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Mean-field theory of sandpile avalanches: From the intermittent- to the continuous-flow regime

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We model the dynamics of avalanches in granular assemblies in partly filled rotating cylinders using a mean-field approach. We show that, upon varying the cylinder angular velocity ω, the system undergoes a hysteresis cycle between an intermittent- and a continuous-flow regime. In the intermittent-flow regime, and approaching the transition, the avalanche duration exhibits critical slowing down with a temporal power-law divergence. Upon adding a white-noise term, and close to the transition, the distribution of avalanche durations is also a power law. The hysteresis, as well as the statistics of avalanche durations, are in good qualitative agreement with recent experiments in partly filled rotating cylinders.

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I. INTRODUCTION

The dynamics of granular materials is not well understood in spite of its widespread scientific and technological interest [1]. Bistability, segregation, arching, hysteresis, instabilities, and other properties found in noncohesive granular assemblies make these systems very difficult to analyze. For over two centuries, most studies have focused on the two extreme regimes: the fluidlike continuous flow and the static compact state. The intermediate quasistatic regime, with intermittent transitions between flowing and static behavior, has recently generated a flurry of activity including several neat experiments [1–5] and novel theoretical proposals (see, e.g., Refs. [6,7]).

Several experiments [2–4] have studied the dynamics of granular assemblies in partly filled rotating drums. In particular, Rajchenbach [3] monitored the following quantities: (i) the statistics of avalanches as a function of the rotating speed of the cylinder ω, (ii) the transition from the discrete to the steady regime, including experimental evidence of hysteresis as a function of ω, and (iii) the dependence of the current of particles j with the angle θ between the top layer and the horizontal. It is the purpose of this work to study these issues by using a simple mean-field model with two dynamical variables. The recent studies on this subject raise many interesting questions in granular transport, such as hysteresis in ω (see, for instance, Fig. 1 of Ref. [3]) and avalanche statistics, and this work addresses some of them.

Our mean-field approach reduces the many degrees of freedom of a granular material in a rotating drum into two: the average velocity $v(\equiv dx/dt \equiv \dot{x})$ of the avalanche and the average angle θ of the granular assembly top surface (of length L, $0 < x \leq L$). The grains move in the "downhill" x direction of motion, which forms an angle θ with the horizontal, and the shear between two layers with a separation d is $\dot{\gamma} = v/d$. An important feature of our approach is that the dynamics of the model used here predicts the occurrence of avalanches as a function of ω. By slowly increasing ω, we obtain the angular velocity ω_+ at which the system undergoes a transition from the discrete flow regime to a continuous flow (see Fig. 1). At this point, if ω is decreased, the reverse transition is observed at ω_-, where $\omega_- < \omega_+$. Thus, our model exhibits hysteresis in ω during the transition between the "intermittent-" and "continuous-" flow regimes.

Bagnold [9] extensively studied the dissipation produced by the grain collisions in several regimes, including the quasistatic intermittent state, finding in the latter a periodic response over time. On the other hand, simple rules for the dynamics [6] do not predict a periodic response over time but the absence of length and time scales. Our model can produce distributions in agreement with recent experiments [3,4]. Furthermore, when $\omega \to \omega_+$, we predict that the system exhibits a power-law critical slowing down.

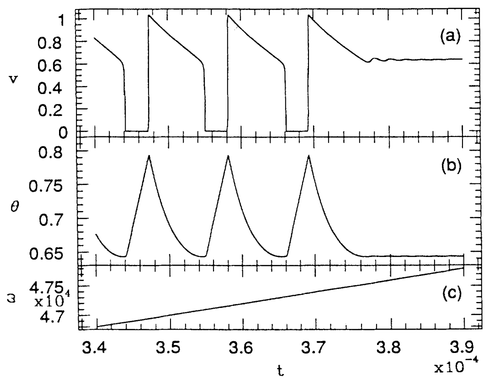

FIG. 1. The time dependence of (a) the flow velocity v, (b) the angle θ of the top layer, and (c) the angular velocity ω of the rotating drum. Throughout this work, we use the following parameters values: $\alpha = 1$, $\alpha_1 = 2$, $\beta = 1/2$, and $\rho = 10^{-3}$. For these ramping rates, we obtain $\omega_+ = 4.76 \times 10^{-4}$ and $\omega_- = 4.66 \times 10^{-4}$ for the $D_0 = 0$ deterministic case.

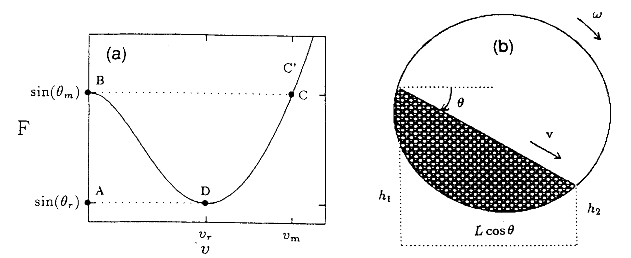

FIG. 2. (a) Schematic plot of the friction force versus velocity v for the case $c > 1$. An initially flat ($\theta = 0$) and stationary ($v = 0$) granular surface is tilted with an angular velocity ω, thus having the following sequence of values for the friction force: $O \to A \to B$, from the $F = v = 0$ origin O. The velocity v is zero in the stick-slip case with infinite slope for the OAB segment. In the AB branch, the loading time τ_A satisfies $\langle \tau_A \rangle = (\theta_m - \theta_r)/\omega$. When the maximum angle of stability is reached in B, an avalanche ($v \neq 0$) starts and the system jumps to C. Different situations, described in the text, can now develop according to the value of ω. (b) Schematic diagram of a rotating drum indicating the extreme heights, the mean downhill velocity v, and the mean angle θ.

II. DISSIPATION IN GRANULAR MEDIA

In granular flow, the friction force depends on v^2 for large values of the velocity. The models of Bagnold [9] and Jenkins and Savage [10] correctly describe this limit. Jaeger *et al.* [8] analyzed how the stress varies as v approaches zero, obtaining the following form for the total friction force: $mv^2/2\lambda + \alpha mg\cos\theta/\kappa$, where $\kappa = 1 + \alpha_1 v^2/gd\cos\theta$, which in dimensionless form can be written as $F = \beta v^2 + \alpha\cos\theta/\kappa$, where now $\kappa = 1 + \alpha_1 v^2/\cos\theta$, $v = v/\sqrt{dg}$, and α, α_1, β, λ are constants. Here the first (second) term dominates the friction force at high (low) velocities and, when $v \to 0$, the second term represents the maximum dry friction due to rubbing forces.

III. DYNAMICS

The equation of motion is $\dot{v} = \sigma - F$, where $F = F(\theta, v)$ is the friction force and σ is the gravitational force ($\propto \sin\theta$). If $c \equiv \alpha\alpha_1/\beta > 1$, then the frictional force decreases as a function of v, until a minimum is reached at v_r, and then increases. We focus on this case $c > 1$, illustrated in Fig. 2(a). The stationary points of F determine the maximum θ_m and minimum (or repose) θ_r angle of stability. Explicitly $\theta_m = \arctan(\alpha)$ and $\theta_r = \arctan[\alpha(2\sqrt{c} - 1)/c]$.

Let us consider a stationary ($v = 0$) granular assembly with a horizontal ($\theta = 0$) top surface contained in a partly filled cylinder. If we slowly rotate the cylinder at an angular velocity $\dot{\theta} = \omega$, the average angle of the granular assembly surface θ grows until it reaches the maximum angle of stability θ_m, where the system suddenly produces avalanches. During this sudden transport of mass, the average velocity of the sand grains v becomes nonzero and θ decreases until it reaches the angle of repose θ_r, where the system relaxes (v=0). At this point the cycle repeats itself.

In order to model this dynamics, it is useful to know the time evolution of the mean angle θ. For this purpose, let us consider the two extreme heights [shown in Fig. 2(b)] of granular vertical columns. Since the column heights are proportional to the mass of the columns, the rate of mass transport is proportional to the height difference ($0 < x \leq L$),

$$\frac{\partial h_1}{\partial t} = -\frac{v}{L}(h_1 - h_2), \quad \frac{\partial h_2}{\partial t} = +\frac{v}{L}(h_1 - h_2)\,. \qquad (1)$$

The proportionality constant is the inverse of the propagation time. The sum of these equations gives the mass conservation condition $\partial_t(h_1 + h_2) = 0$, which is valid in a closed rotating drum, while substracting them gives

$$\dot{\theta} = -2\frac{v}{L}\tan\theta \qquad (2)$$

since $h_1 - h_2 = L\sin\theta$.

In this work, we concentrate our efforts in this $\partial_x\theta = 0$ case. Elsewhere, we will consider the case with local spatial variations in $\theta(x,t)$, relevant to describe the S-shaped curves seen in the continuous flow regime for very large values of ω.

We thus model the dynamics of avalanches with the following equations of motion:

$$\dot{v} = \sin\theta - \beta v^2 - \frac{\alpha\cos\theta}{1 + \alpha_1 v^2/\cos\theta} = \sin\theta - F(\theta, v), \quad (3)$$

$$\dot{\theta} = -\rho v\tan\theta + \omega + \eta(t), \qquad (4)$$

where $\rho = 2\sqrt{dg}/L$, Eq. (3) is Newton's equation of motion, and the very small Gaussian noise term $\eta(t)$ simulates small fluctuations in ω and satisfies $\langle\eta(t)\rangle = 0$ and $\langle\eta(t)\eta(t')\rangle = D_0\,\delta(t - t')$, with $D_0 \ll \omega$.

In the continuous-flow regime, the equation for the steady state ($\dot{v} = 0$, $\dot{\theta} = 0$) is

$$\sin\theta = \beta \left(\frac{\omega}{\rho \tan\theta}\right)^2 + \alpha \cos^2\theta \left[\cos\theta + \alpha_1 \left(\frac{\omega}{\rho\tan\theta}\right)^2\right]^{-1} . \quad (5)$$

This solution corresponds to any point in the branch DC' in Fig. 2(a) with positive slope in $F(v)$. Let us denote by v_r (v_m) the velocity corresponding to θ_r (θ_m) in Eq. (3) when $\dot{v} = 0$. When $\omega \ll \rho v_r \tan\theta_r$, the system has intermittent avalanches, and in each one of them Eq. (4) predicts that θ relaxes quickly to θ_r. When $\omega > \rho v_m \tan\theta_m$, the system is certainly in the continuous flow regime at some point C' above C [see Fig. 2(a)]. If $\rho v_r \tan\theta_r < \omega < \rho v_m \tan\theta_m$, then the system evolves from C to a point between C and D and either remains there (if the stationary solution is stable at this point) or jumps back (i.e., $\theta \to \theta_r$ and $v \to 0$) to the AB segment and the avalanche stops. If $\omega < \rho v_r \tan\theta_r$, then the system certainly decays to the AB segment and has intermittent avalanches evolving through the loop $ABCD$, and so on, since D is unstable.

IV. NUMERICAL RESULTS

Figure 1 shows the time dependence of (a) the flow velocity v, (b) the angle θ of the top layer, and (c) the angular velocity ω of the rotating drum, obtained by numerically solving Eqs. (3) and (4) with $D_0 = 0$. For each avalanche, v has a peak and θ has a sudden decrease from $\theta_m \approx 0.8$ to $\theta_r \approx 0.65$. The $v = 0$ regions in between peaks correspond to the loading time in between avalanches, where the angle increases linearly in time from θ_r to θ_m. For clarity, only three avalanches are shown. They are approaching the transition point ω_+ to the continuous-flow regime, and their periods become longer close to ω_+, i.e., the system exhibits a critical slowing down. The values of both ω_+ and ω_- depend on $\dot{\omega}$ and the constants α, α_1, β, and D_0 in Eqs. (3) and (4). The addition of noise also reduces the hysteresis.

Notice the asymmetric form of the velocity peaks in Fig. 1(a). During a loading time τ_A, the system moves from A to B, on the branch AB in Fig. 2(a). Right after, an avalanche starts and v has a sudden increase lasting a time τ_B [$B \to C$ in Fig. 2(a)], a slow decrease with duration τ_C ($C \to D$), and a final decrease lasting a time τ_D ($D \to A$).

Equations (3) and (4), without a random source term, and for a fixed ω, produce a periodic sequence of avalanches. However, the real many-body system inevitably has *disorder*. When $D_0 \neq 0$, we observe a richer behavior [see Figs. 3 and 4(b)] with broad distributions.

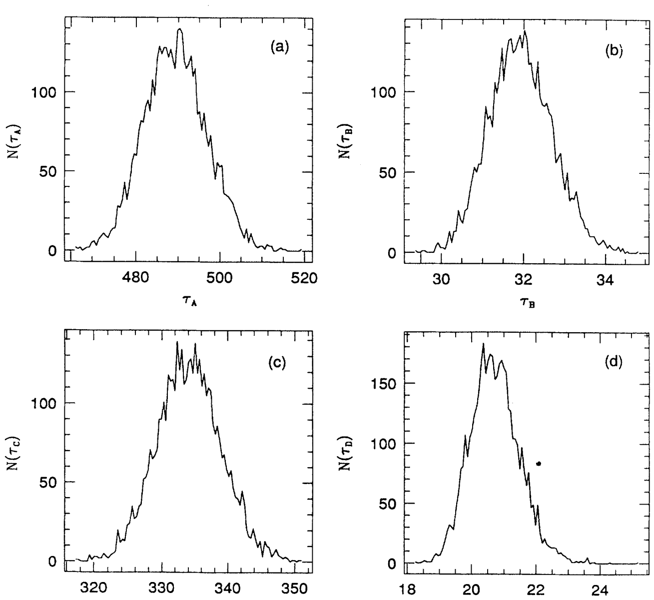

FIG. 3. Statistics of the avalanche durations. Distributions N for (a) τ_A, (b) τ_B, (c) τ_C, and (d) τ_D, for $\omega = 3 \times 10^{-4}$ and $D_0 = 10^{-8}$.

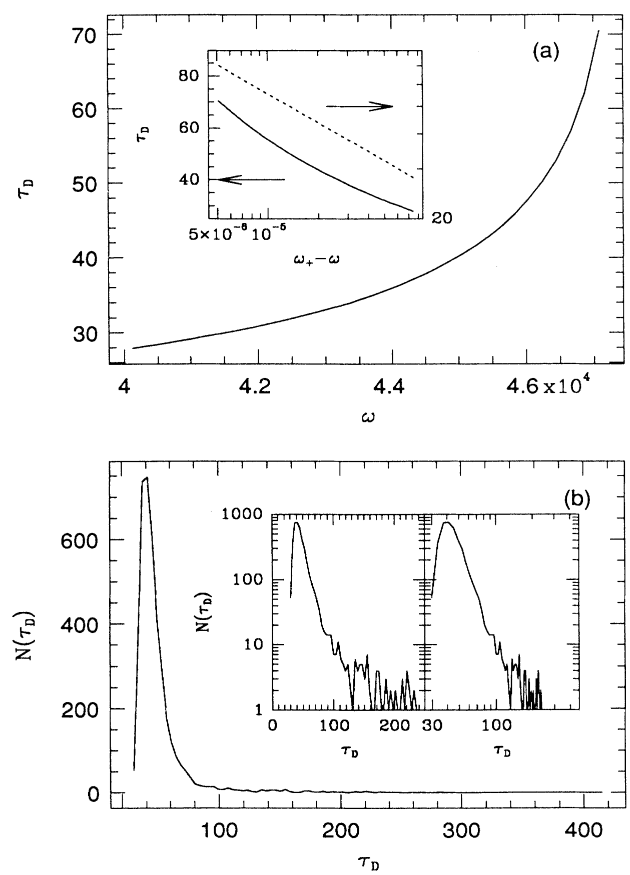

FIG. 4. (a) Time evolution of $\tau_D(\omega)$ when $\omega \longrightarrow \omega_+(D_0 = 0)$. This critical slowing down with a power-law divergence is shown in the inset by the dashed line corresponding to the right vertical logarithmic axis. For comparison purposes, the solid line in the inset, corresponding to the left vertical linear axis, has a small curvature because of the small value of the power ($p \simeq 0.34$). (b) Distribution of τ_D for $\omega = 4.6 \times 10^{-4}$, which is slightly below $\omega_+(D_0 = 0)$. The inset shows the same data in semilog and log-log scales.

The statistics of the avalanche durations are shown in Fig. 3. In order to gain more insight into the several time scales involved in the problem, we show the distributions N for (a) τ_A, (b) τ_B, (c) τ_C, and (d) τ_D, for $\omega = 3\times10^{-4}$ and $D_0 = 10^{-8}$.

Figure 4(a) shows the evolution of τ_D when $\omega \longrightarrow \omega_+$. We have considered two different possibilities for the divergence: logarithmic (left vertical axis of the inset) and power law (right vertical axis). For $D_0 = 0$, the behavior of τ_D at the transition is found to be

$$\tau_D \sim (\omega_+ - \omega)^{-p} \qquad \text{for } \omega < \omega_+ \,. \tag{6}$$

Temporal critical slowing down behavior has also been found in bistable optical systems [11]. From Eq. (6) and recalling that $\omega + \eta$ is a Gaussian random variable, one obtains that $N(\tau_D) \sim \tau_D^{-p}$. This is checked numerically in Fig. 4(b) where we have obtained the distribution of τ_D for fixed $\omega = 4.6\times10^{-3}$, which is slightly below $\omega_+(D_0 = 0)$, with $D_0 = 10^{-8}$.

In the intermittent-flow regime, our dynamical model exhibits an attractive limit cycle around an unstable fixed point. For $\theta < \theta_m$, $v = 0$ is a stable attractor manifold. In the continuum-flow regime, the dynamical system is in an attractive fixed point with nonzero velocity. Here a harmonic approximation to Newton's equation [Eq. (3)] with $\dot{v} = 0$ gives

$$v - v_r \simeq (\theta - \theta_r)^{1/2} \tag{7}$$

for the relationship between the mass current flow and the angle, for $\omega \geq \omega_-$. This result is consistent with experimental results by Rajchenbach [3].

V. LIMITATIONS

Our mean-field equations have only two degrees of freedom for modeling an extremely complex system. Thus, it might need to be extended in order to provide a more complete description of the system. The beauty of the model lies precisely in its simplicity and in the fact that it can naturally describe several dynamical features observed in experiments in granular media.

VI. CONCLUSIONS

We have studied a mean-field model of granular assemblies driven to the threshold of instability, where they produce avalanches. From this simple model, we obtained the distributions of durations of avalanches, the transition between the two flowing regimes, and the hysteresis between them in a natural manner, and with a good qualitative agreement with results recently obtained using experiments in rotating cylinders. Furthermore, when $\omega \rightarrow \omega_+$, the avalanche durations exhibit a critical slowing down with a temporal power-law divergence.

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