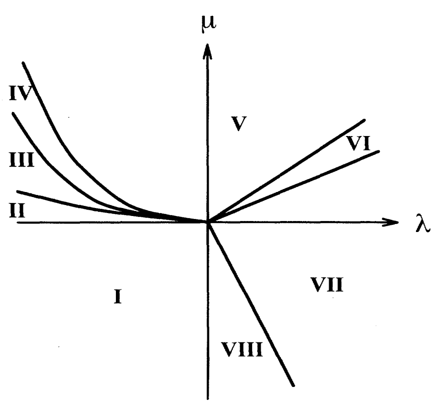

FIG. 2. Existence and stability diagram of eight possible solutions in the dispersionless case $\beta \equiv 0$. In (I) only the trivial solutions exists, whereas the other areas are the stability regions of a simple mode (II), a mixed mode (III), a standing wave (IV), chaos (V), a modulated wave (VI), a traveling wave (VII), and a mixed mode (VIII).

incorrectly, the stability region of (30) would extend to the whole region $\lambda < 0$ and $\mu > 0$. However, the present calculation shows that stability of the simple mode (30) is only true if the growth (at $k_y = 2$) is weaker than the (heavy) damping of the neighboring mode ($k_y = 1$). The borderline is given by (31).

When the simple mode (30) becomes unstable, a quite interesting scenario occurs. In the region III a mixed mode ($k_y = 2$ and $k_y = 1$) appears, which for larger μ values becomes unstable. The new stable state is a standing wave solution (IV). Particularly interesting is the region V where chaos is possible. Before discussing the latter let us have a look on the symmetry between the $k_y = 1$ and $k_y = 2$ modes. In the region VIII the mode $k_y = 1$ is unstable whereas the mode $k_y = 2$ is stable. However, the situation is not the same as in region II (of course with $k_y = 2$ and $k_y = 1$ interchanged). A single complex amplitude equation for the $k_y = 1$ mode is not appropriate in all of region VIII. That means that the nonlinear description of an adjacent damped mode has to be included to find the appropriate saturation. In the bifurcation diagram, Fig. 2, this fact is expressed by the observation that in region VIII a mixed mode ($k_y = 1$ *and* $k_y = 2$) appears. Thus region VIII is comparable to region III where a similar situation occurs. The differences in the nonlinear treatments of the two wave number modes $k_y = 1$ and $k_y = 2$ originates from the asymmetry in the growth rate curve Fig. 1. In the (λ, μ) plane there exists a transition from the mixed mode (VIII) via a traveling wave (VII) and a modulated wave (VI) to chaos (V).

Let us now discuss the chaotic region V in more detail. When the simple modes (30) are hyperbolic fixed points, a heteroclinic connection can occur between two solutions being oppositely situated on the "circle" with radius $|c_2|^2 = -\mu/\mathcal{F}$. Let us consider the specific situation

$$(a_1, a_2, a_3, a_4)_\pm \equiv \left(0, 0, \pm\sqrt{-\frac{\mu}{\mathcal{F}}}, 0\right) \ , \tag{32}$$

which is obtained from (30) for $\vartheta = 0$ and π. If we linearize (29) with respect to (32), we find the eigenvalues

$$\begin{aligned}
\eta_1 = \lambda - \frac{\mathcal{E}}{\mathcal{F}}\mu + \mathcal{A}\sqrt{-\frac{\mu}{\mathcal{F}}} \ , &\quad \eta_2 = \lambda - \frac{\mathcal{E}}{\mathcal{F}}\mu - \mathcal{A}\sqrt{-\frac{\mu}{\mathcal{F}}} \ , \\
\eta_3 = -2\mu \ , &\quad \eta_4 = 0 \ .
\end{aligned} \tag{33}$$

Armbruster, Guckenheimer, and Holmes [27] show for a similar case that for $\eta_1 > 0 > \eta_2$ a heteroclinic connection exists between the (one-dimensional) unstable manifold of $(a_1, a_2, a_3, a_4)_+$ and the (two-dimensional) stable manifold of $(a_1, a_2, a_3, a_4)_-$, on a way lying first totally in the (a_1, a_3) plane and going back in the (a_2, a_3) plane. A sketch of this path is shown in Fig. 3. For the PDE (1) it means that the solution oscillates (randomly in time) between the two $k_y = 2$ states (30) with $\vartheta = 0$ and $\vartheta = \pi$, respectively. A typical result is shown in Fig. 4. The same behavior is obtained from the simplified ODE model (29). We demonstrate this by plotting the solution of (29) in the (a_1, a_2, a_3) space [see Fig. 5(a)] and com-

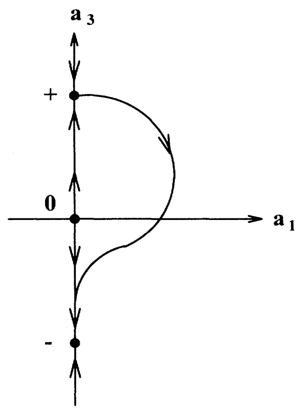

FIG. 3. Sketch of the heteroclinic orbit connecting the $\pm$ states (32).

pare with the appropriate projection of the PDE solution obtained from direct numerical simulation; see Fig. 5(b). As one can see, the agreement is excellent.

In closing this section we want to emphasize that for the dispersionless case $\beta \equiv 0$ the center-manifold theory provides us with an efficient and elegant method to catalog the complicated behavior of the PDE which otherwise would be difficult to disclose.

IV. BIFURCATION PROPERTIES WITH DISPERSION ($\beta \neq 0$)

We have seen that in the dispersionless ($\beta \equiv 0$) case the system (28) simplifies considerably and it becomes tractable. For $\beta \neq 0$ we intend to rewrite (28) in a more transparent variant which allows some general conclusions. We shall try to find coordinate transformations to get the system in normal form which is as simple as possible. For the general method we refer to the excellent and comprehensive presentation of Arnold [28]. In the present case one can prove that the second order terms on the right-hand side of the system (25) can be removed while the third order terms always survive. Instead of showing the details of the strategy to remove the irrelevant terms we present the result in the form of the nonlinear transformation,

$$\tilde{c}_1 := c_1 - \frac{(\mathrm{Re}\,\mathcal{A})}{(\mathrm{Re}\,\mu) + i6\beta}\bar{c}_1 c_2 ,$$
$$\tilde{c}_2 := c_2 + \frac{(\mathrm{Re}\,\mathcal{B})}{(\mathrm{Re}\,\mu) - 2(\mathrm{Re}\,\lambda) + i6\beta}c_1^2 . \tag{34}$$

When using the new variables $\tilde{c}_1$ and $\tilde{c}_2$ instead of c_1 and c_2, the first two equations of (25) read

$$\dot{\tilde{c}}_1 = \lambda\tilde{c}_1 + \tilde{\mathcal{C}}\tilde{c}_1|\tilde{c}_1|^2 + \tilde{\mathcal{E}}\tilde{c}_1|\tilde{c}_2|^2 + O\left(|\tilde{c}|^4\right) ,$$
$$\dot{\tilde{c}}_2 = \mu\tilde{c}_2 + \tilde{\mathcal{D}}\tilde{c}_2|\tilde{c}_1|^2 + \tilde{\mathcal{F}}\tilde{c}_2|\tilde{c}_2|^2 + O\left(|\tilde{c}|^4\right) ; \tag{35}$$

the equations $\dot{a}_5 = \dot{a}_6 = 0$ are unchanged. The new coefficients are

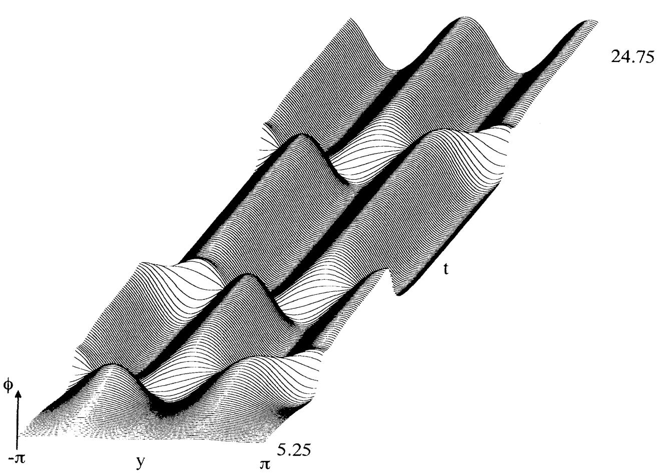

FIG. 4. Space-time plot of ϕ as a solution of Eq. (1) when we are in the region V of Fig. 2 for $\alpha = 5.25$, $\nu = 3.8$, and $\beta = 0$. Note the random oscillations in time between oppositely phased $k_y = 2$ modes.

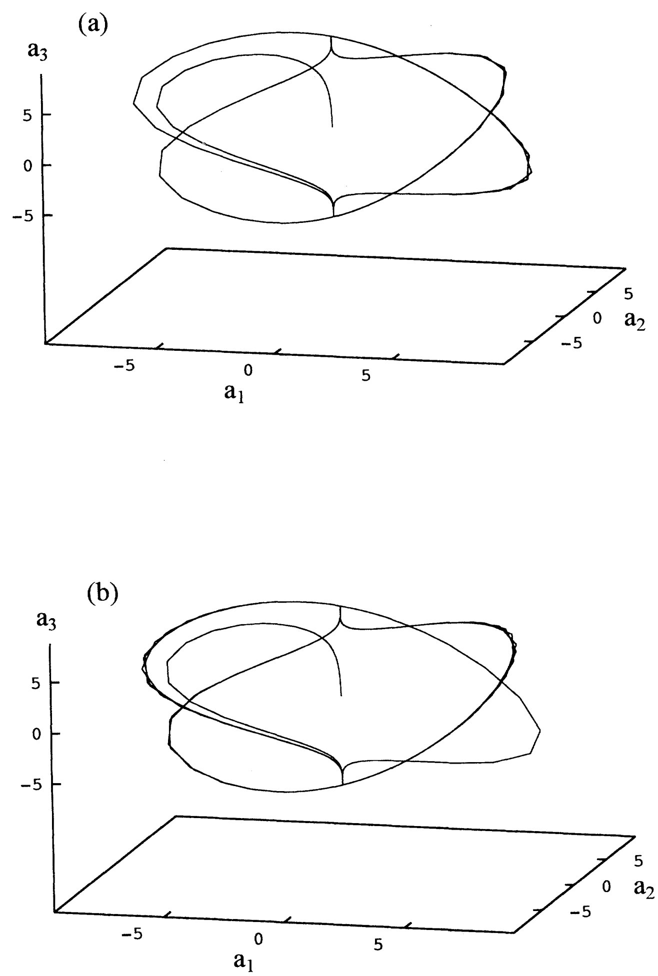

FIG. 5. (a) Trajectory of the solution of the ODE's (29) for the same parameters as in Fig. 4. (b) The corresponding projection of the solution of the PDE (1).

$$\tilde{\mathcal{C}} := -\frac{(\mathrm{Re}\,\mathcal{A})(\mathrm{Re}\,\mathcal{B})}{(\mathrm{Re}\,\mu) + i6\beta}\,,$$
$$\tilde{\mathcal{D}} := \mathcal{D} + \frac{2(\mathrm{Re}\,\mathcal{A})(\mathrm{Re}\,\mathcal{B})}{(\mathrm{Re}\,\mu) - 2(\mathrm{Re}\,\lambda) + i6\beta}\,,$$
$$\tilde{\mathcal{E}} := \mathcal{E} - \frac{(\mathrm{Re}\,\mathcal{A})^2}{(\mathrm{Re}\,\mu) + i6\beta}\,,$$
$$\tilde{\mathcal{F}} := \mathcal{F}\,. \tag{36}$$

One should note that at the marginal point $\mathrm{Re}\,\mu = \mathrm{Re}\,\lambda \equiv 0$ the transformation (34) becomes singular at $\beta = 0$. That is the reason why we treat the cases $\beta \equiv 0$ and $\beta \neq 0$ separately.

Now we can draw some general conclusions. Introducing

$$\tilde{c}_1 := r_1 e^{i\varphi_1}\,, \quad \tilde{c}_2 := r_2 e^{i\varphi_2}\,, \tag{37}$$

the system (35) reads

$$\dot{r}_1 = (\mathrm{Re}\,\lambda)\,r_1 + \left(\mathrm{Re}\,\tilde{\mathcal{C}}\right) r_1^3 + \left(\mathrm{Re}\,\tilde{\mathcal{E}}\right) r_1 r_2^2\,,$$
$$\dot{r}_2 = (\mathrm{Re}\,\mu)\,r_2 + \left(\mathrm{Re}\,\tilde{\mathcal{D}}\right) r_1^2 r_2 + \left(\mathrm{Re}\tilde{\mathcal{F}}\right) r_2^3\,,$$
$$\dot{\varphi}_1 = (\mathrm{Im}\,\lambda) + \left(\mathrm{Im}\,\tilde{\mathcal{C}}\right) r_1^2 + \left(\mathrm{Im}\,\tilde{\mathcal{E}}\right) r_2^2\,,$$
$$\dot{\varphi}_2 = (\mathrm{Im}\,\mu) + \left(\mathrm{Im}\,\tilde{\mathcal{D}}\right) r_1^2 + \left(\mathrm{Im}\,\tilde{\mathcal{F}}\right) r_2^2\,. \tag{38}$$

Note that the equations for the magnitudes r and phases φ decouple. The first two equations of (38) represent a closed two-dimensional system which only can have fixed points and periodic orbits as attracting sets [29]. From that we can conclude that the amplitudes a_1, a_2, a_3, and a_4 cannot show for $\beta \neq 0$ (in the vicinity of the marginal point $\mathrm{Re}\,\mu = \mathrm{Re}\,\lambda \equiv 0$) any chaotic behavior.

When solving the amplitude equations (28) for $\beta \approx 1.49$ we obtain a behavior which is consistent with the above general conclusion. The typical results are summarized in Figs. 6(a)–6(c). We can distinguish three generic types of attracting solutions: (a) A mixed mode (dominated by $k_y = 1$, but with additional contributions from $k_y = 2$) which exists in the shaded area of Fig. 6(a), but is stable only in the darkly hatched area of Fig. 6(a); (b) a single mode ($k_y = 2$) for $\mathrm{Re}\,\mu > 0$ being stable in the darkly hatched area shown in Fig. 6(b); and (c) a mixed mode (dominated by $k_y = 2$) existing in the $(\mathrm{Re}\,\lambda, \mathrm{Re}\,\mu)$ plane above the lower border of the marked area of Fig. 6(c); it is stable in the darkly hatched area of Fig. 6(c). When $\beta \longrightarrow 0$ the point P moves closer to the origin $\mathrm{Re}\,\lambda = \mathrm{Re}\,\mu = 0$.

It is important to note that these predictions by the center-manifold theory through ODE's are confirmed by solutions of the PDE (1). In numerous runs we have found the convincing agreement, but renounce the demonstration by figures due to space limitation.

The dependence of the stability of the attracting solutions on β is shown in Fig. 7 for the particular values $\mathrm{Re}\,\lambda = 0.45$ and $\mathrm{Re}\,\mu = 1.2$. The latter parameters correspond to a point in the chaotic region V of Fig. 2. In Fig. 7 the growth rates γ of the solutions (a), (b), and (c), corresponding to the cases (a)–(c) of Fig. 6, are drawn versus β. We clearly see that for $\beta = 0$ no stable attracting regular solutions exists, and for finite β more and more the situation depicted in Fig. 6 becomes typical.

V. SUMMARY AND OUTLOOK

In this paper we have investigated the nonlinear development of a trapped particle instability. The trapped particle mode is described by a self-consistently driven and damped KdV-type equation. This equation looks simple but contains a rich dynamics. The evolution in

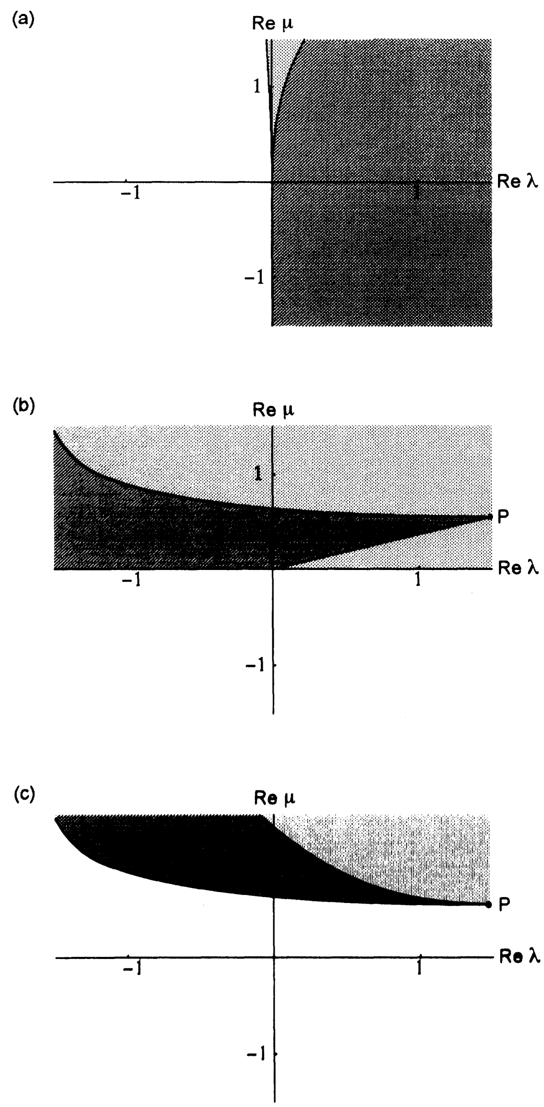

FIG. 6. Typical attracting solutions for $\beta \neq 0$ ($\beta \approx 1.49$). (a) Existence region and stability region of a mixed mode, dominated by $k_y = 1$. (b) Existence and stability region of a single mode with $k_y = 2$. (c) Existence and stability region of a mixed mode, dominated by $k_y = 2$. In all cases the modes exist in the shaded areas, but are stable only in the darkly hatched areas.

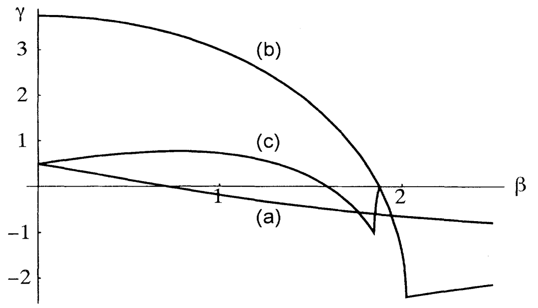

FIG. 7. Growth rates γ of (a) the mixed mode dominated by $k_y = 1$, (b) the single mode with $k_y = 2$, and (c) the mixed mode dominated by $k_y = 2$ are shown vs the dispersion parameter β, respectively.

time can be chaotic, but close to the onset of instability it is determined by a few spatially coherent modes. However, from a systematic treatment with the center-manifold theory very interesting conclusions result, e.g., with respect to the number of modes and their interplay in time. One interesting aspect is that the codimension-two analysis can describe successive bifurcations of one unstable mode which in some cases can lead to chaos in time.

Although these findings have been obtained from a restricted physical model, they have severe consequences for more applied physical situations. We seem to be in a position now to clarify some of the previous problems where saturation by a simple one-envelope ansatz could not be obtained. Works on multidimensional KdV equations are in progress.

ACKNOWLEDGMENTS

This work has been supported by the Deutsche Forschungsgemeinschaft. Many discussions with Dr. E.W. Laedke and his help during the development of the numerical code are gratefully acknowledged. Discussions for applications to the TEXTOR experiment at the KFA Jülich are also acknowledged.

APPENDIX: SUMMARY OF MATHEMATICAL TOOLS

In this appendix we give a short review of the main tools used from the center-manifold theory. First, consider a system of ODE's,

$$\begin{aligned} \dot{a} &= Aa + N(a,b) \ , \\ \dot{b} &= Bb + M(a,b) \ , \end{aligned} \tag{A1}$$

describing the dynamics of the amplitudes $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ of n linear marginal stable modes and m linear stable modes, respectively $[(a_1, \ldots, a_n) =: a$, $(b_1, \ldots, b_m) =: b]$. This implies that the real parts of the eigenvalues of the matrix A vanish and the real parts of the eigenvalues of the matrix B are negative. The functions $N(a,b), M(a,b) \in C^r$ on the right-hand side of Eq. (A1) represent the nonlinear terms. Let E^c be the n-dimensional (generalized) eigenspace of A and E^s the m-dimensional (generalized) eigenspace of B. Under these assumptions the center-manifold theorem provides the following statement [30]:

> There exists an invariant C^r manifold W^s and an invariant C^{r-1} manifold W^c which are tangent at $(a,b) = (0,0)$ to the eigenspaces E^s and E^c, respectively. The stable manifold W^s is unique but the center manifold W^c is not necessarily unique.

Locally the center manifold W^c can be represented as a graph,

$$W^c = \{(a,b) | b = h(a)\} \ , \ h(0) = 0 \ , \ Dh(0) = 0 \ , \tag{A2}$$

where the C^{r-1} function h is defined in a neighborhood of the origin; Dh denotes the Jacobi matrix. Introducing (A2) into Eq. (A1), we obtain

$$\dot{a} = Aa + N(a, h(a)) \ , \tag{A3}$$

$$Dh(a)\,[Aa + N(a,h(a))] = Bh(a) + M(a,h(a)) \ . \tag{A4}$$

The solution h of Eq. (A4) can be approximated by a power series. The ambiguity of the center manifold is manifested by the fact that h is determined only modulo a C^∞ nonanalytic function; so the power series approximation of the function h is unique. The importance of the center-manifold theory is reflected by the following theorem [18]:

> If there exists a neighborhood U^c of $(a,b) = (0,0)$ on W^c, so that every trajectory starting in U^c never leaves it, then there exists a neighborhood U of $(a,b) = (0,0)$ in $\mathbb{R}^n \times \mathbb{R}^m$, so that every trajectory starting in U converges to a trajectory on the center-manifold.

Therefore it is sufficient to discuss the dynamics on the center manifold, described by Eq. (A3), and if all solutions are bounded to some neighborhood of the origin, then we have described all features of the asymptotic behavior of Eq. (A1). In order to fulfill the condition, the function $N(a,h(a))$ has to be developed up to a sufficiently high order. In the main part of this paper we have seen that the third order is adequate.

Our problem contains parameters and is infinite dimensional. We have to generalize the theory presented so far. We take into account the parameters by enlarging Eq. (A1):

$$\begin{aligned} \dot{a} &= A(\Lambda)a + N(a,b,\Lambda) \ , \\ \dot{b} &= B(\Lambda)b + M(a,b,\Lambda) \ , \\ \dot{\Lambda} &= 0 \ , \end{aligned} \tag{A5}$$

where $\Lambda = (a_{n+1}, \ldots, a_{n+l})$ contains l parameters. The center manifold now has dimension $n + l$. The theory is also valid in the infinite-dimensional case, if the spectrum of the linear operator on the right-hand side of the equation (analogous to) (A5) can be split into two parts. The first part contains a finite number of eigenvalues with real parts equal to zero, the second part contains (an infinite number of) eigenvalues with negative real parts which are bounded away from zero. In our case this condition is fulfilled.

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