Explicitly integrable polynomial Hamiltonians and evaluation of Lie transformations

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(Received 14 June 1993)

We have found that any homogeneous polynomial can be written as a sum of integrable polynomials of the same degree, with which each associated polynomial Hamiltonian is integrable, and the associated Lie transformation can be evaluated exactly. An integrable polynomial factorization has thus been developed to convert a sympletic map in the form of a Dragt-Finn factorization into a product of exactly evaluable Lie transformations associated with integrable polynomials. Having a small number of factorization bases of integrable polynomials enables one to consider a factorization with the use of high-order symplectic integrators so that a symplectic map can always be evaluated with the desired accuracy. The results are significant for studying the long-term stability of beams in accelerators.

PACS number(s): 41.85.−p, 29.20.−c, 29.27.Bd, 03.20.+i

I. INTRODUCTION

In large storage rings, charged-particle beams are required to circulate for many hours in the presence of nonlinear perturbations of multipole errors in magnets. Extensive computer simulations are thus necessary to investigate the long-term stabilities. The conventional approach in which trajectories of particles are followed element by element through accelerator structures is, however, slow for these studies. A substantial computational as well as conceptual simplification is to study the stability of particles using one-turn maps [1].

While finding a closed analytical form of a one-turn map is impossible for a large storage ring with thousands of elements, a truncated Taylor expansion of a one-turn map can be easily obtained through concatenating actions of individual elements by means of Lie and differential algebras [2,3]. Even though some successes have been reported by directly using the Taylor maps for tracking [4,5], the truncation inevitably violates the symplectic nature of systems and consequently leads to spurious effects if the maps are used to study the long-term stability [1,6,7]. Increasing the order of the Taylor map can make the nonsymplecticity arbitrarily small, but as its size grows exponentially, the map tracking will lose its advantage in speed to the element-by-element tracking. A reliable long-term tracking study with the Taylor map is therefore possible only if its nonsymplecticity effect can be eliminated without greatly reducing the tracking speed.

The Taylor map extracted from a symplectic system can always be converted into Lie transformations with a Dragt-Finn factorization [8]. A map in the form of Lie transformations is guaranteed to be symplectic, but it generally cannot be used for tracking directly because evaluating a nonlinear map in such a form is equivalent to solving nonlinear Hamiltonian systems, which cannot be done in general. Several methods, such as jolt (kick) factorization [9,10] and monomial factorization [11,12], have been proposed to deal with this difficulty by converting the Lie transformation from its general form into special forms that can be evaluated directly. While these methods seem promising, their applications lead to considerable theoretical and computational complexities, chief of which is the unpredictability of high-order spurious terms that may lead to less than accurate evaluation of the map.

Since a general Lie transformation corresponds to a nonintegrable Hamiltonian system that cannot be evaluated exactly, the challenge here is how to evaluate a Lie transformation approximately without violating the symplecticity and with a controllable accuracy. One way is to divide the nonintegrable system into subsystems that are integrable individually. The set of subsystems of minimum number is the most promising one to serve as the zeroth-order approximation because it would be the closest to the original system and the best starting point for higher-order treatments. For Lie transformations associated with homogeneous polynomials, we shall show in this paper that any polynomial can be written as a sum of integrable polynomials with which each associated Lie transformation can be evaluated exactly. Since the number of integrable polynomials can be much smaller than the number of monomials, a factorization consisting of Lie transformations associated with integrable polynomials will have many fewer terms so that a higher-order factorization becomes practical. In order to achieve an optimization between a desired accuracy and a fast tracking speed, we propose an integrable polynomial factorization, which can also be made symmetric to enhance accuracy with the use of symplectic integrators [13–16].

The paper is organized as follows. In Sec. II, we introduce the Taylor map and its corresponding Lie transformations. A definition of integrable polynomial and guidelines for constructing an optimal set of integrable polynomials are given in Sec. III. In Secs. IV–VII, we construct integrable polynomials for a homogeneous polynomial of degree 3–6. Factorizations with integrable polynomials are discussed in Sec. VIII. Section IX contains a summary and discussion.

II. THE TAYLOR MAP AND LIE TRANSFORMATION

At any "checkpoint" of an accelerator, motions of particles can be described mathematically by a six-dimensional symplectic one-turn map

$$\mathbf{z}'=\mathcal{M}\mathbf{z} , \tag{1}$$

where $\mathbf{z}=(q_1,p_1,q_2,p_2,q_3,p_3)$ is a phase-space vector and p_i is the conjugate momenta of q_i. $\mathcal{M}$ is, in general, a nonlinear functional operator. Because we are usually not interested in transformations that simply translate the origin in phase space, only maps that map the origin to itself ($\mathbf{z}=\mathbf{0}$ is the closed orbit) are considered. Within its analytic domain, $\mathcal{M}$ can be written as a product of Lie transformations with Dragt-Finn factorization [8],

$$\mathcal{M}\mathbf{z}=\mathcal{R}\prod_{i=3}^{\infty}\exp(:f_i:)\mathbf{z} , \tag{2}$$

where $\mathcal{R}$ denotes a linear symplectic transformation, f_i is a homogeneous polynomial in $\mathbf{z}$ of degree i, and $:f_i:$ is the Lie operator associated with f_i, which is defined by the Poisson bracket operation [2]

$$:f_i:=\sum_{l=1}^{3}\left[\frac{\partial f_i}{\partial q_l}\frac{\partial}{\partial p_l}-\frac{\partial f_i}{\partial p_l}\frac{\partial}{\partial q_l}\right] , \tag{3}$$

and the Lie transformation is defined by the exponential series

$$\exp(:f:)=\sum_{n=0}^{\infty}\frac{1}{n!}(:f:)^n . \tag{4}$$

Within its analytic domain, $\mathcal{M}\mathbf{z}$ can also be expanded in a power series of $\mathbf{z}$

$$\mathcal{M}\mathbf{z}=\sum_{i=1}^{\infty}\mathbf{U}_i(\mathbf{z})=\sum_{i=1}^{N}\mathbf{U}_i(\mathbf{z})+\epsilon(N+1) , \tag{5}$$

where $\mathbf{U}_i(\mathbf{z})$ is a vectorial homogeneous polynomial of degree i,

$$\mathbf{U}_i(\mathbf{z})=\sum_{|\sigma|=i}\mathbf{u}_i(\sigma)q_1^{\sigma_1}p_1^{\sigma_2}q_2^{\sigma_3}p_2^{\sigma_4}q_3^{\sigma_5}p_3^{\sigma_6} , \tag{6}$$

and $\epsilon(N+1)$ represents a remainder series consisting of terms higher than degree N. In Eq. (6), σ denotes a collection of exponents $(\sigma_1,\ldots,\sigma_6)$ and $|\sigma|=\sum_{j=1}^{6}\sigma_j$. Truncating the expansion in Eq. (5) at the Nth order results in an Nth-order Taylor map

$$\mathbf{U}\equiv\sum_{i=1}^{N}\mathbf{U}_i . \tag{7}$$

Due to the truncation, $\mathbf{U}$ is no longer exactly symplectic. However, the symplecticity can be recovered by converting $\mathbf{U}$ into a product of Lie transformations with an accuracy up to the truncation order N [2,8]

$$\mathbf{U}(\mathbf{z})=\mathcal{R}\prod_{i=3}^{N}\exp(:f_i:)\mathbf{z}+\epsilon(N+1) , \tag{8}$$

where $\mathcal{R}$ is again the linear symplectic transformation and f_i a homogeneous polynomial in $\mathbf{z}$ of degree i. f_i can be obtained from $\mathbf{U}_i$ in an iterative manner [1,2]. The symplectic map

$$\mathcal{M}_s\mathbf{z}=\mathcal{R}\prod_{i=3}^{N}e^{:f_i:}\mathbf{z} \tag{9}$$

is thus a symplectic approximation to $\mathcal{M}$, which is considered an acceptable approximation for the study of the long-term stability of the original system in the phase-space region of interest. Tracking with $\mathcal{M}_s\mathbf{z}$, however, requires the evaluation of Lie transformations in Eq. (9). Because of the isomorphism property of Lie transformation [2,8], for any function $F(\mathbf{z})$,

$$\exp(:f_i:)F(\mathbf{z})=F(\exp(:f_i:)\mathbf{z}) . \tag{10}$$

A product of Lie transformation of the form $\exp(:f_i:)\exp(:f_j:)\mathbf{z}$ thus requires only evaluations of $\exp(:f_i:)\mathbf{z}$ and $\exp(:f_j:)\mathbf{z}$. On the other hand, for any autonomous Hamiltonian H, the solution of the Hamiltonian equation can be formally written as

$$\mathbf{z}(t)=\exp(-t:H:)\mathbf{z}(0) . \tag{11}$$

A comparison of $\exp(:f_i:)\mathbf{z}$ with Eq. (11) indicates that the problem of evaluating $\exp(:f_i:)\mathbf{z}$ is equivalent to the problem of solving a Hamiltonian with $H=-f_i$ from $t=0$ to 1, which cannot be done in general. The challenge, therefore, is how to evaluate $\exp(:f_i:)\mathbf{z}$ approximately without violating the symplecticity and with a controllable accuracy.

III. INTEGRABLE POLYNOMIALS IN LIE TRANSFORMATION

Definition. The associated Hamiltonian of a polynomial $f_i(\mathbf{z})$ is defined by $H=-f_i$.

Definition. A polynomial in $\mathbf{z}$ is called an integrable polynomial if its associated Hamiltonian is integrable and its associated Lie transformation can be evaluated exactly.

Let $\{g_i^{(k)}|k=1,2,\ldots,N_g\}$ denotes a set of integrable polynomials of degree i. Then any polynomial in $\mathbf{z}$ can be expressed as a sum of integrable polynomials of the same degree, i.e.,

$$f_i(\mathbf{z})=\sum_{|\sigma|=i}a(\sigma)q_1^{\sigma_1}p_1^{\sigma_2}q_2^{\sigma_3}p_2^{\sigma_4}q_3^{\sigma_5}p_3^{\sigma_6}=\sum_{k=1}^{N_g}g_i^{(k)} , \tag{12}$$

where f_i is any homogeneous polynomial of degree i and $a(\sigma)$ are constant coefficients. After factorizing it as a product of Lie transformations associated with integrable polynomials, $\exp(:f_i:)\mathbf{z}$ can be evaluated directly. Since the minimum number of integrable polynomials is much smaller than the number of monomials, the accuracy of factorization with Lie transformations associated with $\{g_i^{(k)}\}$ can be carried out to a suitable order while maintaining a reasonable computational speed in symplectic map tracking.

In order to construct integrable polynomials, we list possible integrable systems with polynomial Hamiltonians: (a) All Hamiltonians with one degree of freedom. In this case, the Hamiltonian is any polynomial in a pair of canonical variables. In order to have closed forms for solutions of Hamiltonian equations, the Hamiltonians have to be limited to certain special forms. (b) All Hamiltonians with linear equations of motion. In this case, the Hamiltonians are homogeneous polynomials in **z** of degree 1 or 2. The former corresponds to a translation in phase space and the latter is a coupled harmonic oscillator. (c) All other systems that can be transformed into their "action-angle" variables, that is, their Hamiltonians depend on "actions" (or "angles") only, e.g., kick Hamiltonians. (d) All nonlinear systems which can be separated into (uncoupled) (a), (b), and (c). For example, a polynomial Hamiltonian is integrable if it consists only of a product of any monomial in one degree of freedom and a homogeneous polynomial of degree 1 or 2 in the other two degrees of freedom. By following these guidelines, one can construct an optimal set of integrable polynomials for a homogeneous polynomial of any degree.

For later use, we give the formula for the number of monomials of degree i in d variables,

$$N(i,d)=C^{i}_{i+d-1}=\frac{(i+d-1)!}{i!(d-1)!} \ . \tag{13}$$

IV. INTEGRABLE POLYNOMIALS IN z OF DEGREE 3

Homogeneous polynomials of degree 3 in six variables consist of 56 monomials, which can be grouped under eight integrable polynomials of degree 3 $\{g_3^{(n)}|n=1,2,\ldots,8\}$:

$$g_3^{(1)}=c^{(1)}_{3,1}q_1^3+c^{(1)}_{3,2}q_1^2p_1+c^{(1)}_{3,3}q_2^3 +c^{(1)}_{3,4}q_2^2p_2+c^{(1)}_{3,5}q_3^3+c^{(1)}_{3,6}q_3^2p_3 \ , \tag{14}$$

$$g_3^{(2)}=c^{(2)}_{3,1}p_1^3+c^{(2)}_{3,2}p_1^2q_1+c^{(2)}_{3,3}p_2^3 +c^{(2)}_{3,4}p_2^2q_2+c^{(2)}_{3,5}p_3^3+c^{(2)}_{3,6}p_3^2q_3 \ , \tag{15}$$

$$g_3^{(2+i)}=q_i h^{(2+i)}_{3,2}(q_j,p_j,q_k,p_k) \ , \tag{16}$$

$$g_3^{(5+i)}=p_i h^{(5+i)}_{3,2}(q_j,p_j,q_k,p_k) \ , \tag{17}$$

where (i,j,k) goes over all cyclic permutations of (1,2,3) and $h^{(n)}_{3,2}$s are homogeneous polynomials of degree 2 in four variables

$$\begin{aligned} h^{(n)}_{3,2}(q_j,p_j,q_k,p_k)=&c^{(n)}_{3,1}q_j^2+c^{(n)}_{3,2}q_jp_j+\tfrac{1}{3}c^{(n)}_{3,3}q_jq_k \\ &+\tfrac{1}{3}c^{(n)}_{3,4}q_jp_k+c^{(n)}_{3,5}p_j^2 \\ &+\tfrac{1}{3}c^{(n)}_{3,6}p_jq_k+\tfrac{1}{3}c^{(n)}_{3,7}p_jp_k \\ &+c^{(n)}_{3,8}q_k^2+c^{(n)}_{3,9}q_kp_k+c^{(n)}_{3,10}p_k^2 \ . \end{aligned} \tag{18}$$

$c^{(n)}_{3,m}$ is the coefficient of the corresponding monomial in f_3. For $n>3$ and $m=3$, 4, 6, or 7, the same monomial appears three times so that $c^{(n)}_{3,m}$ in these three terms are chosen to be identical, for example, $c^{(3)}_{3,3}=c^{(4)}_{3,3}=c^{(5)}_{3,3}$. Such an arrangement is necessary for symmetric forms of $h^{n}_{3,2}$. It should be noted that the decomposition of f_i into integrable polynomials is not unique. $g_3^{(1)}$ and $g_3^{(2)}$ can be further combined into a single integrable polynomial since the Hamiltonian $H=-(g_3^{(1)}+g_3^{(2)})$ is integrable. We chose two separate integrable polynomials instead of the combined one because the solution for the latter cannot be written in a closed form and directly used in tracking.

Since

$$\exp(:g_3^{(1)}:)\mathbf{z}=\prod_{i=1}^{3}\exp(:c^{(1)}_{3,2i-1}q_i^3+c^{(1)}_{3,2i}q_i^2p_i:)\mathbf{z} \ , \tag{19}$$

in order to evaluate $\exp(:g_3^{(1)}:)\mathbf{z}$, we consider a Hamiltonian $H=-c^{(1)}_{3,2i-1}q_i^3-c^{(1)}_{3,2i}q_i^2p_i$. The Hamiltonian equations are

$$\dot{q}_i=-c^{(1)}_{3,2i}q_i^2 \ , \tag{20}$$

$$\dot{p}_i=3c^{(1)}_{3,2i-1}q_i^2+c^{(1)}_{3,2i}q_ip_i \ . \tag{21}$$

These two equations with the initial condition $[q_i(0), p_i(0)]$ have the solution

$$q_i(t)=\frac{q_i(0)}{1+c^{(1)}_{3,2i}q_i(0)t} \ , \tag{22}$$

$$p_i(t)=\frac{1}{c^{(1)}_{3,2i}q_i^2(t)}[H-c^{(1)}_{3,2i-1}q_i^3(t)] \ . \tag{23}$$

Let $t=1$ and $[q_i(0),p_i(0)]=(q_i,p_i)$, then

$$\exp(:g_3^{(1)}:)q_i=\frac{q_i}{1+c^{(1)}_{3,2i}q_i} \ , \tag{24}$$

$$\exp(:g_3^{(1)}:)p_i=\frac{(c^{(1)}_{3,2i-1}q_i+c^{(1)}_{3,2i}p_i)(1+c^{(1)}_{3,2i}q_i)^3-c^{(1)}_{3,2i-1}q_i}{c^{(1)}_{3,2i}(1+c^{(1)}_{3,2i}q_i)} \ , \tag{25}$$

where $i=1$, 2, and 3. A similar calculation with $H=-c^{(2)}_{3,2i-1}p_i^3-c^{(2)}_{3,2i}q_ip_i^2$ yields

$$\exp(:g_3^{(2)}:)q_i=\frac{(c^{(2)}_{3,2i-1}p_i+c^{(2)}_{3,2i}q_i)(1-c^{(2)}_{3,2i}r_i)^3-c^{(2)}_{3,2i-1}p_i}{c^{(2)}_{3,2i}(1-c^{(2)}_{3,2i}p_i)} \ , \tag{26}$$

$$\exp(:g_3^{(2)}:)p_i=\frac{p_i}{1-c^{(2)}_{3,2i}p_i} \ , \tag{27}$$

where $i=1$, 2, and 3.

For $n=3,\ldots,8, \exp(:g_3^{(n)}:)\mathbf{z}$ can also be converted into explicit symplectic maps given in Eqs. (A12), (A13), and (A22) (see the Appendix).

V. INTEGRABLE POLYNOMIALS IN z OF DEGREE 4

For convenience, we define eight kick functions for polynomials in z of degree n:

$$G_1(n)=\sum_{l=1}^{k_1} c_{4,1}^{(1)}q_1^{l_1}q_2^{l_2}q_3^{l_3} , \quad (28)$$

$$G_2(n)=\sum_{l=1}^{k_1} c_{4,l}^{(2)}p_1^{l_1}p_2^{l_2}p_3^{l_3} , \quad (29)$$

$$G_3(n)=\sum_{l=1}^{k_2} c_{4,l}^{(3)}q_1^{l_1}p_2^{l_2}q_3^{l_3} , \quad l_1\neq 0,\ l_2\neq 0 \quad (30)$$

$$G_4(n)=\sum_{l=1}^{k_2} c_{4,l}^{(4)}q_1^{l_1}p_2^{l_2}p_3^{l_3} , \quad l_1\neq 0,\ l_3\neq 0 \quad (31)$$

$$G_5(n)=\sum_{l=1}^{k_2} c_{4,l}^{(5)}q_l^{l_1}q_2^{l_2}p_3^{l_3} , \quad l_2\neq 0,\ l_3\neq 0 \quad (32)$$

$$G_6(n)=\sum_{l=1}^{k_2} c_{4,l}^{(6)}p_1^{l_1}q_2^{l_2}p_3^{l_3} , \quad l_1\neq 0,\ l_3\neq 0 \quad (33)$$

$$G_7(n)=\sum_{l=1}^{k_2} c_{4,l}^{(7)}p_1^{l_1}q_2^{l_2}q_3^{l_3} , \quad l_1\neq 0,\ l_3\neq 0 \quad (34)$$

$$G_8(n)=\sum_{l=1}^{k_2} c_{4,l}^{(8)}p_1^{l_1}p_2^{l_2}q_3^{l_3} , \quad l_2\neq 0,\ l_3\neq 0 , \quad (35)$$

where $l_1+l_2+l_3=n$ and l goes over all permutations of (l_1,l_2,l_3). The numbers of terms in sums can be calculated from Eq. (13) as

$$k_1=C_{n+2}^n=\tfrac{1}{2}(n+2)(n+1) , \quad (36)$$

$$k_2=C_{n+2}^n-2C_{n+1}^n+1=\tfrac{1}{2}n(n-1) . \quad (37)$$

We also define a monomial function for polynomials in z of degree n:

$$G_9(n,l,j)=c_{n,1}^{(j)}q_1^{n-l}p_1^{l}+c_{n,2}^{(j)}q_2^{n-l}p_2^{l}+c_{n,3}^{(j)}q_3^{n-l}p_3^{l} , \quad (38)$$

where $1\le l<n$ and j is an index.

The Lie transformations $\exp[:G_i(n):]\mathbf{z}$ for $i=1,\ldots,8$ can simply be written as kicks on either coordinates or momenta:

$$\exp[:G_i(n):]q_i=q_i-\frac{\partial G_i(n)}{\partial p_i} , \quad (39)$$

$$\exp[:G_i(n):]p_i=p_i+\frac{\partial G_i(n)}{\partial q_i} , \quad i=1,\ldots,8 , \quad (40)$$

and $\exp[:G_9(n,l,j):]\mathbf{z}$ can be evaluated by Lie transformations associated with monomials (see the Appendix, Sec. 1):

$$\exp[:G_9(n,l,j):]q_i =q_i[1+(n-2l)c_{n,i}^{(j)}q_i^{n-l-1}p_i^{l-1}]^{l/(2l-n)} , \quad (41)$$

$$\exp[:G_9(n,l,j):]p_i =p_i[1+(n-2l)c_{n,i}^{(j)}q_i^{n-l-1}p_i^{l-1}]^{(n-l)/(n-2l)} \quad \text{for } n\neq 2l , \quad (42)$$

$$\exp[:G_9(n,l,j):]q_i=q_i\exp[-lc_{n,i}^{(j)}(q_ip_i)^{l-1}] , \quad (43)$$

$$\exp[:G_9(n,l,j):]p_i=p_i\exp[lc_{n,i}^{(j)}(q_ip_i)^{l-1}] \quad \text{for } n=2l . \quad (44)$$

Homogeneous polynomials of degree 4 in six variables consist of 126 monomials, which can be grouped under 20 integrable polynomials of degree 4 $\{g_4^{(n)}|n=1,2,\ldots,20\}$:

$$g_4^{(n)}=G_n(4) , \quad n=1,\ldots,8 \quad (45)$$

$$g_4^{(8+i)}=G_9(4,i,8+i) , \quad (46)$$

$$g_4^{(11+i)}=q_i^2p_ih_{4,1}^{(11+i)}(q_j,p_j,q_k,p_k) , \quad (47)$$

$$g_4^{(14+i)}=q_ip_i^2h_{4,1}^{(14+i)}(q_j,p_j,q_k,p_k) , \quad (48)$$

$$g_4^{(17+i)}=q_ip_ih_{4,2}^{(17+i)}(q_j,p_j,q_k,p_k) , \quad (49)$$

where (i,j,k) goes over all cyclic permutations of (1,2,3) and $h_{4,1}^{(n)}$ and $h_{4,2}^{(n)}$ are homogeneous polynomials in four variables of degree 1 and 2, respectively,

$$h_{m,1}^{(n)}(q_j,p_j,q_k,p_k)=c_{m,1}^{(n)}q_j+c_{m,2}^{(n)}p_j+c_{m,3}^{(n)}q_k+c_{m,4}^{(n)}p_k , \quad (50)$$

$$\begin{aligned} h_{4,2}^{(n)}(q_j,p_j,q_k,p_k)=&c_{4,1}^{(n)}q_j^2+\tfrac{1}{2}c_{4,2}^{(n)}q_jp_j+c_{4,3}^{(n)}q_jq_k\\ &+c_{4,4}^{(n)}q_jp_k+c_{4,5}^{(n)}p_j^2+c_{4,6}^{(n)}p_jq_k\\ &+c_{4,7}^{(n)}p_jp_k+c_{4,8}^{(n)}q_k^2\\ &+\tfrac{1}{2}c_{4,9}^{(n)}q_kp_k+c_{4,10}^{(n)}p_k^2 . \end{aligned} \quad (51)$$

$c_{4,m}^{(n)}$ is the coefficient of the corresponding monomial in f_4. $c_{4,2}^{(18)}=c_{4,9}^{(19)}$, $c_{4,2}^{(19)}=c_{4,9}^{(20)}$, and $c_{4,2}^{(20)}=c_{4,9}^{(18)}$ are again for a symmetric form of $h_{4,2}^{(n)}$. It should be noted that $g_4^{(10)}$ can be further combined with $g_4^{(9)}$ or $g_4^{(11)}$, but the solution of the combined system is more complicated than that of an individual one. Because a considerable high-order factorization for polynomials of degree 4 can be easily achieved (see Sec. VIII), the separation of $g_4^{(9)}$, $g_4^{(10)}$, and $g_4^{(11)}$ is preferred.

All Lie transformations associated with $g_4^{(n)}$ can be written as explicit symplectic maps, which are given in Eqs. (39)–(44) for $n=1,\ldots,11$ and in Eqs. (A12)–(A15), (A20), and (A22) for $n=12,\ldots,20$.

VI. INTEGRABLE POLYNOMIALS IN z OF DEGREE 5

A homogeneous polynomial of degree 5 in six variables consists of 252 monomials, which can be grouped under 42 integrable polynomials of degree 5 $\{g_5^{(n)}|n=1,2,\ldots,42\}$:

$$g_5^{(n)}=G_n(5) , \quad n=1,\ldots,8 \quad (52)$$

$$g_5^{(8+n)}=G_9(5,n,8+n) , \quad n=1,2,3,4 \quad (53)$$

$$g_5^{(9+3n+i)}=q_i^{4-n}p_i^n h_{5,1}^{(9+3n+i)}(q_j,p_j,q_k,p_k),\quad n=1,2,3 \tag{54}$$

$$g_5^{(21+i)}=q_ip_iq_jp_jh_{5,1}^{(21+i)}(q_k,p_k)\ , \tag{55}$$

$$g_5^{(21+3n+i)}=q_i^{3-n}p_i^n h_{5,2}^{(21+3n+i)}(q_j,p_j,q_k,p_k),\quad n=1,2 \tag{56}$$

$$g_5^{(30+i)}=q_ip_ih_{5,3}^{(30+i)}(q_j,q_k)\ , \tag{57}$$

$$g_5^{(33+i)}=q_ip_ih_{5,3}^{(33+i)}(p_j,p_k)\ , \tag{58}$$

$$g_5^{(36+i)}=q_ip_ih_{5,3}^{(36+i)}(q_j,p_k)\ , \tag{59}$$

$$g_5^{(39+i)}=q_ip_ih_{5,3}^{(39+i)}(p_j,q_k)\ , \tag{60}$$

where (i,j,k) goes over all cyclic permutations of (1,2,3). $G_n(5)$ and $G_9(5,n,m)$ are defined in Eqs. (28)–(35) and (38). $h_{5,1}^{(n)}(q_j,p_j,q_k,p_k)$ is a homogeneous polynomial in four variables of degree 1 defined in Eq. (50). $h_{5,2}^{(n)}(q_j,p_j,q_k,p_k)$, $h_{5,1}^{(n)}(q,p)$, and $h_{5,3}^{(n)}(q,p)$ are homogeneous polynomials in four variables of degree 2 and in two variables of degree 1 and 3, respectively,

$$\begin{aligned} h_{m,2}^{(n)}(q_j,p_j,q_k,p_k)=c_{m,1}^{(n)}q_j^2+c_{m,2}^{(n)}q_jp_j+c_{m,3}^{(n)}q_jq_k \\ +c_{m,4}^{(n)}q_jp_k+c_{m,5}^{(n)}p_j^2+c_{m,6}^{(n)}p_jq_k \\ +c_{m,7}^{(n)}p_jp_k+c_{m,8}^{(n)}q_k^2 \\ +c_{m,9}^{(n)}q_kp_k+c_{m,10}^{(n)}p_k^2\ , \end{aligned} \tag{61}$$

$$h_{5,1}^{(n)}(q,p)=c_{5,1}^{(n)}q_j+c_{5,2}^{(n)}p_j\ , \tag{62}$$

$$h_{5,3}^{(n)}(q,p)=\tfrac{1}{2}c_{5,1}^{(n)}q^3+c_{5,2}^{(n)}q^2p+c_{5,3}^{(n)}qp^2+\tfrac{1}{2}c_{5,4}^{(n)}p^3\ . \tag{63}$$

$c_{5,m}^{(n)}$ is the coefficient of the corresponding monomial in f_5. $c_{5,1}^{(30+i)}=c_{5,1}^{(36+i)}$, $c_{5,4}^{(30+i)}=c_{5,4}^{(39+i)}$, $c_{5,1}^{(33+i)}=c_{5,1}^{(39+i)}$, and $c_{5,4}^{(33+i)}=c_{5,4}^{(36+i)}$ result from symmetric forms of $h_{5,3}^{(n)}(q,p)$.

All Lie transformations associated with $g_5^{(n)}$ can be written as explicit symplectic maps, which are given in Eqs. (39)–(44) for $n=1,\dots,12$, in Eqs. (A12)–(A15) and (A20) for $n=13,\dots,21$, and in Eqs. (A12)–(A15) and (A22) for $n=25,\dots,30$. For $n=31,\dots,42$, all monomials in $g_5^{(n)}$ commute so that $\exp(:g_5^{(n)}:)\mathbf{z}$ can be evaluated by Lie transformations associated with monomials (see the Appendix, Sec. 1). By using Eqs. (A7), (A8), and (A20), $\exp(:g_5^{(n)}:)\mathbf{z}$ for $n=22$, 23, and 24 can be written as

$$\exp(:q_ip_iq_jp_jh_{5,1}^{(21+i)}:)q_l=q_l\exp(-q_{l'}p_{l'}h_{5,1}^{(21+i)})\ , \tag{64}$$

$$\exp(:q_ip_iq_jp_jh_{5,1}^{(21+i)}:)p_l=p_l\exp(q_{l'}p_{l'}h_{5,1}^{(21+i)})\ ,\quad (l,l')=(i,j)\text{ or }(j,i) \tag{65}$$

$$\exp(:q_ip_iq_jp_jh_{5,1}^{(21+i)}:)q_k=q_k-c_{5,2}^{(n)}q_ip_iq_jp_j\ , \tag{66}$$

$$\exp(:q_ip_iq_jp_jh_{5,1}^{(21+i)}:)p_k=p_k+c_{5,1}^{(n)}q_ip_iq_jp_j\ . \tag{67}$$

VII. INTEGRABLE POLYNOMIALS IN z OF DEGREE 6

Homogeneous polynomials of degree 6 in six variables consist of 462 monomials, which can be grouped under 79 integrable polynomials of degree 6 $\{g_6^{(n)}|n=1,2,\dots,79\}$:

$$g_6^{(n)}=G_n(6)\ ,\quad n=1,\dots,8 \tag{68}$$

$$g_6^{(8+n)}=G_9(6,n,8+n)\ ,\quad n=1,\dots,5 \tag{69}$$

$$g_6^{(10+3n+i)}=q_i^{5-n}p_i^n h_{6,1}^{(10+3n+i)}(q_j,p_j,q_k,p_k),\quad n=1,2,3,4 \tag{70}$$

$$g_6^{(22+3n+i)}=q_i^{4-n}p_i^n h_{6,2}^{(22+3n+i)}(q_j,p_j,q_k,p_k),\quad n=1,2,3 \tag{71}$$

$$g_6^{(34+i)}=q_ip_iq_jp_jh_{6,2}^{(34+i)}(q_k,p_k)\ , \tag{72}$$

$$g_6^{(34+3n+i)}=q_i^{3-n}p_i^n q_j h_{6,2}^{(34+3n+i)}(q_j,q_k,p_k),\quad n=1,2 \tag{73}$$

$$g_6^{(40+3n+i)}=q_i^{3-n}p_i^n p_j h_{6,2}^{(40+3n+i)}(p_j,q_k,p_k),\quad n=1,2 \tag{74}$$

$$g_6^{(46+3n+i)}=q_i^{3-n}p_i^n q_k h_{6,2}^{(46+3n+i)}(q_k,q_j,p_j),\quad n=1,2 \tag{75}$$

$$g_6^{(52+3n+i)}=q_i^{3-n}p_i^n p_k h_{6,2}^{(52+3n+i)}(p_k,q_j,p_j),\quad n=1,2 \tag{76}$$

$$g_6^{(61+i)}=q_ip_ih_{6,4}^{(61+i)}(q_j,q_k)\ , \tag{77}$$

$$g_6^{(64+i)}=q_ip_ih_{6,4}^{(64+i)}(q_j,p_k)\ , \tag{78}$$

$$g_6^{(67+i)}=q_ip_ih_{6,4}^{(67+i)}(p_j,q_k)\ , \tag{79}$$

$$g_6^{(70+i)}=q_ip_ih_{6,4}^{(70+i)}(p_j,p_k)\ , \tag{80}$$

$$g_6^{(74)}=c_{6,1}^{(74)}q_1^2p_1q_2^2p_2+c_{6,2}^{(74)}q_1^2p_1q_3^2p_3+c_{6,3}^{(74)}q_2^2p_2q_3^2p_3\ , \tag{81}$$

$$g_6^{(75)}=c_{6,1}^{(75)}q_1^2p_1q_2p_2^2+c_{6,2}^{(75)}q_1^2p_1q_3p_3^3+c_{6,3}^{(75)}q_2p_2^2q_3p_3^3\ , \tag{82}$$

$$g_6^{(76)}=c_{6,1}^{(76)}q_1p_1^2q_2p_2^2+c_{6,2}^{(76)}q_1p_1^2q_3p_3^3\ , \tag{83}$$

$$g_6^{(77)}=c_{6,1}^{(77)}q_1p_1^2q_2^2p_2+c_{6,2}^{(77)}q_1p_1^2q_3^2p_3\ , \tag{84}$$

$$g_6^{(78)}=c_{6,1}^{(78)}q_2p_2^2q_3^2p_3\ , \tag{85}$$

$$g_6^{(79)}=c_{6,1}^{(79)}q_2^2p_2q_3p_3^2\ , \tag{86}$$

where (i,j,k) goes over all cyclic permutations of (1,2,3). $G_n(6)$ and $G_9(6,n,m)$ are defined in Eqs. (28)–(35) and (38). $h_{6,1}^{(n)}(q_j,p_j,q_k,p_k)$ and $h_{6,2}^{(n)}(q_j,p_j,q_k,p_k)$ are homogeneous polynomials in four variables of degree 1 and 2 defined in Eqs. (50) and (61), respectively. $h_{6,2}^{(n)}(q,p)$ and $h_{6,2}^{(n)}(x_1,x_2,x_3)$ are homogeneous polynomials of degree 2 in two and three variables, respectively,

$$h_{6,2}^{(n)}(q,p)=c_{6,1}^{(n)}q^2+\tfrac{1}{3}c_{6,2}^{(n)}qp+c_{6,3}^{(n)}p^2\ , \tag{87}$$

$$\begin{aligned} h_{6,2}^{(n)}(x_1,x_2,x_3)=c_{6,1}^{(n)}x_1^2+\tfrac{1}{2}c_{6,2}^{(n)}x_1x_2+\tfrac{1}{2}c_{6,3}^{(n)}x_1x_3 \\ +\tfrac{1}{2}c_{6,4}^{(n)}x_2^2+c_{6,5}^{(n)}x_2x_3+\tfrac{1}{2}c_{6,6}^{(n)}x_3^2\ . \end{aligned} \tag{88}$$

$h_{6,4}^{(n)}(x_1,x_2)$ is a homogeneous polynomial of degree 4 in two variables

$$h_{6,4}^{(n)}(x_1,x_2)=\tfrac{1}{2}c_{6,1}^{(n)}x_1^4+c_{6,2}^{(n)}x_1^3x_2+c_{6,3}^{(n)}x_1^2x_2^2+c_{6,4}^{(n)}x_1x_2^3+\tfrac{1}{2}c_{6,5}^{(n)}x_2^4 \ . \tag{89}$$

$c_{6,m}^{(n)}$ is the coefficient of the corresponding monomial in f_6. Some monomials appear more than once in $g_6^{(n)}$. In order to have symmetric forms of $g_6^{(n)}$, $c_{6,m}^{(n)}$ of those monomials are divided by their number of appearances.

Similarly, by using Eqs. (39)–(44), (A5)–(A8), (A12)–(A15), (A20), and (A22), all Lie transformations associated with $g_6^{(n)}$ can be expressed as explicit symplectic maps that can be used for tracking directly.

VIII. INTEGRABLE-POLYNOMIAL FACTORIZATION

A. Nonsymmetric integrable-polynomial factorization

With integrable polynomials, a symplectic map in Eq. (9) can be rewritten as

$$\mathcal{M}_s\mathbf{z}=\mathcal{R}\prod_{i=3}^{N}\exp\left[\sum_{n=1}^{N_g(i,6)}:g_i^{(n)}:\right]\mathbf{z} \ , \tag{90}$$

where $N_g(i,6)$ is the number of integrable polynomials of degree i in six variables, which is 8, 20, 42, and 79 for $i=3$, 4, 5, and 6, respectively. By means of the Baker-Campbell-Hausdorff (BCH) formula [2], one can, in principle, convert the Lie transformation associated with a sum of integrable polynomials into a product of Lie transformations associated with integrable polynomials. We can first separate $:g_3^{(n)}:$ by changing $:g_i^{(n)}:$ of $i\geq 4$ accordingly [1]:

$$\prod_{i=3}^{N}\exp\left[\sum_{n=1}^{N_g(i,6)}:g_i^{(n)}:\right]=\left[\prod_{n=1}^{8}\exp(:g_3^{(n)}:)\right]\left[\prod_{i=4}^{N}\exp\left[\sum_{n=1}^{N_g(i,6)}:\tilde{g}_i^{(n)}:\right]\right] \ , \tag{91}$$

where $\tilde{g}_i^{(n)}$s are integrable polynomials with all coefficients recalculated. Repeating this process order by order will yield, after a truncation at a certain order, a product of Lie transformations associated with integrable polynomials:

$$\mathbf{U}_s(\mathbf{z})=\mathcal{R}\prod_{i=3}^{N}\left[\prod_{n=1}^{N_g(i,6)}\exp(:g_i^{(n)}:)\right]\mathbf{z} \ , \tag{92}$$

where for simplicity we have used $g_i^{(n)}$ to notate integrable polynomials of degree i which coefficients are recalculated by means of the BCH formula. This factorization scheme is effective in the sense that there are fewer terms in the final form of the map. For example, a six-order symplectic map is a product of 149 Lie transformations associated with integrable polynomials. The separation of low-order $\exp(:g_i^{(n)}:)$ as shown in Eq. (91), however, generates high-order spurious terms that may cascade so large that the truncation becomes invalid. Moreover, it is unclear from Eq. (92) which integrable polynomial of the same degree should be arranged in precedence of the others in the series of Lie transformations. Such a nonsymmetric property actually affects the accuracy of the map by generating larger, high-order spurious terms.

B. Symmetric integrable-polynomial factorization

In construction of symplectic integrators for numerical integration, techniques have been developed to formulate a Lie transformation for a sum of Lie operators by Lie transformations for individual Lie operators with a controllable truncation error [13–16]. A small number of factorization bases with integrable polynomials enables us to utilize these symplectic integrators.

Letting A and B be any Lie operators, the problem is finding a set of coefficients $(d_1,d_2,\ldots,d_{2k})$ such that

$$\exp[\tau(A+B)]=\prod_{i=1}^{k}\exp(d_{2i-1}\tau A)\times\exp(d_{2i}\tau B)+O(\tau^{n+1}) \ , \tag{93}$$

where integer n is the order of the integrator and τ is a small real number used only for tracking the truncation order. It was shown [16] that for any even order of τ, Eq. (93) can be systematically constructed in a symmetric form with exact coefficients. The symmetric feature also greatly suppresses the high-order truncation error. For the second order of τ, the symmetric integrator can be written as

$$e^{\tau(A+B)}=e^{(1/2)\tau A}e^{\tau B}e^{(1/2)\tau A}+O(\tau^3) \ , \tag{94}$$

and the fourth-order symmetric integrator is

$$e^{\tau(A+B)}=e^{d_1\tau A}e^{d_2\tau B}e^{d_3\tau A}e^{d_4\tau B}e^{d_5\tau A}e^{d_6\tau B}e^{d_7\tau A}+O(\tau^5) \ , \tag{95}$$

where [7,9,10]

$$d_1=d_7=\frac{1}{2(2-2^{1/3})} \ , \quad d_2=d_6=\frac{1}{2-2^{1/3}} \ , \quad d_3=d_5=\frac{1-2^{1/3}}{2(2-2^{1/3})} \ , \quad d_4=\frac{-2^{1/3}}{2-2^{1/3}} \ . \tag{96}$$

For $i\geq 5$, since $(:g_i^{(n_1)}:g_i^{(n_2)})$ is a homogeneous polynomial with degree higher than 7, a factorization with up to the seventh order is easily obtained by directly using the first-order integrator,

$$\exp\left[\sum_{n=1}^{N_g(i,6)}:g_i^{(n)}:\right]=\prod_{n=1}^{N_g(i,6)}\exp(:g_i^{(n)}:)+\epsilon(2i-2) \ , \tag{97}$$

where $i\geq 5$ and $\epsilon(2i-2)$ represents the truncated terms, which are homogeneous polynomials with degree higher than $2i-3$. For $i=5$ and 6, the lowest-order truncated term is a homogeneous polynomial of degree 8 and 10, respectively.

For homogeneous polynomials of degree 4, $(:g_4^{(n_1)}:g_4^{(n_2)})$ is a homogeneous polynomial of degree 6. We thus use the second-order integrator in Eq. (94) and obtain

$$\exp\left[\sum_{n=1}^{20} :g_4^{(n)}:\right] = \left[\prod_{i=1}^{19} \exp(:\tfrac{1}{2}g_n^{(n_i)}:)\right] \times \exp(:g_4^{(n_{20})}:) \times \left[\prod_{i=1}^{19} \exp(:\tfrac{1}{2}g_4^{(20-n_i)}:)\right] + \epsilon(8) , \quad (98)$$

where $(n_1, n_2, \ldots, n_{20})$ is any permutation of $(1,2,\ldots,20)$. The lowest-order truncated term in Eq. (98) consists of $(:g_4^{(n_1)}::g_4^{(n_2)}:g_4^{(n_3)})$, which is also a homogeneous polynomial of degree 8. Therefore, the eighth-order factorization for homogeneous polynomials of degree 4 yields a product of 39 Lie transformations associated with integrable polynomials.

The Lie transformation $\exp(:f_3:)$ is the most troublesome one. In order to obtain a sixth-order symplectic map, one has to use the fourth-order integrator in Eq. (95). Applying it once to a Lie transformation associated with eight integrable polynomials of degree 3 will result in a product of seven Lie transformations associated with four integrable polynomials. By applying the fourth-order integrator two more times, we end up with a product of $7^3 = 343$ Lie transformations associated with integrable polynomials:

$$\exp\left[\sum_{n=1}^{8} :g_3^{(n)}:\right] = \prod_{i=1}^{7}\prod_{j=1}^{7}\prod_{k=1}^{7} \exp(:d_i d_j d_k D_{ijk}:) + \epsilon(7) . \quad (99)$$

D_{ijk} is an integrable polynomial of degree 3 that can be chosen according to the following pattern:

$$i=\text{even}\begin{cases} j=\text{even}\begin{cases} k=\text{even}, & D_{ijk}=g_3^{(n_1)} \\ k=\text{odd}, & D_{ijk}=g_3^{(n_2)} \end{cases} \\ j=\text{odd}\begin{cases} k=\text{even}, & D_{ijk}=g_3^{(n_3)} \\ k=\text{odd}, & D_{ijk}=g_3^{(n_4)}, \end{cases} \end{cases}$$

$$i=\text{odd}\begin{cases} j=\text{even}\begin{cases} k=\text{even}, & D_{ijk}=g_3^{(n_5)} \\ k=\text{odd}, & D_{ijk}=g_3^{(n_6)} \end{cases} \\ j=\text{odd}\begin{cases} k=\text{even}, & D_{ijk}=g_3^{(n_7)} \\ k=\text{odd}, & D_{ijk}=g_3^{(n_8)}), \end{cases} \end{cases}$$

where $(n_1,n_2,n_3,n_4,n_5,n_6,n_7,n_8)$ is any permutation of the first eight digits (1, 2, 3, 4, 5, 6, 7, 8). For example, when i, j, and k are even, odd, and even, respectively, $D_{ijk} = g_3^{(n_3)}$. The lowest-order truncated term in Eq. (99) consists of homogeneous polynomials of degree 7 such as $(:g_3^{(n_1)}::g_3^{(n_2)}::g_3^{(n_3)}::g_3^{(n_4)}::g_3^{(n_5)}::g_3^{(n_6)}:g_3^{(n_7)})$.

IX. SUMMARY AND DISCUSSION

We have shown that any polynomial can be written as a sum of integrable polynomials of the same degree. The number of optimized integrable polynomials is much smaller than the number of monomials. For homogeneous polynomials of degree 3 or 6, we have been able to group 56, 126, 252, and 462 monomials into 8, 20, 42, and 79 integrable polynomials, respectively. All Lie transformations associated with these integrable polynomials were translated into simple iterations that can be directly used in tracking. By utilizing the symmetric symplectic integrators, we have developed a factorization scheme based on the integrable polynomials in which Lie transformations associated with homogeneous polynomials are converted into a product of Lie transformations associated with integrable polynomials. A much smaller number of integrable polynomials not only serves a more accurate set of factorization bases but also enables us to use high-order factorization schemes so that the truncation error can be greatly suppressed. The map in the form of Lie transformations associated with integrable polynomials could, therefore, be a reliable model for studying the long-term behavior of symplectic systems in the phase-space region of interest.

It should be noted that integrable polynomials with lower degrees can be completely combined with those of higher degrees. For example, 8 integrable polynomials of degree 3 can be mixed into 20 integrable polynomials of degree 4 so that a sum of homogeneous polynomials of degree 3 and 4 can be written as a sum of these 20 integrable polynomials. The factorization with integrable polynomials can, therefore, be directly applied to the Deprit-type Lie transformation that takes the form $\exp(:\sum_i f_i:)$. Since high-order factorizations are much more difficult to achieve on lower-degree polynomials, a separate treatment of homogeneous polynomials of degree 3 and 4 is favorable, even though the combined one has fewer integrable polynomials.

In Table I, we listed the number of monomials $N(i,6)$ and the number of integrable polynomials $N_g(i,6)$ of degree i in 6 variables. A slow decrease in $N(i,6)/N_g(i,6)$ with increase in i indicates that the advantage of the integrable polynomials diminishes slowly with i. In practical cases, however, it would not be a serious limitation since accelerators are mainly dominated by low-order multipoles.

It should be pointed out that the "solvable maps," for which the Taylor-series expansion in Eq. (4) can be summed explicitly, was proposed to approximate general

TABLE I. The number of monomials $N(i,6)$, number of integrable polynomials $N_g(i,6)$, and effectiveness of grouping the monomials into the integrable polynomials $N(i,6)/N_g(i,6)$ for degree i in six variables.

Degree i	$N(i,6)$	$N_g(i,6)$	$N(i,6)/N_g(i,6)$
3	56	8	7
4	126	20	6.3
5	252	42	6
6	462	79	5.8

Lie transformations [17]. This idea was not fully developed, since directly finding the sum of the expansion is difficult in general. On some simple examples, the solvable maps were obtained [17] as special cases of the Lie transformations associated integrable polynomials. Good agreement with exact numerical integration was found in tracking studies for these examples.

ACKNOWLEDGMENTS

J.S. would like to thank Professor Sho Ohnuma for encouraging his study of this problem and for many stimulating discussions. We thank P. J. Channell, A. W. Chao, A. J. Dragt, E. Forest, and M. J. Syphers for many helpful discussions. This work is supported by TNRLC under Grant No. FCFY9221 and by the U.S. Department of Energy under Grants Nos. DE-FG05-87ER40374 and DE-AC35-89ER40486.

APPENDIX

In this appendix, we shall convert Lie transformations associated with monomials and integrable polynomials $q_i^{\sigma_1}p_i^{\sigma_2}h_{m,l}^{(n)}(q_j,p_j,q_k,p_k)$ for $l=1$ and 2 into symplectic maps that can be directly used for tracking, where $h_{m,l}^{(n)}$ is a homogeneous polynomial in four variables of degree l, and (i,j,k) is any permutation of (1,2,3).

1. Lie transformations associated with monomials

Consider a Hamiltonian

$$H=-aq_1^{\sigma_1}p_1^{\sigma_2}q_2^{\sigma_3}p_2^{\sigma_4}q_3^{\sigma_5}p_3^{\sigma_6} . \tag{A1}$$

The equations of motion are

$$\dot{z}_l=\sigma_{2l}\frac{H}{p_l} , \tag{A2}$$

$$\dot{p}_l=-\sigma_{2l-1}\frac{H}{q_l} , \tag{A3}$$

which give

$$q_l^{\sigma_{2l-1}}p_l^{\sigma_{2l}}=\text{const} , \tag{A4}$$

where $l=1$, 2, and 3. With Eq. (A4), we can solve Eqs. (A2) and (A3) and obtain, for $\sigma_{2l-1}\neq\sigma_{2l}$,

$$\exp(:aq_1^{\sigma_1}p_1^{\sigma_2}q_2^{\sigma_3}p_2^{\sigma_4}q_3^{\sigma_5}p_3^{\sigma_6}:)q_l =q_l\left[1+(\sigma_{2l-1}-\sigma_{2l})\times\frac{aq_1^{\sigma_1}p_1^{\sigma_2}q_2^{\sigma_3}p_2^{\sigma_4}q_3^{\sigma_5}p_3^{\sigma_6}}{q_lp_l}\right]^{\sigma_{2l}/(\sigma_{2l}-\sigma_{2l-1})} , \tag{A5}$$

$$\exp(:aq_1^{\sigma_1}p_1^{\sigma_2}q_2^{\sigma_3}p_2^{\sigma_4}q_3^{\sigma_5}p_3^{\sigma_6}:)p_l =p_l\left[1+(\sigma_{2l-1}-\sigma_{2l})\times\frac{aq_1^{\sigma_1}p_1^{\sigma_2}q_2^{\sigma_3}p_2^{\sigma_4}q_3^{\sigma_5}p_3^{\sigma_6}}{q_lp_l}\right]^{\sigma_{2l-1}/(\sigma_{2l-1}-\sigma_{2l})} , \tag{A6}$$

and for $\sigma_{2l-1}=\sigma_{2l}$,

$$\exp(:aq_1^{\sigma_1}p_1^{\sigma_2}q_2^{\sigma_3}p_2^{\sigma_4}q_3^{\sigma_5}p_3^{\sigma_6}:)q_l =q_l\exp\left[\frac{-\sigma_{2l-1}aq_1^{\sigma_1}p_1^{\sigma_2}q_2^{\sigma_3}p_2^{\sigma_4}q_3^{\sigma_5}p_3^{\sigma_6}}{q_lp_l}\right] , \tag{A7}$$

$$\exp(:aq_1^{\sigma_1}p_1^{\sigma_2}q_2^{\sigma_3}p_2^{\sigma_4}q_3^{\sigma_5}p_3^{\sigma_6}:)p_l =p_l\exp\left[\frac{\sigma_{2l-1}aq_1^{\sigma_1}p_1^{\sigma_2}q_2^{\sigma_3}p_2^{\sigma_4}q_3^{\sigma_5}p_3^{\sigma_6}}{q_lp_l}\right] . \tag{A8}$$

2. Lie transformations associated with $q_1^{\sigma_1}p_i^{\sigma_2}h_{m,l}^{(n)}(q_j,p_j,q_k,p_k)$ for $l=1$ and 2

Consider a Hamiltonian

$$H=-q_i^{\sigma_1}p_i^{\sigma_2}h_{m,l}^{(n)}(q_j,p_j,q_k,p_k) . \tag{A9}$$

The equations of motion for (q_i,p_i) are

$$\dot{q}_i=-\sigma_2q_i^{\sigma_1}p_i^{\sigma_2-1}h_{m,l}^{(n)} , \tag{A10}$$

$$\dot{p}_i=\sigma_1q_i^{\sigma_1-1}p_i^{\sigma_2}h_{m,l}^{(n)} , \tag{A11}$$

which give $q_i^{\sigma_1}p_i^{\sigma_2}=\text{const}$. Since $H=\text{const}$, $h_{m,l}^{(n)}=\text{const}$, and solving Eqs. (A10) and (A11) is similar to solving Eqs. (A2) and (A3). Using Eqs. (A5)–(A8) we obtain, for $\sigma_1\neq\sigma_2$,

$$\exp[:q_i^{\sigma_1}p_i^{\sigma_2}h_{m,l}^{(n)}(q_j,p_j,q_k,p_k):]q_i =q_i[1+(\sigma_1-\sigma_2)q_i^{\sigma_1-1}p_i^{\sigma_2-1}h_{m,l}^{(n)}]^{\sigma_2/(\sigma_2-\sigma_1)} , \tag{A12}$$

$$\exp[:q_i^{\sigma_1}p_i^{\sigma_2}h_{m,l}^{(n)}(q_j,p_j,q_k,p_k):]p_i =p_i[1+(\sigma_1-\sigma_2)q_i^{\sigma_1-1}p_i^{\sigma_2-1}h_{m,l}^{(n)}]^{\sigma_1/(\sigma_1-\sigma_2)} , \tag{A13}$$

and for $\sigma_1=\sigma_2$,

$$\exp[:q_i^{\sigma_1}p_i^{\sigma_2}h_{m,l}^{(n)}(q_j,p_j,q_k,p_k):]q_i =q_i\exp[-\sigma_1(q_ip_i)^{\sigma_1-1}h_{m,l}^{(n)}] , \tag{A14}$$

$$\exp[:q_i^{\sigma_1}p_i^{\sigma_2}h_{m,l}^{(n)}(q_j,p_j,q_k,p_k):]p_i =p_i\exp[\sigma_1(q_ip_i)^{\sigma_1-1}h_{m,l}^{(n)}] . \tag{A15}$$

Let us define

$$\mathbf{r}=(q_j,p_j,q_k,p_k)^T \tag{A16}$$

and

$$\frac{\partial}{\partial\mathbf{r}}=\left[\frac{\partial}{\partial q_j},\frac{\partial}{\partial p_j},\frac{\partial}{\partial q_k},\frac{\partial}{\partial p_k}\right]^T , \tag{A17}$$

where superscript T denotes the transpose. The equations of motion for (q_j,p_j,q_k,p_k) can then be written as

$$\dot{\mathbf{r}}=-q_i^{\sigma_1}p_i^{\sigma_2}\Gamma\frac{\partial}{\partial\mathbf{r}}h_{m,l}^{(n)} , \tag{A18}$$

where Γ is a four-dimensional antisymmetric matrix,

$$\Gamma=\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} . \tag{A19}$$

For $l=1$, $\partial h_{m,l}^{(n)}/\partial \mathbf{r}$ are constants, and solving Eq. (A18) yields

$$\exp[:q_i^{\sigma_1}p_i^{\sigma_2}h_{m,1}^{(n)}(q_j,p_j,q_k,p_k):]\mathbf{r}=\mathbf{r}-q_i^{\sigma_1}p_i^{\sigma_2}\Gamma\frac{\partial h_{m,1}^{(n)}}{\partial \mathbf{r}} . \tag{A20}$$

For $l=2$, Eq. (A18) can be rewritten as

$$\dot{\mathbf{r}}=-q_i^{\sigma_1}p_i^{\sigma_2}\left[\Gamma\frac{\partial}{\partial \mathbf{r}}\left[\frac{\partial}{\partial \mathbf{r}}\right]^T h_{m,2}^{(n)}\right]\mathbf{r} . \tag{A21}$$

By diagonalizing the constant matrix $[\Gamma(\partial/\partial \mathbf{r})(\partial/\partial \mathbf{r})^T h_{m,2}^{(n)}]$, we solve Eq. (A21) and obtain

$$\exp(:q_i^{\sigma_1}p_i^{\sigma_2}h_{m,2}^{(n)}:)\mathbf{r}=U_{m,n}^{-1}\begin{bmatrix} e^{-q_i^{\sigma_1}p_i^{\sigma_2}\lambda_{m,1}^{(n)}} & 0 & 0 & 0 \\ 0 & e^{-q_i^{\sigma_1}p_i^{\sigma_2}\lambda_{m,2}^{(n)}} & 0 & 0 \\ 0 & 0 & e^{-q_i^{\sigma_1}p_i^{\sigma_2}\lambda_{m,3}^{(n)}} & 0 \\ 0 & 0 & 0 & e^{-q_i^{\sigma_1}p_i^{\sigma_2}\lambda_{m,4}^{(n)}} \end{bmatrix}U_{m,n}\mathbf{r} , \tag{A22}$$

where

$$U_{m,n}\left[\Gamma\frac{\partial}{\partial \mathbf{r}}\left[\frac{\partial}{\partial \mathbf{r}}\right]^T h_{m,2}^{(n)}\right]U_{m,n}^{-1}=\begin{bmatrix} \lambda_{m,1}^{(n)} & 0 & 0 & 0 \\ 0 & \lambda_{m,2}^{(n)} & 0 & 0 \\ 0 & 0 & \lambda_{m,3}^{(n)} & 0 \\ 0 & 0 & 0 & \lambda_{m,4}^{(n)} \end{bmatrix} . \tag{A23}$$

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