Solitary waves in a class of generalized Korteweg–de Vries equations

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We study the class of generalized Korteweg–de Vries equations derivable from the Lagrangian $L(l,p) = \int \left[\frac{1}{2}\varphi_x\varphi_t - (\varphi_x)^l/l(l-1) + \alpha(\varphi_x)^p(\varphi_{xx})^2\right] dx$, where the usual fields $u(x,t)$ of the generalized KdV equation are defined by $u(x,t) = \varphi_x(x,t)$. This class contains compactons, which are solitary waves with compact support, and when $l = p+2$, these solutions have the feature that their width is independent of the amplitude. We consider the Hamiltonian structure and integrability properties of this class of KdV equations. We show that many of the properties of the solitary waves and compactons are easily obtained using a variational method based on the principle of least action. Using a class of trial variational functions of the form $u(x,t) = A(t)\exp\left[-\beta(t)\left|x-q(t)\right|^{2n}\right]$ we find solitonlike solutions for all n, moving with fixed shape and constant velocity c. We show that the velocity, mass, and energy of the variational traveling-wave solutions are related by $c = 2rEM^{-1}$, where $r = (p+l+2)/(p+6-l)$, independent of n.

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I. INTRODUCTION

Recently, Rosenau and Hyman [1] have shown that in a particular generalization of the Korteweg–de Vries (KdV) equation, defined by parameters (m,n), namely

$$u_t + (u^m)_x + (u^n)_{xxx} = 0 \quad [K(n,m)] , \tag{1}$$

a new form of solitary wave with compact support and width independent of amplitude exists. For their choice of generalized KdV equations the compactons with $m = n \leq 3$ had the form $[\cos(a\xi)]^{2/(m-1)}$, where $\xi = x - ct$ and for m=2,3 they obtained

$$u_c = \begin{cases} \frac{4c}{3}\cos^2(\xi/4) \quad [K(2,2)] \\ \left(\frac{3c}{2}\right)^{1/2}\cos(\xi/3) \quad [K(3,3)] \end{cases} \tag{2}$$

Unlike the ordinary KdV equation, the generalized KdV equation considered by Rosenau and Hyman was not derivable from a first-order Lagrangian, except for $n = 1$, and did not possess the usual conservation laws of energy and mass that the KdV equation possessed. It is presumed that the generalized KdV equations found by the above authors are not completely integrable, but instead possess only a finite number of conservation laws. Because of this, we were led to consider a different generalization of the KdV equation based on a first-order Lagrangian formulation. That is, we consider

$$L(l,p) = \int \left(\frac{1}{2}\varphi_x\varphi_t - \frac{(\varphi_x)^l}{l(l-1)} + \alpha(\varphi_x)^p(\varphi_{xx})^2\right) dx. \tag{3}$$

This Lagrangian leads to a generalized sequence of KdV equations of the form

$$u_t = u_x u^{l-2} + \alpha[2u_{xxx}u^p + 4pu^{p-1}u_x u_{xx} + p(p-1)u^{p-2}(u_x)^3] \quad [K^*(l,p)] , \tag{4}$$

where

$$u(x) = \varphi_x(x). \tag{5}$$

These equations have the same terms as the equations considered by Rosenau and Hyman, but the relative weights of the terms are quite different leading to the possibility that the integrability properties might be different. [For the purposes of comparison it may be helpful to note that their set (m,n) corresponds to our $(l-1,p+1)$.] The rest of the paper is organized follows: In Sec. II we discuss some exact traveling-wave solutions to (4). In Sec. III we derive the conservation laws and discuss the Hamiltonian structure of these equations. In Sec. IV we apply the time-dependent variational approach to obtaining approximate solitary-wave solutions and compare the variational solutions to the exact ones.

II. EXACT SOLITARY WAVE AND COMPACTON SOLUTIONS

If we assume a solution to (4) in the form of a traveling wave:

$$u(x,t) = f(\xi) = f(x+ct), \tag{6}$$

one obtains for f

$$cf' = f'f^{l-2} + \alpha[2f'''f^p + 4pf^{p-1}f'f'' + p(p-1)f^{p-2}f'^3]. \quad (7)$$

Integrating twice we obtain

$$\frac{c}{2}f^2 - \frac{f^l}{l(l-1)} - \alpha f'^2 f^p = C_1 f + C_2. \quad (8)$$

We seek solutions where the integration constants C_1 and C_2 are zero. This puts lower bounds on l and p: $l > 1$ and $f''f^p \to 0, f'^2 f^{p-1} \to 0$ at edges where $f \to 0$. Then we obtain

$$\alpha f'^2 = \frac{c}{2}f^{2-p} - \frac{f^{l-p}}{l(l-1)}. \quad (9)$$

For finite f' at the edges, we must have $p \leq 2, l \geq p$.

Let us now look at some special examples. (Note that we have chosen signs so that all traveling waves have $u > 0$ and move to the left.) The usual KdV equation has $\alpha = 1/2$, $l = 3$, $p = 0$. For that case one has the well known soliton

$$u = (3c)\text{sech}^2\left[\sqrt{3c/2}\xi\right]. \quad (10)$$

We define the "mass" M via

$$M = \int_{-\infty}^{\infty} dx[u(x,t)]^2. \quad (11)$$

For this solution we find that we can express M and E in terms of c as follows: $M = 24c^{3/2}$, $E = \frac{36}{5}c^{5/2}$ so that

$$c = \frac{10}{3}EM^{-1} = (M/24)^{2/3}. \quad (12)$$

The case $l = p + 2$ is the case relevant for compactons whose width is independent of the velocity c. For $p = 1$, $\alpha = 1/2$ one obtains the compacton solution

$$u_1 = 3c\cos^2(\xi/\sqrt{12}), \quad (13)$$

where $|\xi| \leq \sqrt{3}\pi$. One finds $M = \frac{27}{4}\pi\sqrt{3}c^2$, $E = \frac{9}{4}\sqrt{3}\pi c^3$ so that

$$c = 3EM^{-1} = \left(\frac{4M}{27\pi\sqrt{3}}\right)^{1/2}. \quad (14)$$

There is another compacton solution with $p = 2$, $\alpha = 3$,

$$u_2 = \sqrt{6c}\cos(\xi/6) \quad (15)$$

with $|\xi| \leq 3\pi$. For this compacton, one finds $M = 18\pi c$, $E = \frac{9\pi c^2}{2}$ so that

$$c = 4EM^{-1} = \frac{M}{18\pi}. \quad (16)$$

For the values, $l = 3$, $p = 2$ there is a compacton whose width depends on the velocity. Choosing $\alpha = 1/4$ we find

$$u = 3c - (\xi^2)/6 \quad (17)$$

on the interval

$$|\xi| \leq 3\sqrt{2c}; \quad (18)$$

elsewhere it is zero. For this compacton one finds $M = \frac{144}{5}\sqrt{2}c^{5/2}$, $E = \frac{72}{7}\sqrt{2}c^{7/2}$ so

$$c = \frac{14}{5}EM^{-1} = \left(\frac{5M}{144\sqrt{2}}\right)^{2/5}. \quad (19)$$

Thus, apart from constants we find the same functional form for the compactons for our generalized KdV equations as those found by Rosenau and Hyman in their different generalization of the KdV equation.

III. CONSERVATION LAWS AND CANONICAL STRUCTURE

Equation (4) can be written in canonical form displaying the same Poisson bracket structure as found for the KdV equation:

$$u_t = \partial_x \frac{\delta H}{\delta u} = \{u, H\}, \quad (20)$$

where H is the Hamiltonian obtained from the Lagrangian (3),

$$H = \int [(\pi\dot{\varphi}) - L]\, dx$$
$$= \int \left[\frac{(\varphi_x)^l}{l(l-1)} - \alpha(\varphi_x)^p(\varphi_{xx})^2\right] dx \quad (21)$$
$$= \int \left[\frac{u^l}{l(l-1)} - \alpha u^p(u_x)^2\right] dx. \quad (22)$$

By the usual arguments [2] this is consistent with a Poisson bracket structure

$$\{u(x), u(y)\} = \partial_x\delta(x-y). \quad (23)$$

Let us now show that we have a system of equations which have exactly the same first three conservation laws as the ordinary KdV equation, namely the area, mass, and energy. This is unlike the equations studied by Rosenau and Hyman that did not conserve the mass and energy, but instead had different conserved quantities.

We have

$$u_t = \partial_x \frac{\delta H}{\delta u} \quad (24)$$

so that the "area" under $u(x,t)$ is conserved:

$$l\int u(x,t)dx \equiv H_0 . \quad (25)$$

Multiplying by $u(x,t)$ we find

$$\partial_t\left(\frac{u^2}{2}\right) = \partial_x\left[\frac{u^l}{l} + \alpha\{(p-1)u^p u_x^2 + 2u^{p+1}u_{xx}\}\right], \quad (26)$$

which leads to the conservation of mass

$$(1/2)\int u^2(x,t)dx = (1/2)M \equiv H_1 \ . \tag{27}$$

For the KdV equation H_1 was a second Hamiltonian under a second Poisson bracket structure. From Lagrange's equations we immediately get a third conservation law, the energy

$$H = \int \left[\frac{u^l}{l(l-1)} - \alpha u^p (u_x)^2\right] dx \equiv H_2. \tag{28}$$

The energy provided the first Poisson bracket structure: Considering the mass as a second Hamiltonian, the KdV equation has a second Poisson bracket structure using H_1. Assuming

$$u_t = \{u, H_1\} = \int dy \{u(x), u(y)\}_1 \frac{\delta H_1}{\delta u(y)}, \tag{29}$$

one finds for the KdV equation that

$$\{u(x), u(y)\}_1 = \left(D^3 + \frac{1}{3}(Du + uD)\right)\delta(x-y) \ , \tag{30}$$

where $D = \partial_x$. With this assumed Poisson bracket structure one again recovers the KdV equation. This Poisson bracket structure is identical to the Virasoro algebra with a specific central charge. This fact enables one to show that there is an infinite number of conservation laws in the KdV equation and it is an exactly integrable system [2].

For the generalized KdV equations we find that we can write

$$u_t = \left(\alpha(D^2 u^p D + Du^p D^2) + \frac{1}{l}(Du^{l-2} + u^{l-2}D)\right) u \tag{31}$$

so that there is a chance for a second Hamiltonian if the Jacobi identity is satisfied. One can postulate that the second Poisson bracket structure is given by

$$\{u(x), u(y)\}_1 = \Bigg(\alpha(D^2 u^p D + Du^p D^2) + \frac{1}{l}(Du^{l-2} + u^{l-2}D)\Bigg)\delta(x-y). \tag{32}$$

So we need to show for what l, p this bracket structure obeys the Jacobi identity, where the bracket is defined by

$$\{F[u], G[u]\} = \int_{-\infty}^{\infty} dx dy \frac{\delta F}{\delta u(x)} \{u(x), u(y)\}_1 \frac{\delta G}{\delta u(y)}. \tag{33}$$

One can show immediately that the Hamiltonians H_1 and H_2 commute using either Poisson Bracket structure (23) or (32).

We have that

$$\{H_2[u], H_1[u]\} = \int_{-\infty}^{\infty} dx dy \frac{\delta H_2}{\delta u(x)} \{u(x), u(y)\}_1 \frac{\delta H_1}{\delta u(y)}. \tag{34}$$

For the usual bracket structure (23) we can rewrite (34) as

$$\begin{aligned}\{H_2[u], H_1[u]\} &= \int_{-\infty}^{\infty} dx u_t(x) \left(\frac{\delta H_1}{\delta u(x)}\right) \\ &= \frac{1}{2}\partial_t \int_{-\infty}^{\infty} dx u^2(x,t) = 0.\end{aligned} \tag{35}$$

For the second bracket structure (32) we have instead

$$\begin{aligned}\{H_2[u], H_1[u]\}_1 &= \int_{-\infty}^{\infty} dx u_t(x) \left(\frac{\delta H_2}{\delta u(x)}\right) \\ &= \int_{-\infty}^{\infty} dx \frac{\delta H_2}{\delta u(x)} \partial_x \left(\frac{\delta H_2}{\delta u(x)}\right) \\ &= \frac{1}{2}\int_{-\infty}^{\infty} dx \partial_x \left(\frac{\delta H_2}{\delta u(x)}\right)^2 = 0.\end{aligned} \tag{36}$$

Encouraged by this result we have attempted to repeat the induction proof of the existence of an infinite number of conservation laws, assuming as in the KdV equation that one has the conservation laws obey the recursion relations:

$$\left(\alpha(D^2 u^p D + Du^p D^2) + \frac{1}{l}(Du^{l-2} + u^{l-2}D)\right)\frac{\delta H_{n-1}}{\delta u(x)} = D\frac{\delta H_n}{\delta u(x)}. \tag{37}$$

Starting with H_0 defined by (25) we get the candidate Hamiltonian:

$$H_1 = \int \frac{u^{l-1}(x,t)}{(l-1)} dx \tag{38}$$

instead of (27). If we now ask if this is conserved by considering the equation

$$\frac{dH_1}{dt} = \{H_1, H_2\} \tag{39}$$

using the first Poisson bracket structure, we find that the right hand side of (39) is not a total divergence unless $l = 3$. For $l = 3$ one has

$$\left(\alpha(D^2 u^p D + Du^p D^2) + \frac{1}{3}(Du + uD)\right)\frac{\delta H_1}{\delta u(x)} = D\frac{\delta H_2}{\delta u(x)}. \tag{40}$$

However, if we iterate one more time (with $l = 3$) we obtain

$$\left(\alpha(D^2 u^p D + Du^p D^2) + \frac{1}{3}(Du + uD)\right)\frac{\delta H_2}{\delta u(x)} = DF_3(x) \tag{41}$$

and we find by explicit construction that $F_3(x)$ is not the variational derivative of a local Hamiltonian unless $p = 0$. Thus this bi-Hamiltonian method of finding an infinite number of conservation laws only works for the original KdV equation. We surmise that (32) is not a valid bracket structure and that

$$\{H[u],\{F[u],G[u]\}\} + \{G[u],\{H[u],F[u]\}\} + \{F[u],\{G[u],H[u]\}\} = 0 \quad (42)$$

is *not* satisfied for the postulated second bracket. Thus we have not succeeded in showing that these alternative equations are exactly integrable, and we are in the same situation, in spite of having a first-order Lagrangian, as for the generalized KdV equations of Rosenau and Hyman [1].

We have not as yet performed numerical simulations of the scattering of our alternative compacton solutions. For Rosenau and Hyman such numerical experiments produced behavior very similar to, but not exactly the same as, that observed in completely integrable systems, namely stability and preservation of shape. They find that elastic collisions are accompanied by the production of low amplitude compacton-anticompacton pairs [1].

IV. VARIATIONAL APPROACH

Our time-dependent variational approach for studying solitary waves is related to Dirac's variational approach to the Schrödinger equation [5,6]. In our previous work [3,4], we introduced a post-Gaussian variational approximation, a continuous family of trial variational functions more general than Gaussians, which can still be treated analytically. Assuming a variational ansatz of the form $u(x,t) = A(t)\exp\left[-\beta(t)\left|x-q(t)\right|^{2n}\right]$, we will extremize the effective action for the trial wave functional and determine the classical dynamics for the variational parameters. We will find that for all (l,p) the dynamics of the variational parameters lead to solitary waves moving with constant velocity and constant amplitude. For the special case of $l = p+2$ we find immediately that the width of the soliton is independent of the amplitude, and velocity. Correct functional relations between energy, mass, amplitude, and velocity are obtained very quickly from the variational method, although one does not find that the $l = p+2$ variational solitons have compact support. We will find that most of the properties of the single "soliton" solutions to these equations can be obtained by using this very simple trial wave function ansatz and extremizing the action.

The starting point for the variational calculation is the action

$$\Gamma = \int L dt, \quad (43)$$

where L is given by (3).

Just as we did in our study of the KdV equation we choose a trial wave function of the form

$$u_v(x,t) = A(t)\exp\left[-\beta(t)|x-q(t)|^{2n}\right], \quad (44)$$

where n is an arbitrary continuous, real parameter.

The variational parameters have a simple interpretation in terms of expectation values with respect to the "probability"

$$P(x,t) = \frac{[u_v(x,t)]^2}{M(t)}, \quad (45)$$

where the mass M is defined as above

$$M(t) \equiv \int [u_v(x,t)]^2\, dx. \quad (46)$$

(Here we allow M to be a function of t, even though M is conserved.)

Since $\langle x - q(t)\rangle = 0$, $q(t) = \langle x\rangle$. From (46) and (44) we have

$$A(t) = \frac{M^{1/2}(2\beta)^{1/4n}}{\left[2\Gamma\left(\frac{1}{2n}+1\right)\right]^{1/2}}. \quad (47)$$

The inverse width β is related to

$$G_{2n} \equiv \langle |x-q(t)|^{2n}\rangle = \frac{1}{4n\beta}. \quad (48)$$

Following our approach in [4], we find that the action for the trial wave function (44) is given by

$$\Gamma(q,\beta,M,n) = \int\left(-\frac{1}{2}M\dot{q} - C_1(n)\beta^{(l-2)/4n}M^{l/2} + C_2(n)M^{1+p/2}\beta^{(p+4)/4n}\right)dt \equiv \int L_1(q,\dot{q},M,\beta)dt, \quad (49)$$

where

TABLE I. Comparison of variational and exact solutions.

l	p	α	n	$E_{\rm var}$	$E_{\rm exact}$	$c_{\rm var}$	$c_{\rm exact}$
3	0	1/2	0.877	0.035999	0.0360562	0.119995	0.120187
3	2	1/4	1.423	0.0803831	0.0810735	0.225073	0.227006
3	1	1/2	1.154	0.054888	0.055002	0.164666	0.165006
4	2	3	1.283	0.00436284	0.00442097	0.017451	0.0176839

$$C_1(n) = \frac{1}{l(l-1)}\left(\frac{2^l}{l^2}\right)^{1/4n}\left[2\Gamma\left(\frac{1}{2n}+1\right)\right]^{(2-l)/2}, \tag{50}$$

$$C_2(n) = 4\alpha n(2)^{(p+2)/4n}(2+p)^{\frac{1}{2n}-2}\frac{\Gamma\left(2-\frac{1}{2n}\right)}{\left[2\Gamma\left(\frac{1}{2n}+1\right)\right]^{1+p/2}}.$$

We eliminate the variable of constraint β (using $\delta\Gamma/\delta\beta = 0$) and find

$$\beta = [d(n)]^{4n}\, M^{2n(p+2-l)/(l-p-6)}, \tag{51}$$

where

$$d(n) = \left[\frac{(p+4)C_2(n)}{(l-2)C_1(n)}\right]^{1/(l-p-6)}. \tag{52}$$

From (51) we see that when

$$l = p+2, \tag{53}$$

the width of the soliton β does not depend on M and thus is independent of the amplitude or velocity. This special case is precisely the case when the exact solution is a compacton.

We now eliminate β in favor of M, and symmetrizing the Lagrangian (3) we obtain [7]

$$L_2 = \frac{1}{4}\left(q\dot{M} - \dot{q}M\right) - H(M), \tag{54}$$

where

$$H(M) = \left(C_1 d^{(l-2)} - C_2 d^{(p+4)}\right)M^r, \tag{55}$$

where $r = (p+l+2)/(p+6-l)$. Extremizing the action yields

$$\dot{M} = 0 \Longrightarrow M = \text{const}\ ,$$
$$\dot{M} = 0 \Longrightarrow \beta = \text{const}\ , \tag{56}$$

and

$$\dot{q} = -2r\left(C_1 d^{(l-2)} - C_2 d^{(p+4)}\right)M^{r-1}, \tag{57}$$

as well as a conserved energy

$$E = \left(C_1 d^{(l-2)} - C_2 d^{(p+4)}\right)M^r\ . \tag{58}$$

Thus the velocity of the solitary wave is constant and can be related to the conserved energy via

$$\dot{q} = -c = -2rEM^{-1}. \tag{59}$$

This is precisely the form we obtained for the exact solutions.

We have not yet extremized the action with respect to the variational parameter n, which is equivalent to extremizing the energy with respect to n. We perform this extremization graphically for each value of l, p. The explicit form of the trial wave function is

$$u_v(x,t) = d[n,p,l]M^{2/(p+6-l)}2^{1/4n}\left[2\Gamma\left(1+\frac{1}{2n}\right)\right]^{-1/2}\exp\left[-d^{4n}M^{2n(l-p-2)/(p+6-l)}|x+ct-x_0|^{2n}\right], \tag{60}$$

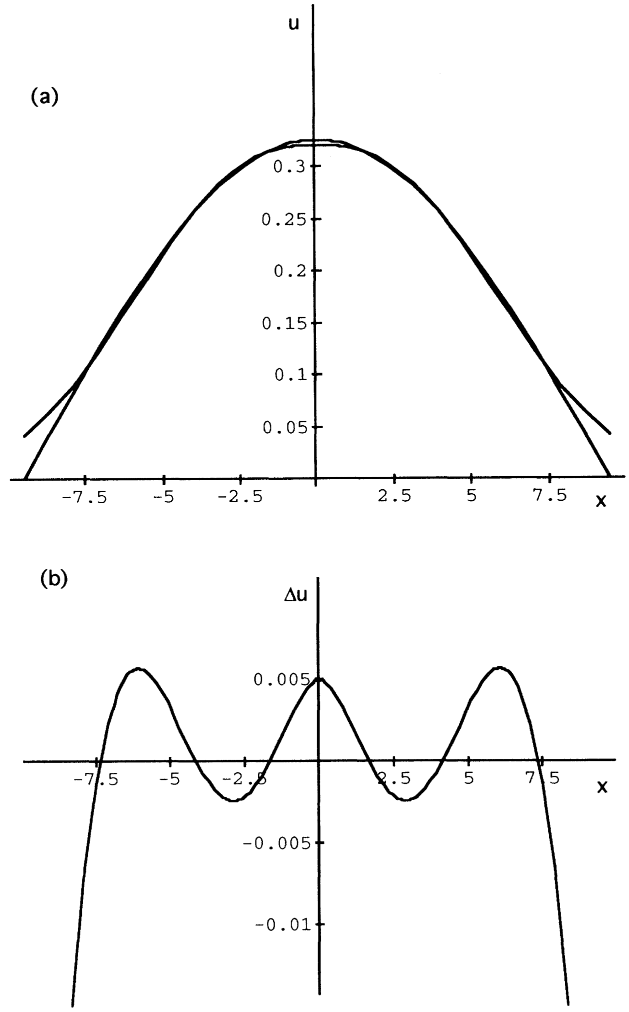

FIG. 1. (a) u_{var} with $n = 1.283$ and u_{exact} for $M = 1$ given by Eq. (15), and (b) $\Delta u = u_{\text{var}} - u_{\text{exact}}$, as a function of x for the case $l = 4$, $p = 2$, $\alpha = 3$.

where d is given by (52).

Now let us see how these trial wave functions and the energy and velocity compare with the exact answers for special cases. Since we explicitly know the M dependence of the answer, we can set M=1 as our normalization for both the variational and exact solitons.

In the Table I we summarize results for the four (l,p) cases described in Sec. II. For each case we list l, p, α, the n value that extremizes the energy (or action), and compare the variational and exact values of the energy and velocity (with $M = 1$).

In Figs. 1(a) and 1(b) we illustrate the global accuracy of our variational solution by displaying the exact and variational solutions for the $l = 4$, $p = 2$ case. (Other cases look very similar.) We note that the global accuracy is a few percent, except near the place where the true compacton goes to zero.

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