Wave interaction with a fractal layer

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We investigate the reflection coefficient of a fractal layer in the small-wavelength limit. Taking the layer structure simulated by a devil's staircase on the Cantor support as an example, we find numerically that the decay of the reflection coefficient with the wave number can be approximated by a power law. We explain this phenomenon analytically based on the two-scale method.

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I. INTRODUCTION

One of the main results in the scattering theory is that the reflection coefficient of a layer, whose permittivity is described by an infinitely differentiable function, exponentially goes to zero with the wave number (see, e.g., Refs. [1, 2], and also Ref. [3] related to reflectionless potentials). However, existence of boundaries, where the function describing the medium properties is not continuous, can give rise to the so-called resonance effects, and in this case, the reflection coefficient does not go to zero with the wave number (see, e.g., Ref. [4]).

It is interesting to consider an intermediate case in which the permittivity function is continuous, but its first derivative has an infinite number of discontinuous points. An example of such a function is given by the fractal devil's staircase [5, 6]. The aim of the present paper is to investigate the behavior of the reflection coefficient of a layer described by such a fractal function.

There are at least two reasons why this problem is interesting. First, an extent transition between two media having different dielectric permittivities may have irregular structure and a rather rich spectrum of inhomogeneities. The classical way to simulate such transitions is to describe them by random functions [7]. Meanwhile, there is a great number of processes characterized by structures with power dependent spectra and fractal geometry configuration [8]. In this sense, introducing models in which the media is described by different fractal function can essentially diversify physical systems that may also be interesting to experimentalists. Second, most of the previous considerations on wave interaction with one-dimensional fractal objects [9–13] deal with the scattering caused by layer "boundaries." In that case the reflection coefficients are mainly determined by a structure, whose characteristic scales are much larger than the length of the incident wave. In other words, the resonant interaction was discussed. As it has been shown in Refs. [11,13], the relative contribution of small-scale structure to the scattering data (in comparison with the large-scale one) grows either with the slab length or with the wave number. The model in the present paper allows us to exclude "boundary" effects and to concentrate only on the small-scale scattering.

At the first sight, there is not an evident small parameter in the problem of the scattering by a fractal layer, if the strength of inhomogeneities is not assumed to be rather small *a priori*. However, as it has been shown in Refs. [11, 13] an effective small parameter related to a small wavelength can be introduced if the wavelength is small enough. Thus it is possible to employ the two-scale method to construct some perturbation (or asymptotic) expansion for the scattering data. In this way, we can investigate the properties of the scattering by fractal structures in the small wavelength limit.

In the case of resonant scattering, the leading order of the expansion mentioned above is determined by an effective large-scale component of inhomogeneities. However, the dimension of the fractal depends on a small-scale part if the fractal is generated by the infinite splitting of a structure [5, 6], thus the results of the resonant scattering contain no information about fractal dimension. To obtain such information we must take into account the next order of the expansion.

The paper is organized as follows. In Sec. II we give a brief description of the model under consideration and present our numerical results of the reflection coefficients for different parameters of the fractal layer. Explanation of the behavior of the reflection coefficient on the basis of two parameter expansion is given in Sec. III. Finally in Sec. IV we discuss the numerical and the analytical outcomes, and draw the conclusions.

II. THE MODEL AND NUMERICAL RESULTS

Let us consider the one-dimensional Helmholtz equation,

$$\frac{d^2\psi}{dx^2} + k^2[1 - \epsilon_L(x)]\psi = 0, \tag{1}$$

where k plays the role of the wave number, and $\epsilon_L(x)$ is defined by an integral,

$$\epsilon_L(x) = \int_{-\infty}^{x} \mu_L(\xi)d\xi. \tag{2}$$

The function $\mu_L(\xi)$ is assumed to be a multifractal based on the Cantor support, and it is constructed in the following way [6]. Starting with a function $\mu_0(x) = \mu_0$

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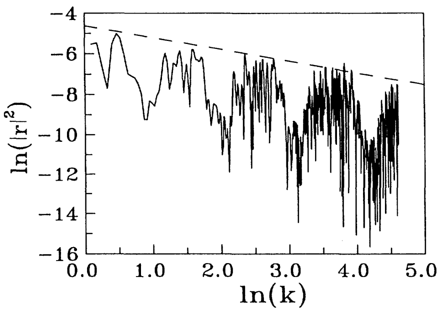

FIG. 1. $\ln(|r|^2)$ vs $\ln(k)$, where r is the reflection coefficient and k is the wave number. Dashed line has the slope, $-\beta$, given by formula (23). Here $\mu_0 = 0.05$.

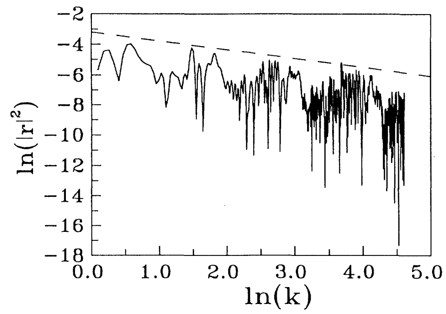

FIG. 2. The same as in Fig. 1, but $\mu_0 = 0.07$.

(=const) for $x \in [0, L]$ and $\mu_0(x) = 0$ for $x \notin [0, L]$, we put $\mu(x)$ equal to zero in the middle third interval $(L/3, 2L/3)$, and we increase the value of $\mu(x)$ in the rest intervals to $\frac{3}{2}\mu_0$. Then we obtain a function $\mu_1(x) = \frac{3}{2}\mu_0$ for $x \in [0, L/3] \cup [2L/3, L]$ and $\mu_1(x) = 0$ otherwise. In the second step we repeat the same process with each of the intervals $[0, L/3]$ and $[2L/3, L]$. Then $\mu_L(x)$ is the limit of the infinite number of these steps. The pictures of both $\epsilon_L(x)$ and $\mu_L(x)$ related to the model under discussion can be found, for instance, in the book by Feder [6]. Here we point out that $\epsilon_L(x) = 0$ for $x < 0$ and $\epsilon_L(x) = \epsilon_L = \mu_0 L$ for $x > L$, i.e., L plays a role of a layer width.

We only consider the case $\epsilon_L(x) < 1$ (i.e., with no turning points), which requires the condition

$$\mu_0 L < 1. \tag{3}$$

First we study the Eq. (1) numerically. In order to get an accurate value of the reflection coefficient we use the following representation:

$$|r|^2 = \frac{[C'(L)/k_L]^2 + C^2(L) + k^2 S(L)^2 + (k/k_L)^2 [S'(L)]^2 - 2k/k_L}{[C'(L)/k_L]^2 + C^2(L) + k^2 S(L)^2 + (k/k_L)^2 [S'(L)]^2 + 2k/k_L}, \tag{4}$$

where $k_L = k\sqrt{1-\epsilon_L}$, $C(x)$ and $S(x)$ being solutions of the equation (1) with "initial" condition $C(0) = S'(0) = 1$ and $C'(0) = S(0) = 0$.

Because we are interested in the small wavelength limit ($k \to \infty$,) we must keep the smallest scale of a fractal much less than the wavelength (otherwise, the conventional resonant scattering will play the main role which has no relation with the scattering by a fractal). This means that $l_N = L/3^N \ll k^{-1}$, N being the number of stages of the fractal generation [note that l_N is the width of an interval in which $\mu_N(x) \neq 0$]. In the numerical simulations we take k from 1 to 100 with a step size $\delta k = 0.1$, and we actually take $L = 10$ and $N = 12$ which satisfies the condition $l_{12} = 10/3^{12} \approx 1.88 \times 10^{-5} \ll 0.01$. We observe that the reflection coefficient will *not* change (within the acceptable accuracy) if a larger number of stages is used.

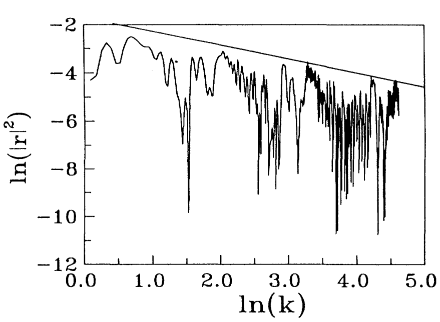

FIG. 3. The same as in Fig. 1, but $\mu_0 = 0.09$.

The numerical results are presented in Figs.1, 2, and 3 for $\mu = 0.05, 0.07, 0.09$, respectively. It is observed that the reflection coefficient does not decay in accordance with an exponential law; instead we find that the decay of the upper bound value of the reflection coefficient may be characterized by a *power* law. This fact does not depend on the value μ_0 as the slopes of the dashed lines are the same for all figures. The small-scale structures of the curves look rather irregular, but they appear quite similar to each other, especially in the neighborhood of the abscissa origin. The change of the value μ_0 only leads qualitatively to changes in the absolute value of the reflection coefficient and in the location of the transparency windows (i.e., regions where r is vanishingly small). "Frequency" of the last ones grows with k.

III. ANALYTICAL ESTIMATES

Now we will try to explain the phenomena observed in the numerical calculation of the preceding section. To accomplish this we use the idea of the two-scale method [11, 13].

Let us introduce an average permittivity,

$$\langle\epsilon_L(x)\rangle = \int_{-\infty}^{\infty} dz\, \delta_\kappa(x-z)\epsilon_L(z), \tag{5}$$

where the function $\delta_\kappa(x)$ is given by

$$\delta_\kappa(x) = \frac{1}{\pi}\frac{\sin \kappa x}{x}. \tag{6}$$

It is clear that $\lim_{\kappa\to\infty}\delta_\kappa(x) \to \delta(x)$, where $\delta(x)$ is the Dirac δ. The physical meaning of such averaging is also evident. It is easy to verify that $\langle\epsilon_L(x)\rangle$ has the same spectrum as $\epsilon_L(x)$ but with cutoff in the points $\pm\kappa$. In other words, the averaging (5) corresponds to the averaging over all spatial scales less than κ^{-1}. Let us also assume that $\kappa = k^\beta\lambda^{\beta-1}$ and find possible values of a constant exponent β such that the requirement

$$\left|\frac{d\langle\epsilon_L(x)\rangle}{dx}\right| \ll k(1-\epsilon_L) \tag{7}$$

is satisfied when $k\to\infty$ and λ is a parameter of unity order. The sense of the requirement (7) is also clear. In fact, in the absence of the small-scale part,

$$\Delta\epsilon_L(x) = \epsilon_L(x) - \langle\epsilon_L(x)\rangle, \tag{8}$$

of the permittivity, the contribution of $\langle\epsilon_L(x)\rangle$ to the scattering data can be described within the framework of the WKB expansion.

We define n_0 through the relation $\kappa^{-1}\sim l_{n_0} = L/3^{n_0}$, then $\delta_\kappa(x)$ is localized in the region $[-\kappa^{-1},\kappa^{-1}]$ which is much smaller than L, so we get the estimation

$$\left|\frac{1}{\kappa}\int_0^L dz\, \mu_L(z)\delta_\kappa(z-x)\right| \sim \left|\int_{x-\kappa^{-1}}^{x+\kappa^{-1}} dz\, \mu(z)\right| = O\left(\frac{\epsilon_L}{2^{n_0}}\right). \tag{9}$$

Since $2^{n_0}\sim(\kappa L/2)^\alpha$ with $\alpha = \frac{\log 2}{\log 3}\approx 0.63$ (note that α is just the fractal dimension of the middle third Cantor set [5]), we finally obtain the relation,

$$\frac{1}{k}\left|\frac{d\langle\epsilon_L(x)\rangle}{dx}\right| = O(\chi_1(kL)^{\beta(1-\alpha)-1}), \tag{10}$$

where $\chi_1 = \epsilon_L 2^{-\alpha}(\frac{L}{\lambda})^{(\beta-1)(\alpha-1)}$ does not depend on k.

Since we are interested in the limit $k\to\infty$, we can conclude that the requirement (7) is satisfied if

$$1+\beta(\alpha-1) > 0. \tag{11}$$

Thus we have determined a small parameter of the problem under consideration: it is related to the large scales and can be defined as $\nu_1 \sim \frac{1}{k}|\frac{d\langle\epsilon_L(x)\rangle}{dx}|$. This parameter will be *modified* later in order to achieve the desired accuracy in our estimation.

On the other hand, the approach based on two-scale method has sense only if the second small parameter exists. It must be defined through a small-scale part of the permittivity. Now we rewrite $\Delta\epsilon_L(x)$ through the spectrum of the inhomogeneities,

$$\Delta\epsilon_L(x) = \frac{i}{\sqrt{2\pi}}\left[\int_{-\infty}^{-\kappa} + \int_\kappa^\infty\right]\frac{dq}{q}e^{iqx}\hat{\mu}(q), \tag{12}$$

where (see, e.g., [14])

$$\hat{\mu}(q) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{iqx}\mu(x) = \frac{\mu_0 L e^{iqL/2}}{\sqrt{2\pi}}\prod_{n=1}^{\infty}\cos\left(\frac{qL}{3^n}\right). \tag{13}$$

Taking into account that the oscillating terms in the integrand of Eq. (12) do not have stationary phase points, the integral in the right-hand side of Eq. (12) can be estimated for $\kappa\to\infty$ by integration by parts. Straightforward algebra yields

$$\Delta\epsilon(x) = O(\chi_2(kL)^{-\beta}) = \chi_2\nu_2, \tag{14}$$

where $\chi_2 \sim \epsilon_L(L/\lambda)^{\beta-1}$ is considered as a constant. Thus the second small parameter exists if $\beta > 0$ and is given by Eq. (14).

Suppose that $\psi(x)$ represents either $C(x)$ or $S(x)$ defined in Eq. (4). As far as we have two effective small parameters that are different in general, we can search for a solution of the Helmholtz equation using the expansion of $\psi(x)$ with respect to the two parameters. Since the main order of the permittivity is given by $\langle\epsilon_L(x)\rangle$ it is natural to provide the expansion of the solution near $\psi_0(x)$ that solves the equation

$$\frac{d^2\psi_0}{dx^2} + k^2[1-\langle\epsilon_L(x)\rangle]\psi_0 = 0. \tag{15}$$

In other words we use the expansion $\psi(x) = \psi_0(x) + \psi_1(x)$, where $\psi_1(x)$ is caused by $\Delta\epsilon_L(x)$.

Though $\langle\epsilon_L(x)\rangle$ is a slowly varying function in comparison with the wavelength [see Eq.(7)], rigorous mathematical results valid for the conventional WKB approximation [1, 2] are not valid here due to dependence of $\langle\epsilon_L(x)\rangle$ on the wave number. The parameter ν_1 cannot be used as a "uniform" parameter of the formal WKB series [1]. In particular, the derivative of the averaged permittivity grows with k (but more slowly than the first power of k). To show this, let us consider formal expansion of $\psi_0(x)$ [$\psi_0(x) = \psi_0^{(0)} + \psi_0^{(1)} + \cdots$] using k^{-1} as a small parameter and estimate the relation between leading WKB term and the next one. The well-known formulas give [1, 2]

$$C_0^{(0)}(x) = q^{-1/4}(x)\cos[kS(0,x)],$$

$$S_0^{(0)}(x) = k^{-1}q^{-1/4}(x)\sin[kS(0,x)], \tag{16}$$

with

$$q(x) = 1-\langle\epsilon_L(x)\rangle, \tag{17}$$

and

$$S(0,x) = \int_0^x dz\sqrt{1-\langle\epsilon_L(z)\rangle}. \tag{18}$$

For the estimation of the next order we have to use the relation

$$|\psi_0^{(1)}| \sim \frac{1}{k}\left|\int_0^x \left(\frac{1}{8}\frac{q''(z)}{q^{3/2}(z)} - \frac{5}{32}\frac{[q'(z)]^2}{iq^{5/2}(z)}\right) dz\right|. \tag{19}$$

Let us require the integral in the right-hand side of Eq. (19) to be much smaller than unity. For the estimate of the second term in the integrand, we can use the relation (10). The first term in the integrand can be shown to be much smaller than the second one (this fact is evident since $q''(x) = d\langle\mu_L(x)\rangle/dx$ contains rapidly oscillating terms due to $\delta_\kappa(x)$ and, hence, integration by parts gives additional smallness). Finally, the first order contribution is estimated as follows:

$$|\psi_0^{(1)}| = O[\chi_1^2(kL)^{2\beta(1-\alpha)-1}] = \epsilon_0^2\bar{\nu}_1, \tag{20}$$

where a small parameter $\bar{\nu}_1$ is introduced. Thus the smallness of $|\psi_0^{(1)}|$ is available only if

$$1 + 2\beta(\alpha - 1) > 0. \tag{21}$$

The inequality (21) is stronger than (11), and hence it is $\bar{\nu}_1$ (instead of ν_1) that should be considered as a (true) small parameter. In this sense, the requirement (7) is just a preliminary and necessary condition to make the relation between the conventional WKB approximation and two-scale expansion evident. Therefore it is sufficient to use only two small parameters, $\bar{\nu}_1$ and ν_2, in order to apply the two-scale method.

Returning to the reflection coefficient (4), after simple expansion one can write

$$|r|^2 = O(C_0^{(1)}(x); S_0^{(1)}(x); C_1(x); S_1(x)), \tag{22}$$

because the insertion of (16) into Eq. (4) gives a value of order $\nu_1(\ll \nu_2)$. For arbitrary β, the small parameters ν_2 and $\bar{\nu}_1$ may have arbitrary relations, so we must hold the lowest term in the right-hand side of Eq. (22). Generally speaking we have to evaluate exactly the order of the corresponding term and the next one due to the lack of a uniformly small parameter. Nevertheless, when both the parameters are of the same order, i.e.,

$$\beta = \frac{1}{3-2\alpha} \approx 0.575, \tag{23}$$

the estimate can be obtained in a general form [see Eqs. (14) and (20)]. From the relation (22) we can get the estimate for the reflection coefficient,

$$|r|^2 = O((kL)^{(2\alpha-3)^{-1}}) \equiv O((kL)^{-\beta}). \tag{24}$$

The associated slope is presented in Figs. 1, 2, and 3 where we can observe good agreement between the numerical results and the analytical predictions. The upper bound of all graphs can be approximated by a line with the slope, $-\beta \approx -0.575$, that is independent of μ_0.

IV. CONCLUSION

We have found that the upper bound of the reflection coefficient of a fractal layer decays with the wave number in accordance with a power law. The small-scale structure of the reflection coefficient has oscillations which manifest some resonant effects. It is unlike the scattering data of either a layer described by a well-differentiable function (the conventional WKB approximation) or a slab described by a function having points of discontinuity (the resonant scattering). The smooth component of the permittivity results in intricate variations of a field phase [see Eqs.(16)–(18)]. The physical nature of the phenomena seems to lie in the spectrum of the inhomogeneities which has a nonexponential decay with wave number [Eq.(13)].

The behavior observed can be explained within the framework of a two-scale method. By means of this approach the reflection coefficient can be expressed in quadratures in a simple form. In fact, by straightforward algebra one can evaluate both the addendum $\psi_1(x)$ caused by the small-scale part of the dielectric permittivity and the term $\psi_0^{(1)}$ of the WKB approximation. However, from the viewpoint of computations the corresponding expression seems not easier than the original Helmholtz equation. In this sense, the two-scale expansion needs to be modified in order to simplify the final expressions and to get more information about the small-scale structure of the scattering data.

Mathematical justification of the two parameter expansion is still an open question. Such an expansion is a hybrid of the formal asymptotic WKB series (characterized by a small parameter $\bar{\nu}_1$) and direct perturbation expansion (with respect to ν_2). In principle, these two expansions have different behavior. On the other hand, even WKB aproximation itself is not well defined in the sense of complex dependence of the averaged permittivity on the wave number (see the above discussion about the relation between ν_2 and $\bar{\nu}_1$). Nevertheless, good agreement between predictions of the two-scale method and direct numerical calculations allows us to believe that the proposed expansion reflects the main physical properties of scattering by a fractal layer.

Finally, we would like to point out that the present treatment is quite general and it is also valid for a devil's staircase on any other fractal supports, which may not necessarily have self-similarity property. As it follows from our analytical estimations, the power decay of the reflection coefficient has to be observed for a layer described by any staircase on a fractal support. The slope of the decay must be related to the dimension of the fractal support rather than to its particular structure. Here we would like to mention that power decay of the contribution of a small-scale structure to the reflection coefficient of a fractal layer described by a Weierstrass function has been reported in Ref. [11]. Thus it is believed to be a general property of one-dimensional scattering by fractals.

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