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Vacancy effects on the force distribution in a two-dimensional granular pile

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The effect of vacancies on the distribution of forces within a two-dimensional pyramid-shaped granular pile has been investigated. With a hexagonal array of vacancies the pile is stable in the absence of friction, and the force distribution can be calculated exactly. This shows that the vertical component of load acting on each grain of the bottom layer is identical.

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The stress analysis of solids (both elastic and plastic) and liquids is a relatively mature subject, dating back over a century. Granular materials, on the other hand, have received much less attention. Knowledge of the distribution of stress within a powder due to applied loads is important for practical applications (e.g., for the design of storage vessels), and when understanding the bulk mechanical properties, both under quasistatic and dynamic conditions. One problem of scientific interest is the stress distribution within a conical sandpile. Intuitively, one might expect the stress on the floor to be proportional to the height of material above it, and therefore to be also of conical shape with a maximum under the peak of the heap. However, experiments have shown that the stress is a local minimum at this point [1]. It has been suggested that this behavior may be due to redistribution of the load by internal arches [2]. Computational [3] and (more recently) exact analytical [4,5] results have been presented for the idealized case of a two-dimensional hexagonal array of disks. These show that for an equilateral pile (angle of repose $=60°$) the vertical component of the force is uniform across the base. There are several significant features of this model which are not present in the real sandpile: (i) friction is assumed to be either zero [4,5] or to be determined by a fixed friction coefficient [3] at all contacts; (ii) packing is regular rather than random; (iii) no vacancies are present; and (iv) the forces are constrained to two dimensions. It is therefore of interest to investigate the effect of relaxing these assumptions in an attempt to explain the experimental observations. References [6–8] describe numerical studies in which random packing was introduced; these appeared to produce stress profiles with a dip in the middle. However, other numerical experiments involving random packing [4] did not result in a reproducible local pressure minimum.

In this Brief Report we present exact results for the same two-dimensional granular pile reported in Refs. [4,5], but modified by the presence of a regular array of vacancies, as shown in Fig. 1. This particular vacancy distribution is easier to analyze than other possibly more realistic arrangements (e.g., a few isolated vacancies), since it is stable in the absence of frictional forces. It also forces the pile to form successive layers of arches, each layer two disks deep. Three typical arches are outlined in Fig. 1. The formation of arches is important in the behavior of real powders, for example, it can cause the blockage of hoppers and pipes. The result of the analysis, however, is that the vertical component of the force acting on each grain in the bottom layer is identical, i.e., the same results as for the vacancy-free model.

The model shown in Fig. 1 consists of $2p-1$ layers of disks ($p=1,2,3\ldots$), with vacancies on alternate sites on alternate layers. $p=6$ in this example. A nonorthogonal coordinate system (i,j) with axes along the surface diagonals will be used in the analysis. Vacancies are present at sites $(i,j)=(2m,2n)$ $(m,n=1,2,3\ldots)$. The horizontal spacing of the disks is chosen to be large enough for horizontal contacts not to occur, except between the two disks forming an arch over a vacancy. Contact is required at these points, otherwise the sand heap will collapse. The sand heap is only stable if the floor on which it is standing is rough, and thereby capable of providing a horizontal force inwards to the center of the heap. For this model, we assume the roughness to be due to asperities of the same spacing and diameter as the disks. This avoids the need for friction at the contacts; in effect, the floor is indistinguishable from another row of disks to the bottom

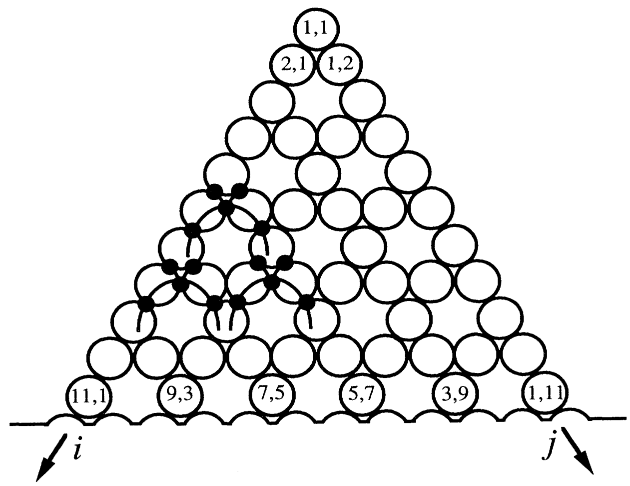

FIG. 1. Hexagonally packed granular pile with vacancies. The pile consists of $2p-1$ layers ($p=6$ in this example) supported on a rough floor. Three typical arches formed by the vacancies are indicated, together with the contact points (●).

layer of the granular pile.

The unit cell (m,n) comprises three disks as shown in Fig. 2(a), occupying sites $(i,j)=(2m-1,2n-1)$ (upper disk), $(2m,2n-1)$ (lower left), and $(2m-1,2n)$ (lower right). The centers form an equilateral triangle with angle $\theta=\pi/3$ for a hexagonal-close-packed array. The weight of each disk is W. Friction is assumed to be zero everywhere, so that the contact forces act along the i and j axes. $I(i,j)$ and $J(i,j)$ will be used to denote the forces on the particle (i,j) from above, along the respective axes. Figure 2(b) shows all the forces acting on the particles within the unit cell. H is the horizontal force between the two lower disks, which is needed for the stability of the arch above the vacancy at site $(2m,2n)$. For equilibrium of the unit cell, the total force in the horizontal and vertical directions on each disk must be zero. For the top disk, this gives

$$I(2m,2n-1)+J(2m-1,2n-1)-I(2m-1,2n-1)-J(2m-1,2n)=0\ , \quad (1)$$

$$I(2m,2n-1)-J(2m-1,2n-1)-I(2m-1,2n-1)+J(2m-1,2n)=W/\sin\theta\ . \quad (2)$$

Adding and subtracting Eqs. (1) and (2) results in

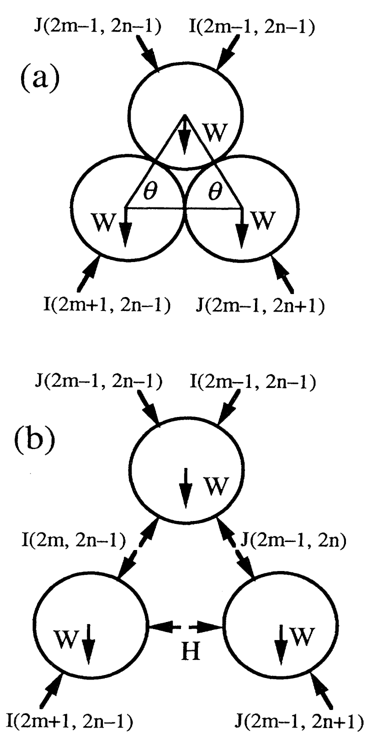

FIG. 2. (a) Unit cell for the pile. Cell (m,n) has disks at coordinates $(i,j)=(2m-1,2n-1)$ (upper), $(2m,2n-1)$ (lower left), and $(2m-1,2n)$ (lower right), with a vacancy at $(2m,2n)$. (b) Force diagram for the unit cell. I and J are forces acting along the i and j axes, respectively.

$$I(2m,2n-1)=I(2m-1,2n-1)+W/2\sin\theta\ , \quad (3)$$

$$J(2m-1,2n)=J(2m-1,2n-1)+W/2\sin\theta\ . \quad (4)$$

For the lower left disk, we get

$$I(2m+1,2n-1)=I(2m,2n-1)+H/\cos\theta\ , \quad (5)$$

$$I(2m+1,2n-1)=I(2m,2n-1)+W/\sin\theta\ , \quad (6)$$

and for the lower right disk

$$J(2m-1,2n+1)=J(2m-1,2n)+H/\cos\theta\ , \quad (7)$$

$$J(2m-1,2n+1)=J(2m-1,2n)+W/\sin\theta\ . \quad (8)$$

Equations (5) and (7) are satisfied provided $H=W\cos\theta/\sin\theta$. The recurrence relations of Eqs. (3), (4), (6), and (8) can be combined with the stress-free surface condition $I(1,j)=J(i,1)=0$ to calculate the force distribution throughout the pile. In particular, the disks on the bottom layer of the pile occupy sites $(2m-1,2n-1)$, which are supported by forces

$$I(2m,2n-1)=(3m-2)W/2\sin\theta\ , \quad (9)$$

$$J(2m-1,2n)=(3n-2)W/2\sin\theta\ . \quad (10)$$

These can be resolved vertically and horizontally to give the normal (N) and transverse (T) forces:

$$N=(3m+3n-4)W/2\ , \quad (11)$$

$$T=3(m-n)W\cos\theta/2\sin\theta\ . \quad (12)$$

The coordinates for the bottom layer of disks $(i,j)=(2m-1,2n-1)$ satisfy the relation

$$i+j=2p\ . \quad (13)$$

By combining Eqs. (11) and (13) it can be seen that

$$N=(3p-1)W/2\ , \quad (14)$$

i.e., that the normal force is uniform across the heap. There are p disks in the lower layer, so that the total normal force on the floor, N_T, is given by

$$N_T=(3p-1)pW/2\ . \quad (15)$$

It is easily verified that there are $(3p-1)p/2$ disks in the pile, the total weight of which therefore equals N_T, as expected. The transverse force [Eq. (12)] is zero on the axis of the pile ($m=n$), and reaches a maximum value

$$|T_m|=3(p-1)W\cos\theta/2\sin\theta \quad (16)$$

at the edge, where $(m,n)=(p,1)$ or $(1,p)$. For large heaps, the ratio $|T_m|/N$ reaches the limiting value of $\cot\theta$, or $1/\sqrt{3}$ for this geometry.

In conclusion, the distribution of forces within a hexagonal-close-packed granular pile containing a regular array of vacancies have been calculated. The presence of the vacancies results in a series of arches which were found to modify the stress distribution locally but not globally. In particular, the normal force on each grain in the bottom layer of the pile is identical, i.e., the same result as for the vacancy-free case. This suggests that the pressure dip at the center of a granular pile, which is observed experimentally, is due to either friction, random packing, or three-dimensional effects.

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