PHYSICAL REVIEW E VOLUME 48, NUMBER 6 DECEMBER 1993

Harmonic oscillators driven by colored noise: Crossovers, resonances, and spectra

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(Received 30 July 1993)

We study second-order properties of linear oscillators driven by exponentially correlated noise. We focus our attention on dynamical exponents and crossovers and also on resonance phenomena that appear when the driving noise is dichotomous. We also obtain the power spectrum and show its different behaviors according to the color of the noise.

PACS number(s): 05.40.+j, 02.50.Ey

I. INTRODUCTION

In two recent papers [1,2] we have studied free second-order processes (also called inertial processes) of the form

$$\ddot{X} + \beta\dot{X} = F(t) \ , \tag{1.1}$$

where the driving random force $F(t)$ is zero-centered dichotomous Markov noise alternately taking on values $\pm a$ with an exponential switch probability density function of the form

$$\psi(t) = \lambda e^{-\lambda t}, \tag{1.2}$$

where λ^{-1} is the average time between switches. If we assume that the initial values of $F(t)$ are equally likely,

$$\text{Prob}\{F(0) = \pm a\} = 1/2,$$

then the above form of $\psi(t)$ implies that $F(t)$ is a stationary colored noise with the following correlation function:

$$\langle F(t)\, F(t')\rangle = a^2\, e^{-|t-t'|/\tau_c}, \tag{1.3}$$

where

$$\tau_c \equiv \frac{1}{2\lambda} \tag{1.4}$$

is the correlation time.

Our focus in [1,2] has been on the joint density, $p(x,y,t)$, for the probability that the position $X(t)$ lies between x and $x+dx$ and that the velocity $\dot{X}(t)$ lies between y and $y+dy$. We have also obtained exact equations for the marginal probability density, $p(y,t)$, of the velocity and the marginal probability density, $p(x,t)$, of the position. The equation satisfied by $p(y,t)$ is a telegrapher's equation with state-dependent coefficients while the equation satisfied by $p(x,t)$ is a telegrapher's equation with time-dependent coefficients. In the Gaussian-white-noise limit, this latter equation reduces to a Fokker-Planck equation with time-dependent coefficients which in turns leads to anomalous "superdiffusive" motion of the form

$$\langle X^2(t)\rangle \sim t^3$$

for $\beta t \ll 1$ and a crossover to ordinary diffusive motion

$$\langle X^2(t)\rangle \sim t$$

for $\beta t \gg 1$.

In the undamped case $\beta = 0$ [1], the results of this analysis have been recently applied [3] to the problem of chemical reactions in constrained geometries where, in some cases, the kinetics of the reaction is "anomalous" in low dimensions in the sense of being different from the results of mass action [4].

One of the main goals of this paper is to study the effect of a potential on the anomalous diffusive behavior of the system. One of the simplest potentials, albeit of great relevance mainly due to its countless applications, is the quadratic potential. We will thus consider the classical harmonic oscillator driven by colored noise:

$$\ddot{X} + \beta\dot{X} + \omega^2 X = F(t), \tag{1.5}$$

where ω is the angular frequency of the oscillator and $F(t)$ is dichotomous Markov noise. Since the pioneering work of Uhlenbeck and Ornstein [5] there has been much literature on the problem of linear oscillators driven by white noise (see [6] and references therein). Nevertheless, there are very few results available when the driving noise is colored. We should mention here the work of Pawula [7] on second-order Butterworth filters (a special kind of linear oscillators driven by the random telegraph signal) where analytical results on stationary moments and some Monte Carlo simulations have been obtained.

Herein we will only consider second-order properties of $X(t)$, i.e., the properties related with the mean-square displacement $\langle X^2(t)\rangle$. Incidentally we note that, from the point of view of probability theory, most of the second-order properties of $X(t)$ depend solely on the correlation function $\langle F(t)F(t')\rangle$ of the driving force, being then independent of the probabilistic nature (dichotomous, Gaussian, etc,...) of $F(t)$. Therefore, we will see that many of our results not only apply to dichotomous Markov noise [with the correlation function given by Eq. (1.3)] but also to Ornstein-Uhlenbeck noise, that is, Gaussian colored noise with the correlation function

$$\langle F(t)\, F(t')\rangle = \frac{D}{\tau_c}\, e^{-|t-t'|/\tau_c}. \tag{1.6}$$

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Note that both dichotomous Markov noise and Ornstein-Uhlenbeck noise have the same correlation function provided that we set

$$a^2 = 2\lambda D = \frac{D}{\tau_c}. \quad (1.7)$$

The paper is organized as follows. In Sec. II we evaluate the variance and the dynamical exponent of the linear oscillator. In Sec. III we study the behavior of the variance which leads to several crossovers for the dynamical exponent. In Sec. IV we present the classical resonances associated with dichotomous driving noise. The power spectrum and its behavior are shown in Sec. V. Conclusions are drawn in Sec. VI and the spectral analysis of the oscillator is given in the Appendix.

II. VARIANCE AND DYNAMICAL EXPONENT

We will now obtain an explicit analytical expression of the variance

$$\sigma^2(t) \equiv \langle [X(t) - \langle X(t)\rangle]^2 \rangle$$

of our noisy linear oscillator. The first step consists in writing the formal solution to the equation

$$\ddot{X} + \beta\dot{X} + \omega^2 X = F(t), \quad (2.1)$$

with (deterministic) initial conditions

$$X(0) = x_0, \quad \dot{X}(0) = \dot{x}_0. \quad (2.2)$$

If we take into account that the driving noise is zero-centered, that is $\langle F(t)\rangle = 0$, and use the Green function of the homogeneous initial problem, it is straightforward to find that the solution to Eqs. (2.1) and (2.2) reads

$$X(t) = \langle X(t)\rangle + \int_0^t dt' G(t-t')F(t'), \quad (2.3)$$

where

$$\langle X(t)\rangle = e^{-\beta t/2}\left[\left(\dot{x}_0 + \frac{1}{2}\beta x_0\right)\frac{\sin\Omega t}{\Omega} + x_0\cos\Omega t\right], \quad (2.4)$$

$$G(t) = e^{-\beta t/2}\frac{\sin\Omega t}{\Omega}, \quad (2.5)$$

and

$$\Omega = \left(\omega^2 - \frac{\beta^2}{4}\right)^{1/2}. \quad (2.6)$$

Note that in the overdamped case, $\beta > 2\omega$, the mean value $\langle X(t)\rangle$ and the Green function $G(t)$ are conveniently written as

$$\langle X(t)\rangle = e^{-\beta t/2}\left[\left(\dot{x}_0 + \frac{1}{2}\beta x_0\right)\frac{\sinh\alpha t}{\alpha} + x_0\cosh\alpha t\right] \quad (2.7)$$

and

$$G(t) = e^{-\beta t/2}\frac{\sinh\alpha t}{\alpha}, \quad (2.8)$$

where

$$\alpha = \left(\frac{\beta^2}{4} - \omega^2\right)^{1/2}. \quad (2.9)$$

In the critical or aperiodic case, $\beta = 2\omega$, we have

$$\langle X(t)\rangle = \left[\left(\dot{x}_0 + \frac{1}{2}\beta x_0\right)t + x_0\right]e^{-\beta t/2} \quad (2.10)$$

and

$$G(t) = t\, e^{-\beta t/2}. \quad (2.11)$$

We can now evaluate the variance of the oscillator. From Eq. (2.3) we have

$$\sigma^2(t) = \int_0^t ds \int_0^t ds' G(t-s)G(t-s')C(s-s'), \quad (2.12)$$

where [cf. Eq. (1.3)]

$$C(s-s') \equiv \langle F(s)F(s')\rangle = a^2 e^{-2\lambda|s-s'|}. \quad (2.13)$$

Due to the symmetry of the correlation function $C(s-s')$ we can write the variance in the following more convenient form:

$$\sigma^2(t) = 2\int_0^t ds G(s)\int_0^s ds' G(s')C(s-s'). \quad (2.14)$$

Now the substitution of Eq. (2.5) into Eq. (2.14) yields

$$\sigma^2(t) = \frac{2a^2}{\Delta_-}\left\{\frac{1}{\Delta_+} - \frac{1}{\Delta_+}e^{-(2\lambda+\beta/2)t}\left[\left(2\lambda + \frac{\beta}{2}\right)\frac{\sin\Omega t}{\Omega} + \cos\Omega t\right] + \frac{\lambda - \beta/2}{\omega^2\beta}\left(1 - e^{-\beta t}\right) - \frac{e^{-\beta t}}{4\omega^2}\left[(2\lambda - \beta)\frac{\sin 2\Omega t}{\Omega} + (2\lambda\beta - \beta^2 + 2\omega^2)\frac{\sin^2\Omega t}{\Omega^2}\right]\right\}, \quad (2.15)$$

where

$$\Delta_\pm \equiv \omega^2 + 2\lambda(2\lambda \pm \beta). \quad (2.16)$$

In the overdamped case the same expression holds just by replacing $\cos\Omega t$ by $\cosh\alpha t$ and $\sin\Omega t/\Omega$ by $\sinh\alpha t/\alpha$. In the critical regime we have $\Omega = 0$ and Eq. (2.15) reads

$$\sigma^2(t) = \frac{2a^2}{\Delta_-}\left\{\frac{1}{\Delta_+} - \frac{1}{\Delta_+}e^{-(2\lambda+\beta/2)t}\left[1+\left(2\lambda+\frac{\beta}{2}\right)t\right] + \frac{\lambda-\beta/2}{\omega^2\beta}\left(1-e^{-\beta t}\right) - \frac{e^{-\beta t}}{4\omega^2}\left[2(2\lambda-\beta)t + (2\lambda\beta-\beta^2+2\omega^2)t^2\right]\right\}. \quad (2.17)$$

Finally in the Gaussian-white-noise limit:

$$a \to \infty, \quad \lambda \to \infty, \quad D \equiv \frac{a^2}{2\lambda} < \infty,$$

Eq. (2.15) reduces to the well known result [12]

$$\sigma^2(t) = \frac{D}{\omega^2\beta}\left\{1 - e^{-\beta t}\left[1+\frac{\beta}{2\Omega}\left(\sin 2\Omega t + \frac{\beta}{\Omega}\sin^2\Omega t\right)\right]\right\} . \quad (2.18)$$

We will now discuss some asymptotic properties of the variance $\sigma^2(t)$. We first observe that, in the case of a linear oscillator driven by colored noise, there are three time scales involved. These time scales are the relaxation time $\tau_r = \beta^{-1}$ and the period $\tau = 2\pi\omega^{-1}$ of the undriven oscillator and the correlation time of the driving noise $\tau_c = (2\lambda)^{-1}$. In the next section we will show that $\sigma^2(t)$ has a different behavior depending on (i) whether the observation time t is longer than some of these time scales, and (ii) the order of magnitude of τ_r, τ_c, and τ. In other words, the behavior of $\sigma^2(t)$ depends on whether the orders of magnitude of τ_r, τ_c, τ, and t are similar or not comparable. In any case we can easily see that when $t \to \infty$, i.e., when t is much longer than any time scale, $\sigma^2(t)$ goes to the following stationary value [cf. Eqs. (2.15) and Eq. (2.16)]:

$$\sigma_s^2 \equiv \lim_{t\to\infty}\sigma^2(t) = \frac{a^2(2\lambda+\beta)}{\omega^2\beta(4\lambda^2+2\lambda\beta+\omega^2)}. \quad (2.19)$$

For the special choice of parameters that corresponds to Butterworth filters, i.e., $a = \omega^2 = \beta^2/2$, Eq. (2.19) agrees with previous results [7].

Another extreme asymptotic behavior of the variance refers to the case $t \to 0$, that is, when the observation time is much smaller than any time scale. In this case and after some lengthy calculation we get from Eq. (2.15) that

$$\sigma^2(t) \approx \frac{1}{4}a^2t^4, \quad (t \to 0). \quad (2.20)$$

We can obtain this result in a much simpler way. Indeed, since t is much smaller than β^{-1} and ω^{-1} then the so called "dominant-balance technique" [8] tells us that $|\ddot{X}| \gg \beta|\dot{X}|$ and $|\ddot{X}| \gg \omega^2|X|$. Hence Eq. (2.1) can be approximated by

$$\ddot{X} \approx F(t)$$

($t \ll \beta^{-1}, \omega^{-1}$). On the other hand, the observation time t is also much smaller than the correlation time $(2\lambda)^{-1}$ of the noise. This means that at time t the probability that the noise has switched to another value is very small, that is, $F(t) = F(0)$ (in probability). Therefore

$$X(t) \approx \langle X(t)\rangle + \frac{1}{2}F(0)\ t^2$$

$[t \ll \beta^{-1}, \omega^{-1}, (2\lambda)^{-1}]$ and Eq. (2.20) holds.

Let us now analyze in detail the behavior of $\sigma^2(t)$. One of the quantities which has been proved to be very suitable for this analysis is the so called "dynamical exponent" $\nu(t)$ of the process [2]. We define this exponent in an analogous way as is done in fractal theory to define the differential fractal dimension, a quantity that characterizes the random motion of particles [9–11]:

$$\nu(t) \equiv \frac{d\ln\sigma^2(t)}{d\ln t} = t\ \frac{\dot{\sigma}^2(t)}{\sigma^2(t)}, \quad (2.21)$$

where $\dot{\sigma}^2(t) \equiv d\sigma^2(t)/dt$. Note that in the regions where this quantity does not appreciably vary, the variance $\sigma^2(t)$ can be written as

$$\sigma^2(t) \simeq t^{\nu(t)}. \quad (2.22)$$

Let $X(t)$ be a random process defined by an equation of the form [cf. Eq. (2.3)]

$$X(t) = \langle X(t)\rangle + \int_0^t dt' G(t-t')F(t') ,$$

where $G(t)$ is an integrable function and $F(t)$ is any wide-sense stationary noise with a given correlation function $C(t)$. In this case we have shown that the variance of $X(t)$ is given by Eq. (2.14) and, consequently, the dynamical exponent is given by

$$\nu(t) = \frac{t\ G(t)\displaystyle\int_0^t ds G(s)C(t-s)}{\displaystyle\int_0^t ds\ G(s)\int_0^s ds' G(s')C(s-s')}. \quad (2.23)$$

When $F(t)$ is white noise (not necessarily Gaussian) then $C(t) = 2D\delta(t)$, Eq. (2.23) reads

$$\nu(t) = \frac{t\ G^2(t)}{\displaystyle\int_0^t ds\ G^2(s)}, \quad (2.24)$$

and $\nu(t)$ is always positive (or zero). Nevertheless, when $F(t)$ is colored noise the sign of $\nu(t)$ will depend on the specific forms of the functions $G(t)$ and $C(t)$.

Finally for the linear oscillator (1.5) with an exponentially correlated driving noise $F(t)$ both Gaussian or dichotomous, the dynamical exponent is given by Eq. (2.21) with $\sigma^2(t)$ given by Eq. (2.15) and

$$\dot{\sigma}^2(t) = \frac{2a^2}{\Delta_-}\left[e^{-\beta t}\left\{\left(2\lambda - \frac{\beta}{2}\right)\frac{\sin\Omega t}{\Omega} - \cos\Omega t\right\} + e^{-(2\lambda+\beta/2)t}\right]\frac{\sin\Omega t}{\Omega}. \quad (2.25)$$

III. CROSSOVERS

We have mentioned in Sec. II that the behavior of the variance of the linear oscillator depends on whether the observation time t, the relaxation time $\tau_r = \beta^{-1}$, the correlation time $\tau_c = (2\lambda)^{-1}$, and the period $\tau = 2\pi\omega^{-1}$ are similar or not comparable. Moreover we have shown in Sec. II that when t is much smaller than any other time scale, the variance is proportional to t^4. Therefore, in the early stages of its evolution the dynamical exponent of the linear oscillator is

$$\nu(t) = 4 \quad (t \to 0).$$

Note that this exponent corresponds to ballistic motion. We have also shown that when $t \to \infty$ the variance goes to a stationary value and hence

$$\nu(t) = 0 \quad (t \to \infty).$$

Let us now investigate the intermediate-time behavior of $\nu(t)$ between these two extreme situations. We first assume that τ_c, τ_r, and τ are not comparable. In this situation we have six different cases.

(1) $\tau_c \ll \tau_r \ll \tau$. In this case the movement is overdamped and we distinguish three different time regimes. (a) The observation time is much longer than the correlation time but still smaller than the relaxation time: $\tau_c \ll t < \tau_r \ll \tau$. In this regime $F(t)$ acts as white noise and by means of Eqs. (2.8) and (2.24) we see that

$$\nu(t) = 3.$$

(b) The observation time is much longer than the relaxation time but still smaller than the period: $\tau_c \ll \tau_r \ll t < \tau$. We define the dimensionless time $t' = t/\tau_r$. Equation (1.5) is thus transformed to

$$X'' + X' + 4\pi^2(\tau_r/\tau)^2 X = \eta(t'),$$

where the primes denote derivatives with respect to t' and $\eta(t')$ is white noise. Since $\tau_r/\tau \ll 1$ we can neglect the last term on the right hand side of this equation. In addition $t' \gg 1$ and the dominant balance technique tells us that

$$|X''| \ll |X'|.$$

Therefore the time evolution of the oscillator can be approximated by

$$X' = \eta(t'),$$

which implies that $\sigma^2(t) \sim t$ [1], whence

$$\nu(t) = 1.$$

(c) The observation time is much longer than any other time scale: $\tau_c \ll \tau_r \ll \tau \ll t$. This corresponds to the stationary regime and hence

$$\nu(t) = 0.$$

Case (1) is shown in Fig. 1 where, as in the following figures of this section, we plot the exact dynamical exponent $\nu(t)$ calculated from Eqs. (2.21) and (2.15). We observe that Fig. 1 totally agrees with the above discussion on crossovers.

(2) $\tau_r \ll \tau_c \ll \tau$. In this case the movement is also overdamped and we again distinguish three different time regimes. (a) The observation time is much longer than the relaxation time but still smaller than the correlation time: $\tau_r \ll t < \tau_c \ll \tau$. Proceeding as in case (1)-(b) we can easily show that the evolution equation can be approximated by

$$\dot{X} = \beta^{-1}F(t).$$

We have shown elsewhere [13] that in this case the variance is proportional to t^2 and

$$\nu(t) = 2.$$

(b) The observation time is longer than the correlation time but still smaller than the period: $\tau_r \ll \tau_c \ll t < \tau$. Now $F(t) \approx \eta(t)$, where $\eta(t)$ is white noise and

$$\nu(t) = 1.$$

(c) The observation time is longer than any other time scale. In this regime the oscillator has already reached the stationary state and $\nu(t) = 0$ (Fig. 2).

(3) $\tau \ll \tau_r \ll \tau_c$. In this case we first observe that τ_c is the longest time involved, this means that the driving noise $F(t)$ cannot be approximated by white noise [except when $t \gg \tau_c$, but in this regime the system has reached the stationary state and $\nu(t) = 0$]. The fact that

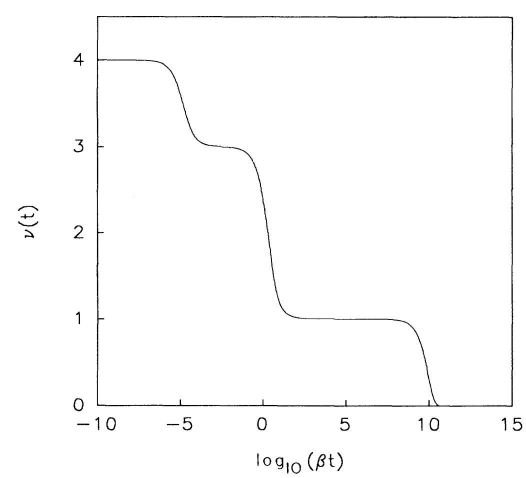

FIG. 1. Dynamical exponent of the linear oscillator. Parameter values: $\beta = 1$, $\omega = 10^{-5}$, $\lambda = 10^5$, and $a = 1$.

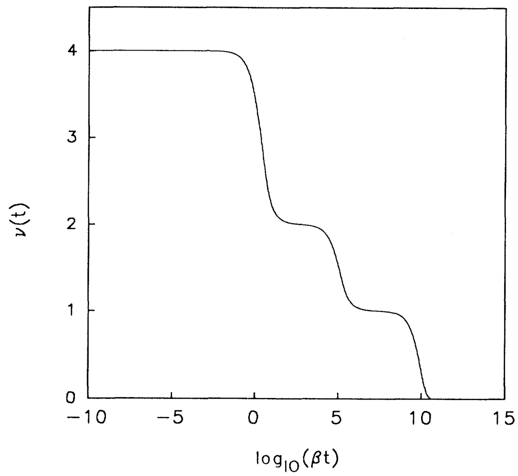

FIG. 2. Dynamical exponent of the linear oscillator. Parameter values: $\beta = 1$, $\omega = \lambda = 10^{-5}$, and $a = 1$.

τ_c is the longest time scale also means that $F(t) = F(0)$ (in probability) which implies an almost deterministic behavior [i.e., $\nu(t) = 4$] for a longer time interval than before. On the other hand, the movement of the oscillator is now underdamped and, for intermediate observation times, we can expect an oscillatory behavior of $\dot{\sigma}^2(t)$ alternately taking on positive and negative values. This leads to a crossover from $\nu(t) = 4$ to $\nu(t) = 0$ with a damped oscillation around 0. These considerations are clearly confirmed by the exact result plotted in Fig. 3.

(4) $\tau \ll \tau_c \ll \tau_r$. This case is very similar to that of the preceding case and there exists an oscillatory crossover from $\nu(t) = 4$ to $\nu(t) = 0$. Since now the relaxation time τ_r is the longest time involved, we expect that the oscillations of $\nu(t)$ around 0 will have a much bigger amplitude than in case (3). This is confirmed by the exact calculation, although due to the big size of the oscillations we have not plotted it.

(5) $\tau_c \ll \tau \ll \tau_r$. (a) In the time regime $\tau_c \ll t < \tau \ll \tau_r$, the evolution of $X(t)$ is approximately given by

$$\ddot{X} = \eta(t),$$

where $\eta(t)$ is white noise. Therefore $\nu(t) = 3$. (b) When $\tau_c \ll \tau \ll t < \tau_r$, the evolution of $X(t)$ is approximately given by

$$\ddot{X} + \omega^2 X = \eta(t)$$

and the behavior of the dynamical exponent is described by an oscillatory crossover from $\nu(t) = 3$ to $\nu(t) = 0$. However, in this case the noise is white and $\nu(t)$ is always positive [cf. Eq. (2.24)]. Therefore the oscillations of the dynamical exponent will be *towards* $\nu(t) = 0$ but not *around* $\nu(t) = 0$ (see Fig. 4).

(6) $\tau_r \ll \tau \ll \tau_c$. In this case we see that, as in case (3), out of the stationary regime the driving noise cannot be approximated by white noise and this rules out the possibility of having $\nu(t) = 3$. On the other hand the movement is now overdamped and no oscillatory behavior will be shown by $\nu(t)$. (a) In the time regime $\tau_r \ll t < \tau \ll \tau_c$, we have $|\ddot{X}| \ll \beta|\dot{X}|$ and the evolution equation can be approximated by

$$\beta\dot{X} + \omega^2 X = F(0),$$

and $\nu(t) = 2$. (b) When $\tau_r \ll \tau \ll t < \tau_c$, we have

$$|\ddot{X}| \ll \beta|\dot{X}| \ll \omega^2|X|.$$

Therefore, $X(t)$ has practically reached its stationary value and hence $\nu(t) = 0$ (Fig. 5).

We finally observe that when all time scales are similar,

$$\tau_c \sim \tau_r \sim \tau,$$

we are only able to distinguish two extreme asymptotic regimes: $t \to 0$ and $t \to \infty$. This leads to the crossover

$$\nu(t) = 4 \longrightarrow \nu(t) = 0,$$

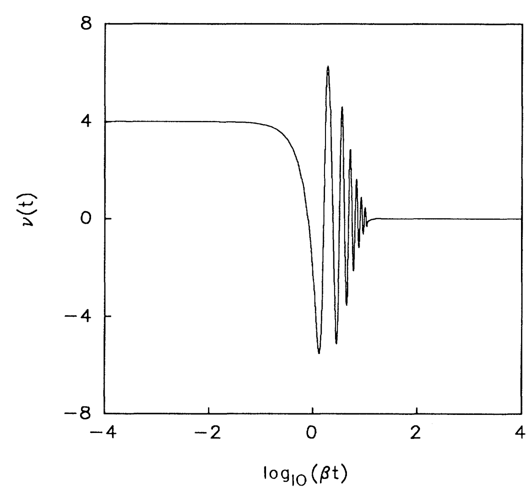

FIG. 3. Dynamical exponent of the linear oscillator. Parameter values: $\beta = 1$, $\omega = 4$, $\lambda = 0.01$, and $a = 1$.

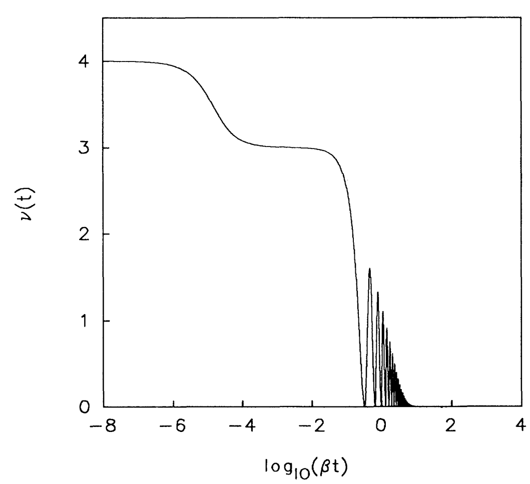

FIG. 4. Dynamical exponent of the linear oscillator. Parameter values: $\beta = 1$, $\omega = 10$, $\lambda = 10^5$, and $a = 1$.

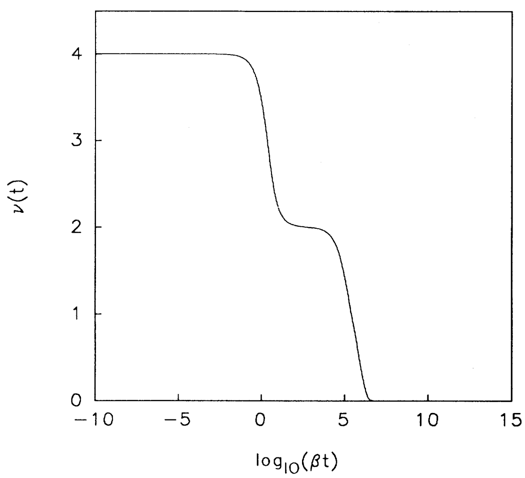

FIG. 5. Dynamical exponent of the linear oscillator. Parameter values: $\beta = 1$, $\omega = 10^{-3}$, $\lambda = 10^{-5}$, and $a = 1$.

without any intermediate value (Fig. 6). Note that if in this case $\tau_r \leq \tau$ the movement can be underdamped. In this situation there exist slight oscillations around $\nu(t) = 0$. In fact, Fig. 6 corresponds to an underdamped case although these oscillations are so small that they do not show in the graphic.

As a final remark we observe that all the results of this section are valid for exponentially correlated driving noises regardless of their probabilistic nature. Thus, although we have carried out the calculations and figures for dichotomous Markov driving noise, *all of the above results also hold for Ornstien-Uhlenbeck driving noise.*

IV. RESONANCES

As is well known from intermediate physics, forced linear oscillators of the form

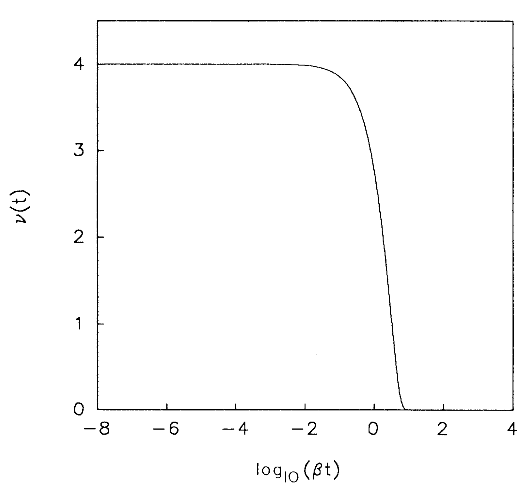

FIG. 6. Dynamical exponent of the linear oscillator. Parameter values: $\beta = 1, \omega = 0.4, \lambda = 1.5$, and $a = 1$.

$$\ddot{X} + \beta\dot{X} + \omega^2 X = F\cos\overline{\omega}t \tag{4.1}$$

present two kinds of resonance according to whether the amplitude or the power absorption have a maximum for certain values of $\overline{\omega}$. Thus, when

$$\overline{\omega} = \sqrt{\omega^2 - \beta^2/2}$$

the average amplitude (or, equivalently, the average potential energy) has a maximum, where the average is taken over one cycle of the undriven motion. On the other hand, the average power delivered by the periodic force has a maximum for $\overline{\omega} = \omega$.

One of the aims of this section is to elucidate whether linear oscillators driven by colored noise present this resonant behavior. As we will see, it is precisely at this point where the probabilistic nature of the driving noise shows its importance. Indeed, dichotomous noise can be viewed as a random square wave with a "random frequency" whose average is given by the inverse of the correlation time 2λ. Nevertheless, one cannot give such an interpretation for Ornstein-Uhlenbeck noise. This is the reason why dichotomous driving noise presents resonances while Ornstein-Uhlenbeck driving noise does not.

For the stochastic oscillator given by Eq. (1.5) we may define the square root of the variance as the average amplitude of the oscillations. In the stationary regime we will have [cf. Eq. (2.19)]

$$\sigma_s = \frac{a}{\omega}\sqrt{\frac{\overline{\omega} + \beta}{\beta(\overline{\omega}^2 + \beta\overline{\omega} + \omega^2)}}, \tag{4.2}$$

where

$$\overline{\omega} \equiv 2\lambda. \tag{4.3}$$

We see from Eq. (4.2) that σ_s (as a function of $\overline{\omega}$) has a maximum for $\overline{\omega} = \overline{\omega}_a$, where

$$\overline{\omega}_a = \omega - \beta. \tag{4.4}$$

This resonant value of the inverse of the correlation time will exist provided that $\omega > \beta$ implying that the oscillator is in the underdamped regime.

Note that initial conditions have no influence on the stationary state. We may thus assume, without loss of generality, that

$$X(0) = \dot{X}(0) = 0. \tag{4.5}$$

In this case the mean potential energy of our oscillator is given by

$$\langle U \rangle_s = \frac{1}{2}\omega^2\sigma_s^2,$$

and $\langle U \rangle_s$ also has a maximum for $\overline{\omega} = \omega - \beta$.

Let us now obtain the mean energy of the oscillator,

$$\langle E(t) \rangle = \frac{1}{2}\langle \dot{X}^2(t) \rangle + \frac{1}{2}\omega^2\langle X^2(t) \rangle, \tag{4.6}$$

and the resonance associated with it. Taking into account Eq. (4.5) we have

$$\sigma^2(t) = \langle X^2(t) \rangle.$$

Hence

$$\dot{\sigma}^2(t) = 2\langle X(t)\dot{X}(t)\rangle$$

and

$$\ddot{\sigma}^2(t) = 2\langle \dot{X}^2(t)\rangle - \beta\dot{\sigma}^2(t) - 2\omega^2\sigma^2(t) + 2\langle X(t)F(t)\rangle.$$

The substitution of these equations into Eq. (4.6) yields

$$\langle E(t)\rangle = \frac{1}{4}[\ddot{\sigma}^2(t) + \beta\dot{\sigma}^2(t)] + \omega^2\sigma^2(t) - \frac{1}{2}\langle X(t)F(t)\rangle. \tag{4.7}$$

In the stationary state $\ddot{\sigma}^2(t) = \dot{\sigma}^2(t) = 0$ and Eq. (4.7) reads

$$\langle E\rangle_s = \omega^2\sigma_s^2 - \frac{1}{2}\varphi_s, \tag{4.8}$$

where

$$\varphi_s \equiv \lim_{t\to\infty}\langle X(t)F(t)\rangle. \tag{4.9}$$

From Eqs. (2.3) and (4.9) we obtain

$$\varphi_s = \int_0^\infty G(t)C(t)dt$$

and using Eqs. (2.5) and (2.13) we finally get

$$\varphi_s = \frac{a^2}{4\lambda^2 + 2\lambda\beta + \omega^2}. \tag{4.10}$$

The substitution of Eqs. (2.19) and (4.10) into Eq. (4.8) yields

$$\langle E\rangle_s = \frac{a^2(2\overline{\omega} + \beta)}{2\beta(\overline{\omega}^2 + \beta\overline{\omega} + \omega^2)}, \tag{4.11}$$

where $\overline{\omega}$ is given by Eq. (4.3). In Fig. 7 we have plotted the mean energy for the overdamped, underdamped, and critical regimes. We observe from the figure that in the underdamped regime $\langle E\rangle_s$ has a relative maximum. Indeed, we see from Eq. (4.11) that the mean energy has a maximum for $\overline{\omega} = \overline{\omega}_e$, where

$$\overline{\omega}_e = \frac{1}{2}\left(-\beta + \sqrt{4\omega^2 - \beta^2}\right). \tag{4.12}$$

This resonant frequency exists if and only if $\omega > \beta/\sqrt{2}$ which corresponds to the underdamped regime.

The average power delivered by the driving noise to the oscillator is defined by

$$\langle P(t)\rangle \equiv \langle \dot{X}(t)F(t)\rangle. \tag{4.13}$$

From Eqs. (2.3) and (2.5) we see that the velocity $\dot{X}$ can be written as

$$\dot{X}(t) = -\frac{\beta}{2}X(t) + \int_0^t e^{-\beta(t-t')/2}F(t')\cos\Omega(t-t')\,dt'.$$

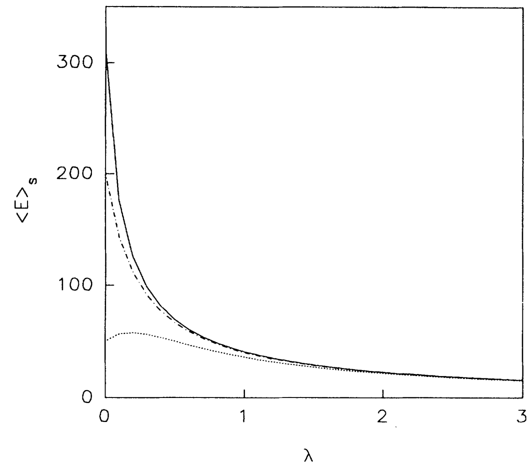

FIG. 7. Mean energy of the linear oscillator in the stationary state as a function of λ. Parameter values: $a = 10$ and $\beta = 1$. Solid line corresponds to $\omega = 0.4$ (overdamped case), dot-dashed line corresponds to $\omega = 0.5$ (critical case), dotted line corresponds to $\omega = 1$ (underdamped case).

Thus

$$\langle P(t)\rangle = -\frac{\beta}{2}\langle X(t)F(t)\rangle + \int_0^t e^{-\beta t'/2}C(t')\cos\Omega t'\,dt'.$$

In the stationary state we get [cf. Eq. (2.13) and Eqs. (4.9) and (4.10)]

$$\langle P\rangle_s = \frac{a^2\overline{\omega}}{\overline{\omega}^2 + \beta\overline{\omega} + \omega^2}, \tag{4.14}$$

where $\overline{\omega}$ is given by Eq. (4.3). The mean power delivered by the driving noise has a maximum for $\overline{\omega} = \overline{\omega}_n$, where

$$\overline{\omega}_n = \omega. \tag{4.15}$$

This resonant frequency exists for all values of the frequency ω of the oscillator (Fig. 8). It is interesting to note that $\overline{\omega}_n$ coincides with the resonant frequency of the average power delivered by the periodic force in the deterministic case (4.1).

The so called "resonant width" associated with $\langle P\rangle_s$ is defined as the size of the frequency range within which $\langle P\rangle_s$ is greater than half its maximum value. The values of $\overline{\omega}$ at half height are

$$\overline{\omega}_\pm = 2\omega + \frac{\beta}{2} \pm \sqrt{\left(2\omega + \frac{\beta}{2}\right)^2 - \omega^2},$$

and the resonant width $\Delta \equiv \overline{\omega}_+ - \overline{\omega}_-$ reads

$$\Delta = \sqrt{(4\omega + \beta)^2 - 4\omega^2}. \tag{4.16}$$

This width depends on both the angular frequency ω of the undriven oscillator and the damping coefficient β. Nevertheless, in the light-damping limit $\beta \ll \omega$, we have

$$\Delta = 2\omega\sqrt{3}\left[1 + \frac{\beta}{3\omega} + O\left(\frac{\beta^2}{\omega^2}\right)\right]. \tag{4.17}$$

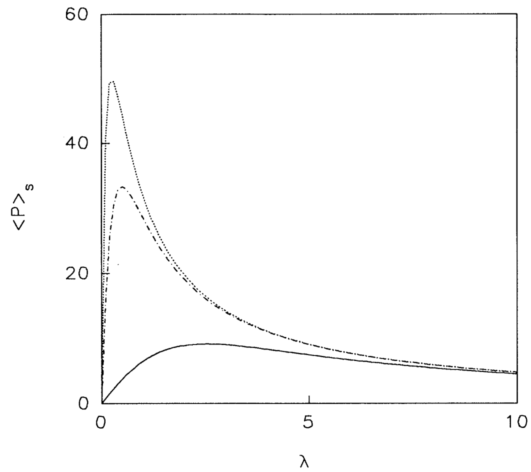

FIG. 8. Average power delivered by the driving noise in the stationary state as a function of λ. Parameter values: $a = 10$ and $\beta = 1$. Solid line corresponds to $\omega = 5$ (underdamped case), dot-dashed line corresponds to $\omega = 0.5$ (critical case), dotted line corresponds to $\omega = 0.4$ (overdamped case).

In this case $\Delta \approx 2\omega\sqrt{3}$ and the resonant width is controlled by the angular frequency of the undriven oscillator.

In the case of heavy damping $\beta \gg \omega$, Eq. (4.16) can be written as

$$\Delta = \beta \left[1 + \frac{4\omega}{\beta} + O\left(\frac{\omega^2}{\beta^2}\right) \right]. \tag{4.18}$$

Now $\Delta \approx \beta$ and the resonant width is controlled by the damping coefficient. This case parallels that of deterministic oscillators driven by harmonic periodic forces where the resonant width is independent of ω and exactly equals β [14].

From Eqs. (1.5) and (4.6) we see that the mean energy decay is given by

$$\left\langle \frac{dE(t)}{dt} \right\rangle = -\beta\langle \dot{X}^2(t)\rangle + \langle \dot{X}(t)F(t)\rangle. \tag{4.19}$$

In the stationary state we have

$$\left\langle \frac{dE(t)}{dt} \right\rangle = 0,$$

and from Eq. (4.19) we see that the mean kinetic energy $\langle T\rangle \equiv \dot{X}^2/2$ is proportional to the power absorption, that is,

$$\langle T\rangle_s = \frac{1}{2\beta}\langle P\rangle_s. \tag{4.20}$$

Therefore, in the stationary state the kinetic energy has the same resonant frequency than power absorption.

It is not difficult to convince oneself that the resonant frequencies discussed above obey the following order relation:

$$\overline{\omega}_n > \overline{\omega}_e > \overline{\omega}_a, \tag{4.21}$$

and in the weak-damping limit we have

$$\overline{\omega}_a \simeq \overline{\omega}_e \simeq \overline{\omega}_n = \omega.$$

When $F(t)$ is Ornstein-Uhlenbeck noise we have

$$a^2 = 2\lambda D$$

and from Eqs. (4.2), (4.11), and (4.14) we obtain

$$\sigma_s = \frac{D^{1/2}}{\omega}\sqrt{\frac{\overline{\omega}(\overline{\omega}+\beta)}{\beta(\overline{\omega}^2+\beta\overline{\omega}+\omega^2)}}, \tag{4.22}$$

$$\langle E\rangle_s = \frac{D\overline{\omega}(2\overline{\omega}+\beta)}{2\beta(\overline{\omega}^2+\beta\overline{\omega}+\omega^2)}, \tag{4.23}$$

and

$$\langle P\rangle_s = \frac{D\overline{\omega}^2}{\overline{\omega}^2+\beta\overline{\omega}+\omega^2}. \tag{4.24}$$

These quantities are monotonic functions of $\overline{\omega}$ and *there are no resonances for Orsntein-Uhlenbeck driving noise.*

V. POWER SPECTRUM

The spectral analysis is a central problem in the study of second-order properties of random functions [15,16]. This analysis is mainly carried out through the spectral representation both of the random process itself and of its autocorrelation function. This latter is especially important because it often carries the sense of an energy and, in any case, represents a simple measure of the intensity of the process.

The power spectrum $S(\alpha)$ of a stationary process $Z(t)$ is the Fourier transform of its autocorrelation:

$$S(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\alpha\tau}K(\tau)d\tau, \tag{5.1}$$

where

$$K(\tau) = \langle Z(t+\tau)Z(t)\rangle .$$

We will now derive an explicit expression for the power spectrum of the linear oscillator driven by colored noise (1.5). There are standard derivations, all of them based on the Wiener-Khinchine theorem, of the power spectrum of stationary linear processes. Nevertheless, in the Appendix we present an alternative derivation (also based on this theorem) that is especially suited for the initial-value problem (2.1) and (2.2). Thus we show that for our linear oscillator in the stationary regime the power spectrum reads

$$S(\alpha) = \tilde{C}(\alpha)\left| \int_0^{\infty} e^{-i\alpha z}G(z)dz \right|^2, \tag{5.2}$$

where $G(z)$ is given by Eq. (2.5) and $\tilde{C}(\alpha)$ is the power spectrum of the driving noise $F(t)$:

$$\tilde{C}(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\alpha\tau}C(\tau)d\tau. \tag{5.3}$$

For exponentially correlated driving noise we have

$$\tilde{C}(\alpha) = \frac{2\lambda a^2}{\pi(4\lambda^2 + \alpha^2)}. \quad (5.4)$$

The substitution of Eqs. (2.5) and (5.4) into Eq. (5.2) yields

$$S(\alpha) = \frac{2\lambda a^2}{\pi(4\lambda^2 + \alpha^2)[(\alpha^2 - \omega^2)^2 + \beta^2\alpha^2]}. \quad (5.5)$$

The qualitative behavior of the power spectrum $S(\alpha)$ obviously depends on the time scales involved. We have two cases.

(1) The correlation time of the driving noise is smaller than the relaxation time:

$$\tau_c < \tau_r.$$

Now depending on the period $\tau = 2\pi\omega^{-1}$ we have three different situations.

$$\text{(a)} \quad \tau < 2\pi\tau_c \left[1 + \sqrt{1 - (\tau_c/\tau_r)^2}\right]^{-1/2}.$$

In this case $S(\alpha)$ has three maxima located at

$$\alpha = 0, \qquad \alpha = \pm\sqrt{(\zeta + \rho)/3}, \quad (5.6)$$

and two minima located at

$$\alpha = \pm\sqrt{(\zeta - \rho)/3}, \quad (5.7)$$

where

$$\zeta \equiv 2\omega^2 - \beta^2 - 4\lambda^2 \quad (5.8)$$

and

$$\rho \equiv \sqrt{\omega^4 + 4(2\lambda^2 - \beta^2)\omega^2 + 4\lambda^2(4\lambda^2 - \beta^2) + \beta^4}. \quad (5.9)$$

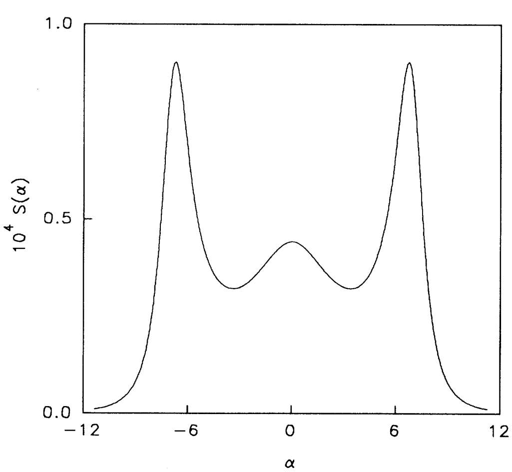

FIG. 9. Power spectrum of the linear oscillator as a function of the spectral frequency α. Parameter values: $a = 1, \beta = 2, \lambda = 1.5$, and $\omega = 7$.

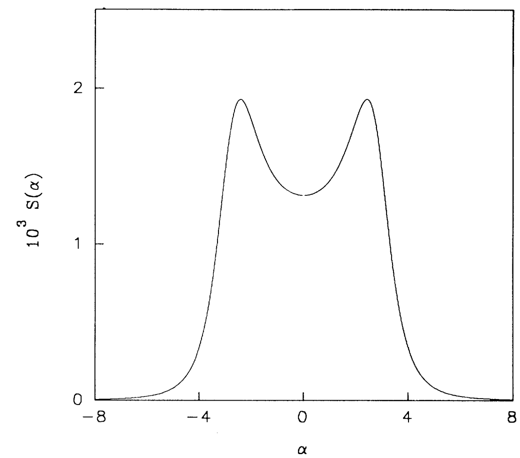

FIG. 10. Power spectrum of the linear oscillator as a function of the spectral frequency α. Parameter values: $a = 1, \beta = 2, \lambda = 1.5$, and $\omega = 3$.

This case is plotted in Fig. 9.

$$\text{(b)} \quad 2\pi\tau_c \left[1 + \sqrt{1 - (\tau_c/\tau_r)^2}\right]^{-1/2} < \tau < 2\pi\tau_c \left[1 - \sqrt{1 - (\tau_c/\tau_r)^2}\right]^{-1/2}.$$

In this situation the power spectrum has one minimum for $\alpha = 0$ and two maxima for

$$\alpha = \pm\sqrt{(\zeta + \rho)/3}$$

(Fig. 10).

$$\text{(c)} \quad \tau > 2\pi\tau_c \left[1 - \sqrt{1 - (\tau_c/\tau_r)^2}\right]^{-1/2}.$$

In this case $S(\alpha)$ only presents one maximum for $\alpha = 0$ (Fig. 11).

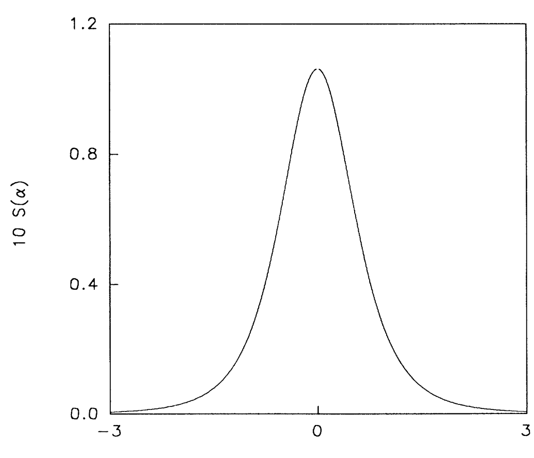

FIG. 11. Power spectrum of the linear oscillator as a function of the spectral frequency α. Parameter values: $a = 1, \beta = 2, \lambda = 1.5$, and $\omega = 1$.

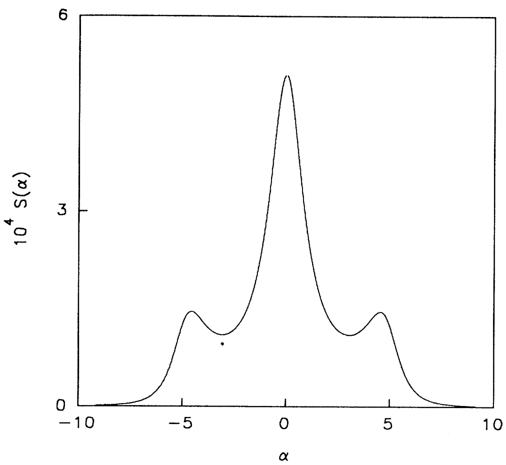

FIG. 12. Power spectrum of the linear oscillator as a function of the spectral frequency α. Parameter values: $a = 1, \beta = 2, \lambda = 0.5$, and $\omega = 5$.

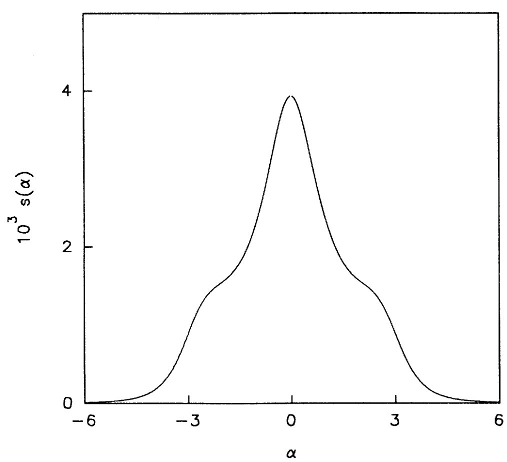

FIG. 14. Power spectrum of the linear oscillator as a function of the spectral frequency α. Parameter values: $a = 1, \beta = 2, \lambda = 0.5$, and $\omega = 3$.

(2) The correlation time of the driving noise is longer than the relaxation time:

$$\tau_c > \tau_r.$$

We have two different regimes:

(a) $\tau < 2\pi\tau_r\{2 - (\tau_r/\tau_c)^2 + \sqrt{3[1-(\tau_r/\tau_c)^2]}\}^{-1/2}$.

In this situation the power spectrum behaves as in case (1)(a) and it has three maxima and two minima [cf. Eqs. (5.6) and (5.7)] (Fig. 12).

(b) $\tau > 2\pi\tau_r\{2 - (\tau_r/\tau_c)^2 + \sqrt{3[1-(\tau_r/\tau_c)^2]}\}^{-1/2}$.

This situation is similar to that of case (1)(c) and the spectral density only has one maximum for $\alpha = 0$ (Fig. 13). However, the numerical analysis of Eq. (5.5) shows

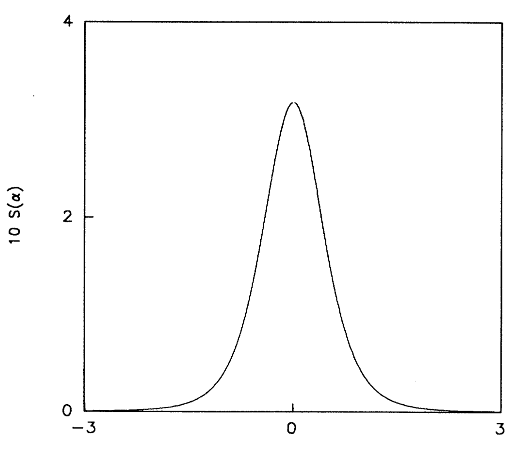

FIG. 13. Power spectrum of the linear oscillator as a function of the spectral frequency α. Parameter values: $a = 1, \beta = 2, \lambda = 0.5$, and $\omega = 1$.

that in this case there exists a range of period values such that $S(\alpha)$ has two symmetrical inflection points around the maximum (Fig. 14).

For Ornstein-Uhlenbeck driving noise the power spectrum is also given by Eq. (5.5) with $a^2 = 2\lambda D$. Therefore, $S(\alpha)$ presents the same behavior as discussed above.

We finally mention that in the Gaussian-white-noise limit we obtain from Eq. (5.5) the well known result [12]

$$S(\alpha) = \frac{D}{\pi[(\alpha^2 - \omega^2)^2 + \beta^2\alpha^2]}. \tag{5.10}$$

As one can expect, the behavior of the power spectrum is similar to that of cases (1)(b) and (c) above. Indeed, now $\tau_c = 0$ and this only leads to cases (1)(b) and (c). Thus, when

$$\beta < \omega\sqrt{2},$$

the power spectrum has one maximum at $\alpha = 0$ and two minima at

$$\alpha = \pm\sqrt{\omega^2 - \beta^2/2}.$$

If the above condition does not hold, then the two maxima merge into a single maximum at $\alpha = 0$.

VI. CONCLUSIONS

We have studied the second-order properties of linear oscillators driven by exponentially correlated noise. We have shown that the dynamical exponent $\nu(t)$ associated with the random position $X(t)$ of the oscillators presents several crossovers from $\nu(t) = 4$, accounting for ballistic motion, to $\nu(t) = 0$, corresponding to the stationary regime. We have also shown that $\nu(t)$ can take on the intermediate values 3, 2, and 1 depending on the fundamental time scales τ_c, τ_r, and τ.

We observe that one can easily obtain numerical estimates for the duration both of "transient regimes" where $\nu(t)$ is not constant and of "stable regimes" where $\nu(t)$

is approximately constant. Therefore, our method can provide quantitative data that can be of interest for experimental work. Moreover numerical analysis (see the figures in Sec. III) seems to indicate that, in several cases and regardless the order of magnitude of the fundamental time scales, these transient regimes have much shorter duration than stable regimes. This might indicate that in many situations the random dynamical evolution of the oscillator goes through well defined regimes. One can argue that the short duration of transients is a direct consequence of a wide separation of time scales. Nevertheless, Fig. 6 shows a case where all time scales are similar and transients are much shorter than stable regimes. Evidently, all of this is just a conjecture that we have not yet been able to prove.

Another interesting feature, which is only associated to dichotomous driving noise, consists in the appearance of resonances for the average values of the amplitude, the energy, and power dissipation. Resonance provides amplification. We thus see that, for certain values of the correlation time, the addition of dichotomous Markov noise enhances the system response in a similar way as is done by deterministic harmonic forces. These resonant phenomena have no counterpart for Gaussian colored noise. We should mention that these phenomena are in the sense of classical resonance in linear systems and they do not refer to the so called "stochastic resonance." The latter is related to nonlinear systems subject to both periodic and random forcing [17,18].

Finally, we have studied the spectrum of the oscillator in the stationary state. In this case the behavior of the spectrum can be substantially altered by color. Therefore, for certain regimes either the energy or the intensity of the oscillator is strongly dependent on the correlation time of the input noise regardless of its probabilistic nature.

ACKNOWLEDGMENTS

This work has been supported in part by Dirección General de Investigación Científica y Técnica under Contract No. PB90-0012 and by Societat Catalana de Física (Institut d'Estudis Catalans).

APPENDIX: DERIVATION OF EQ. (5.2)

If we assume that $X(0) = \dot{X}(0) = 0$, then the solution of an initial-value linear problem such as (2.1) and (2.2) can be written as [cf. Eq. (2.3)]

$$X(t) = \int_0^t dt' G(t') F(t-t').$$

Let us now evaluate the autocorrelation function of $X(t)$. We have

$$\langle X(t+\tau)X(t)\rangle = \int_0^{t+\tau} dt' G(t') \int_0^t dt'' G(t'') \times \langle F(t+\tau-t')F(t-t'')\rangle. \tag{A1}$$

$F(t)$ is a stationary noise and this allows us to write

$$\langle F(t+\tau-t')F(t-t'')\rangle = C(\tau - t' + t'') = \int_{-\infty}^{\infty} d\alpha e^{i\alpha(\tau-t'+t'')}\tilde{C}(\alpha), \tag{A2}$$

where $\tilde{C}(\alpha)$ is the spectral density of the driving noise. Hence

$$\langle X(t+\tau)X(t)\rangle = \int_{-\infty}^{\infty} d\alpha e^{i\alpha\tau}\tilde{C}(\alpha) \int_0^{t+\tau} dt' e^{-i\alpha t'} G(t') \times \int_0^t dt'' e^{i\alpha t''} G(t''). \tag{A3}$$

We now let $t \to \infty$. Thus

$$K(\tau) \equiv \lim_{t\to\infty} \langle X(t+\tau)X(t)\rangle = \int_{-\infty}^{\infty} d\alpha e^{i\alpha\tau}\tilde{C}(\alpha) \left| \int_0^{\infty} e^{-i\alpha z} G(z) dz \right|^2, \tag{A4}$$

and Eq. (5.2) holds.

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