# Coupled-mode theory of Langmuir space-charge waves for general electron-beam and waveguide cross sections

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An alternative approach for the analysis of the electromagnetic field and plasma-wave propagation in a waveguide filled with an electron ($e$) beam is presented. The analysis is based on a formal exact expansion of the total electromagnetic field in terms of waveguide modes. We subsequently use linear fluid plasma equations and electromagnetic coupled-mode theory to find the dispersion relation for the eigenmodes of the beam (plasma) loaded waveguide. The proposed method enables one to solve for the Langmuir space-charge waves in an $e$ beam with an arbitrary transverse geometry and density distribution, moving along any uniform-cross-section waveguide at constant average velocity. The use of the method is demonstrated by presenting a calculation of the dispersion curve and the plasma frequency reduction factor of plasma modes in a practical case of a circular beam drifting along a rectangular waveguide.

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## I. INTRODUCTION

Much effort has been directed towards the investigation of self-field effects in charged beams and plasmas [1–6]. In the early work of Tonks and Langmuir [1], the oscillation of uniformly distributed electrons, which are neutralized by positive ions, was studied. It was shown that the plasma oscillation frequency of an infinite electron cloud is given by $\omega_p = \sqrt{n_0 e^2/\epsilon_0 m}$, where $n_0$ is the density of the electrons and $e$ and $m$ are the electron charge and mass, respectively.

Excitation and propagation of electromagnetic and space-charge waves in electron beams drifting at a constant velocity play an important role in many electronic devices, especially electron microwave tubes. The analysis of waves in a plasma is based on the solution of Maxwell's equations, together with the beam-flow equations under appropriate boundary conditions. In a one-dimensional (1D) model, where an infinite beam with uniform density distribution is assumed, the electric field remains entirely within the plasma and only two ("fast" and "slow") longitudinal-field plasma waves are found to propagate. The 1D case is an idealized model, which may not be useful for practical cases, where a finite-cross-section beam is moving along a partially filled metallic tube or waveguide. A more accurate model should take into account the finite size of the beam and the effect of the conducting walls of the drift tube.

Plasma waves in a finite cross-section electron ($e$) beam were first analyzed by Hahn [2] and Ramo [3]. A distinction between "plasma waves" and "field waves" was suggested [3]. For the first type, the phase velocity is close to the beam velocity and the electrons are carrying most of the wave energy. Such waves (slow and fast waves of different transverse profiles) can also propagate in the plasma at frequencies below the cutoff frequency of the waveguide. When the phase velocity is much greater than the drift velocity of the electrons, the wave is termed "field wave," for which the energy is mainly in the electromagnetic field.

Contrary to the case of a transversely infinite electron beam, where the electric field of a density-modulated space charge is purely longitudinal, in the case of a finite-cross-section beam, there is a fringing field near the edges of the beam. Fringing is further increased by the presence of conducting walls. The modification of the space-charge forces due to the finite cross section of the beam and the conducting walls surrounding the beam leads to a reduction of the plasma oscillation frequency in the axial direction. The reduction of the frequency of oscillation for a given plasma mode, relative to the plasma frequency $\omega_p$ of an infinite beam, is an important design parameter for all types of electron tubes. In [4] the plasma frequency reduction factor was calculated for a variety of uniform-density beam shapes and drift tubes with circular symmetry.

Renewed interest in excitation and propagation of plasma waves in electron beams emerged recently in connection with free-electron lasers (FEL's). Such devices involve density-modulated $e$ beams in which space-charge waves are excited in addition to the electromagnetic radiation of the signal. Space-charge forces between the bunches become significant when the FEL utilizes an intense beam with a high electron density (Raman regime FEL) [7]. Collective effects in FEL's are expected to affect the gain, bandwidth, and efficiency of energy transfer from the electron beam to the radiation field [8,9]. A 1D description of the space-charge field in a FEL is insufficient because of the finite transverse dimensions of the $e$ beam and the effect of the conducting walls. In typical experiments the transverse (fringing) field of the excited space-charge waves may be comparable to the axial field [10]. Thus it is necessary to use a more elaborate model which takes three-dimensional aspects of the plasma-wave propagation.

Several papers discussed the problem of space-charge effects in the framework of 3D models of FEL's. Most of these works [11–14] treated the plasma waves in the framework of a 2D model (planar or circular symmetry), where the eigenmodes of the $e$ beam could be found by solving a dispersion relation emanating from solving the fields inside and outside of the beam and matching boundary conditions at the beam and conducting wall boundaries. Such analytical calculations may be difficult for the general case in which the transverse dimensions cannot be reduced by symmetry into a single dimension (for example, the practical case of circular beam in a rectangular waveguide). Another semianalytical approach was suggested in [15], based on Fourier transformation of the Maxwell and Boltzmann equations. The field is represented as a superposition of plane waves, for which a matrix dispersion relation is found. This approach can be applied to general $e$ beam cross sections, but is limited to the cases of free-space or rectangular-waveguide propagation.

In this paper we develop a general theory for analyzing the space-charge field in waveguides with arbitrary transverse geometries. The total electric field in the plasma-loaded waveguide is expressed as a sum of the transverse eigenmodes of the empty waveguide. The expansion leads to a set of coupled-mode equations, which are solved to produce the dispersion equation of the plasma waves in the beam-loaded waveguide. Such an approach eliminates the need for solving a two-dimensional problem, in which conditions on the transverse boundaries of the $e$ beam and of the waveguide must be considered.

Our formulation should be especially useful for analysis of FEL and other microwave tubes, where there is no need to solve the dispersion relation of the plasma waves separately from the dispersion equation of the device. However, in the present article we confine our analysis to the solution of the plasma waves in a nonradiating structure, namely without coupling to field waves (as in the FEL). In particular we show how this formulation can be applied to solve problems with mixed symmetry (e.g., a circular $e$ beam in a rectangular waveguide).

## II. EXCITATION OF WAVEGUIDE MODES

The excitation of an electromagnetic field in a waveguide is calculated by expansion of the total field in terms of eigenmode solutions of the empty waveguide, following the network formalism developed in [16,17]. This enables us to transform the inhomogeneous steady-state Maxwell vector field equations into scalar transmission-line equations. The equations describe the amplitude growth of each waveguide mode excited by a harmonic current source $\mathbf{J}(r,t)=\mathrm{Re}\{\tilde{\mathbf{J}}(r)e^{-j\omega t}\}$ distributed along the axis of propagation. The results will be used later for calculation of the total electromagnetic and space-charge fields in a plasma waveguide.

For a perfectly conducting waveguide, filled with an isotropic medium, it is possible to write the transverse electric field as a linear superposition of a complete set of transverse waveguide eigenmodes:

$$\tilde{\mathbf{E}}_{\perp}(r)=\sum_q V_q(z)\tilde{\mathcal{E}}_{q\perp}(x,y) ,\quad \tilde{\mathbf{H}}_{\perp}(r)=\sum_q I_q(z)\tilde{\mathcal{H}}_{q\perp}(x,y) . \tag{1}$$

$V_q(z)$ and $I_q(z)$ are the scalar amplitude of the mode and $\tilde{\mathcal{E}}_{q\perp}(x,y)$ and $\tilde{\mathcal{H}}_{q\perp}(x,y)$ are complex vectors representing the transverse profile and polarization of the electric and the magnetic fields of waveguide mode $q$, respectively. The above summation consists of both TE and TM mode profile functions. $r$ represents the $(x,y,z)$ coordinates, where $(x,y)$ are the transverse coordinates and $z$ is the longitudinal axis of propagation. This modal expansion can be carried out for any kind of waveguide with a uniform cross section, and it is an exact representation of the field since $\{\tilde{\mathcal{E}}_{q\perp}(x,y)\}$ (including cutoff modes) is a complete set.

From Maxwell's equations, by imposing the boundary conditions on the waveguide walls it can be shown that the evolution of the electric and magnetic mode amplitudes $V_q(z)$ and $I_q(z)$ is described by two coupled first-order differential equations [17]:

$$\begin{aligned}-\frac{d}{dz}V_q(z)&=-jk_{zq}Z_qI_q(z)\\&\quad+Z_q^*\int\int \tilde{J}_z(r)\tilde{\mathcal{E}}_{qz}^*(x,y)dx\,dy ,\\-\frac{d}{dz}I_q(z)&=-jk_{zq}\frac{1}{Z_q}V_q(z)\\&\quad+\int\int \tilde{\mathbf{J}}_{\perp}(r)\cdot\tilde{\mathcal{E}}_{q\perp}^*(x,y)dx\,dy .\end{aligned} \tag{2}$$

$k_{zq}=\sqrt{k^2-k_{\perp q}^2}$ is the longitudinal wave number of the mode, $k=\omega\sqrt{\epsilon\mu}$, and $Z_q$ is the mode impedance given by $Z_{\mathrm{TE}q}=\omega\mu/k_{zq}$ for TE mode and $Z_{\mathrm{TM}q}=k_{zq}/\omega\epsilon$ for TM mode. From this transmission-line formulation, a differential equation of the second order for the amplitude of the electric-field modes $V_q(z)$ can be derived easily:

$$\begin{aligned}\frac{d^2}{dz^2}V_q(z)+k_{zq}^2V_q(z)&\\=-jk_{zq}Z_q\int\int \tilde{\mathbf{J}}_{\perp}(r)\cdot\tilde{\mathcal{E}}_{q\perp}^*(x,y)dx\,dy&\\-Z_{\mathrm{TM}q}^*\int\int\frac{d\tilde{J}_z(r)}{dz}\tilde{\mathcal{E}}_{\mathrm{TM}_{qz}}^*(x,y)dx\,dy&\ .\end{aligned} \tag{3}$$

The normalization of each mode is chosen in this derivation to satisfy the orthornormality relation:

$$\int\int \tilde{\mathcal{E}}_{q\perp}(x,y)\cdot\tilde{\mathcal{E}}_{q'\perp}^*(x,y)dx\,dy=\delta_{qq'} \tag{4}$$

and consequently the longitudinal components fulfill

$$\int\int \tilde{\mathcal{E}}_{qz}(x,y)\tilde{\mathcal{E}}_{q'z}^*(x,y)dx\,dy=\frac{k_{\perp q}^2}{|k_{zq}|^2}\delta_{qq'} . \tag{5}$$

## III. THE ELECTRON-BEAM FLUID EQUATIONS

To describe the small signal oscillation in a cold-electron beam, we use the standard plasma moment equations. In this linear fluid model, the ac perturbations in the space-charge density and in the electron axial velocity are assumed to oscillate at a single angular frequency $\omega$. The total density of electrons in the beam is given by

$$n(r,t)=n_0(x,y)+\mathrm{Re}\{\tilde{n}_i(r)e^{-j\omega t}\} , \tag{6}$$

where $n_0(x,y)$ is the dc part of the beam density and $\tilde{n}_i(r)$ is the first-order perturbation of the density modulation oscillating at $\omega$. Similarly, we expand the axial velocity of the electrons to its first order:

$$v_z(r,t)=v_{z0}+\mathrm{Re}\{\tilde{v}_z^i(r)e^{-j\omega t}\} . \quad (7)$$

The longitudinal ac part of the current density is

$$\tilde{J}_z^i(r)=-e[n_0(x,y)\tilde{v}_z^i(r)+v_{z0}\tilde{n}_i(r)] . \quad (8)$$

The connection between the oscillation of the space-charge density and the modulation in velocity is found from the continuity equation. In a magnetically confined beam, transverse components of the electrons velocities are sufficiently small compared to the longitudinal ones. This allows a common approximation in which transverse variation of the current density is neglected ($|\nabla_\perp\cdot\tilde{\mathbf{J}}_\perp^i| \ll |\partial J_z^i/\partial z|$) in the equation of continuity

$$\frac{d\tilde{J}_z^i(r)}{dz}=-j\omega e\tilde{n}_i(r) . \quad (9)$$

The axial velocity $\tilde{v}_z^i(r)$ is found from the relativistic axial force equation [9]:

$$\frac{d}{dz}\tilde{v}_z^i(r)-j\frac{\omega}{v_{z0}}\tilde{v}_z^i(r)=-\frac{e}{\gamma_0\gamma_{z0}^2 m v_{z0}}\tilde{E}_z^{\mathrm{sc}}(r) . \quad (10)$$

$\gamma_0=(1-\beta_0^2)^{-1/2}$ is the Lorentz factor and $\gamma_{z0}\equiv(1-\beta_{z0}^2)^{-1/2}$. These parameters may be different from each other when a monoenergetic electron beam is transported in a focusing magnetic structure (like a wiggler in FEL). Otherwise $\gamma_{z0}=\gamma_0$. The forcing term $\tilde{E}_z^{\mathrm{sc}}(r)$ is the longitudinal component of the ac electric field of the electron beam, oscillating at $\omega$.

The total ac space-charge field, which is caused by the density modulation, can be found from the Poisson equation:

$$\nabla\cdot\tilde{\mathbf{E}}_{\mathrm{sc}}(r)=-\frac{e}{\epsilon_0}\tilde{n}_i(r) . \quad (11)$$

From Eqs. (8)–(11), a differential equation of the second order for the density bunching is derived

$$\frac{d^2}{dz^2}\tilde{n}_i(r)-2j\frac{\omega}{v_{z0}}\frac{d}{dz}\tilde{n}_i(r)+\frac{\omega_p^2(x,y)-\omega^2}{v_{z0}^2}\tilde{n}_i(r) =-\frac{\omega_p^2(x,y)}{v_{z0}^2}\frac{\epsilon_0}{e}\nabla_\perp\cdot\tilde{\mathbf{E}}_\perp^{\mathrm{sc}}(r) , \quad (12)$$

where we define $\omega_p^2(x,y)\equiv(e^2/\gamma_0\gamma_{z0}^2\epsilon_0 m)n_0(x,y)$. This "beam equation" fully describes the evolution of the space-charge density modulation along the axis of propagation.

Note that in a 1D model, where an infinite beam with a uniform density distribution is considered, there are no transverse variations in the self-field. Thus Eq. (12) becomes homogeneous and the two well-known solutions of slow and fast plasma waves can be found easily. It is exactly the forcing term on the right-hand side that brings up the 3D effect of the fringing field. In order to solve the complete 3D problem, it is necessary to find the transverse component $\tilde{\mathbf{E}}_\perp^{\mathrm{sc}}(r)$ and substitute it into the beam equation.

Now we use the modal expansion, presented previously, and substitute Eq. (1) into the beam equation (12), remembering that in a waveguide each mode satisfies $\nabla_\perp\cdot\tilde{\mathscr{E}}_{q\perp}=-jk_{zq}\tilde{\mathscr{E}}_{qz}$:

$$\frac{d^2}{dz^2}\tilde{n}_i(r)-2j\frac{\omega}{v_{z0}}\frac{d}{dz}\tilde{n}_i(r)+\frac{\omega_p^2(x,y)-\omega^2}{v_{z0}^2}\tilde{n}_i(r) =j\frac{\omega_p^2(x,y)}{v_{z0}^2}\frac{\epsilon_0}{e}\sum_q k_{zq}V_q(z)\tilde{\mathscr{E}}_{qz}(x,y) . \quad (13)$$

The amplitude $V_q(z)$ of the excited modes can be expressed in terms of the density bunching $\tilde{n}_i(r)$ by substituting Eq. (9) into Eq. (3):

$$\frac{d^2}{dz^2}V_q(z)+k_{zq}^2V_q(z)=j\frac{e}{\epsilon_0}k_{zq}^*\int\int\tilde{n}_i(r)\tilde{\mathscr{E}}_{qz}^*(z,y)dx\,dy . \quad (14)$$

Equation (13), together with the set of Eqs. (14), describes the density modulation in the electron beam and the amplitude of each mode excited in the waveguide.

The only modes that can be excited by the longitudinal ac current are the TM modes which are observed to be coupled by the density bunching $\tilde{n}_i(r)$. While this set of equations includes only the amplitudes of the electric-field-mode profiles $V_q(z)$, clearly in the general case also the magnetic field is excited. The magnetic field can be calculated in principle from the solution of (13) and (14) by substituting in (1).

## IV. THE DISPERSION EQUATION OF THE SPACE-CHARGE WAVES

The eigensolutions of the above linear set of homogeneous equations are found by assuming behavior of the form $e^{sz}$ and substituting the mode amplitudes $V_q(s)$ from Eq. (14) into the beam Eq. (13). We find

$$\left[\left[s-j\frac{\omega}{v_{z0}}\right]^2+\frac{\omega_p^2(x,y)}{v_{z0}^2}\right]\tilde{n}_i(s,x,y) =-\frac{\omega_p^2(x,y)}{v_{z0}^2}\sum_q\frac{|k_{zq}|^2}{s^2+k_{zq}^2}\tilde{\mathscr{E}}_{qz}(x,y) \times\int\int\tilde{n}_i(s,x,y)\tilde{\mathscr{E}}_{qz}^*(x,y)dx\,dy . \quad (15)$$

The profile functions $\tilde{\mathscr{E}}_{qz}(x,y)$ of the longitudinal component of the TM modes including cutoff modes are a complete set of orthogonal functions which can be used to expand the density bunching in a linear combination:

$$\tilde{n}_i(s,x,y)=\sum_q A_q(s)\tilde{\mathscr{E}}_{qz}(x,y) . \quad (16)$$

Substitution of this expansion into Eq. (15) and scalar multiplication of both sides by $\tilde{\mathscr{E}}_{q'z}^*(x,y)$ gives

$$\left[s-j\frac{\omega}{v_{z0}}\right]^2 A_{q'}(s)+\sum_q A_q(s)\theta^2_{p\,q'q}(s,\omega)=0\ , \tag{17}$$

where

$$\theta^2_{p\,q'q}(s,\omega)\equiv\left[1+\frac{k^2_{\perp q}}{s^2+k^2_{zq}}\right]\times\frac{\int\int\frac{\omega_p^2(x,y)}{v_{z0}^2}\tilde{\mathcal{E}}_{qz}(x,y)\tilde{\mathcal{E}}^*_{q'z}(x,y)dx\,dy}{\int\int|\tilde{\mathcal{E}}_{q'z}(x,y)|^2dx\,dy}\ . \tag{18}$$

This constitutes a set of algebraic equations for the expansion coefficients $A_q(s)$ of the density modulation. This homogeneous system can be presented in a compact matrix form:

$$\left[\left[s-j\frac{\omega}{z_{z0}}\right]^2\underline{\underline{I}}+\underline{\underline{\Theta}}_p^{(2)}(s,\omega)\right]\underline{A}(s)=0\ . \tag{19}$$

The coupling between the coefficients $A_q(s)$ is expressed by the matrix $\underline{\underline{\Theta}}_p^{(2)}$, whose elements are given in (18).

The condition for nontrivial solution for $A_q(s)$ is that the matrix determinant is null. This yields a characteristic equation from which the propagation constants $\beta=\text{Im}\{s\}$ of the eigenwaves (both field waves and plasma waves) can be found:

$$\left|\left[s-j\frac{\omega}{v_{z0}}\right]^2\underline{\underline{I}}+\underline{\underline{\Theta}}_p^{(2)}(s,\omega)\right|=0\ . \tag{20}$$

## V. SOLUTION OF THE SPACE-CHARGE MODES

Without any restriction on generality, consider now an electron beam having a space-charge density which is uniformly distributed over its finite cross section. The characteristic Eq. (20) can be written in a convenient form in terms of the dimensionless parameters $X\equiv(c/\omega_p)\beta=(c/\omega_p)\text{Im}\{s\}$ and $Y\equiv\omega/\omega_p$:

$$|(\beta_{z0}X-Y)^2\underline{\underline{I}}-\underline{\underline{R}}^{(2)}(X,Y)|=0\ . \tag{21}$$

The elements of matrix $\underline{\underline{R}}^{(2)}$ are given by

$$r^2_{q'q}(X,Y)=\frac{1}{1+\frac{(ck_{\perp q}/\omega_p)^2}{X^2-Y^2}}\times\frac{\int\int_{e\text{ beam}}\tilde{\mathcal{E}}_{qz}(x,y)\tilde{\mathcal{E}}^*_{q'z}(x,y)dx\,dy}{\int\int_{\text{WG}}|\tilde{\mathcal{E}}_{q'z}(x,y)|^2dx\,dy}\ , \tag{22}$$

where integration is over the electron beam or the waveguide (WG).

When a uniform density, magnetically confined electron beam completely fills up the waveguide, there is no coupling between the modes. The off-diagonal terms in $\underline{\underline{R}}^{(2)}$ vanish due to the orthogonality relation and the matrix $\underline{\underline{R}}^{(2)}$ is naturally diagonal. Namely the profiles $\tilde{\mathcal{E}}_{qz}(x,y)$ of the unloaded waveguide TM modes are also the profiles of the space-charge eigenmodes of the beam for any uniformly loaded geometry. Their propagation constants $\beta$ are found from the well-known characteristic equation:

$$Y=\beta_{z0}X\pm r_q(X,Y)\ . \tag{23}$$

In non-normalized terms this dispersion relation spells $\omega=v_{z0}\beta\pm r_q\omega_q$, which helps us to identify Eq. (23) as the well-known modified 1D plasma dispersion equation, where $r_q(X,Y)$ is the so-called *plasma-frequency reduction factor* of the space-charge mode $q$. The signs + or − stand for fast or slow modes, respectively.

The fact that the space-charge waves propagate with a velocity which is nearly equal to that of the beam permits us to approximate the propagation constant by $\beta=\text{Im}\{s\}\approx\omega/v_{z0}$ (see Ref. [18]). Using this approximation, we can obtain the expression for the reduction factor of the plasma frequency in a confined beam, which fills the entire space between the waveguide walls:

$$r_q=\left[1+\left[\gamma_{z0}\beta_{z0}\frac{k_{\perp q}}{k}\right]^2\right]^{-1/2}\ . \tag{24}$$

This expression is identical (except for the relativistic correction) with the well-known expression which is derived in standard text books for waveguides of cylindrical symmetry [18]. Our general derivation proves its validity for uniformly beam-loaded waveguides of an arbitrary cross section.

For the general case, where the beam partially fills the waveguide, the matrix $\underline{\underline{R}}^{(2)}$ is not diagonal and the dispersion Eq. (21) must be usually solved numerically. Also in these cases it is convenient to make an approximation analogous to the previous case and substitute in $\underline{\underline{R}}^{(2)}$ $\beta=\text{Im}\{s\}\approx\omega/v_{z0}$ (or $X=\omega/\omega_p$). Equation (21) then looks like an algebraic eigenvalue problem $|\underline{\underline{R}}^{(2)}-\lambda\underline{\underline{I}}|=0$, and the plasma-frequency reduction factor of the $q$th space-charge mode is given by $r_q=\sqrt{\lambda_q}$. Using this simple approximation, we now demonstrate the use of the general coupled-mode formulation to calculate the plasma-dispersion relation curves in a

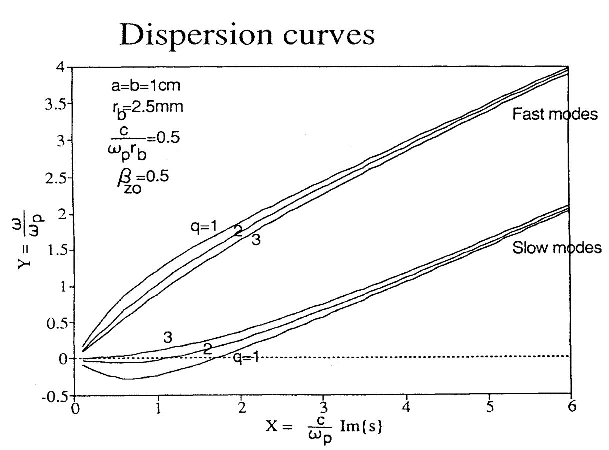


FIG. 1. Dispersion curves of three fundamental space-charge modes of a circular $e$ beam propagating along a rectangular waveguide.

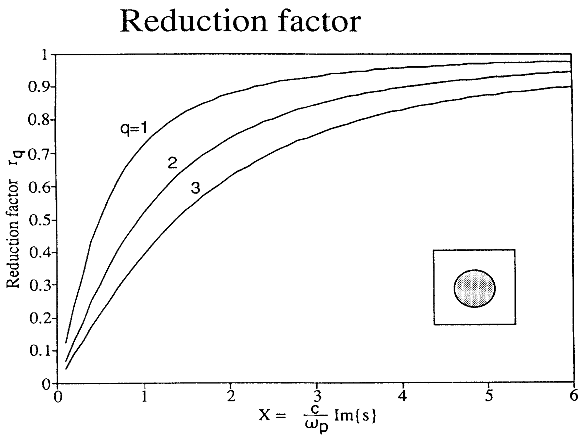


FIG. 2. Plasma frequency reduction factor vs the propagation constant of the three fundamental space-charge waves.

configuration that has not been analyzed previously.

In various practical devices (such as microwave electron devices and FEL's), a circular electron beam propagates along conducting pipes, such as rectangular or curved parallel plates waveguides. The calculation of the space-charge waves in this case cannot be done by solving the plasma modes in a direct analytical way.

We calculated the plasma-dispersion curves of a circular $e$ beam of radius $r_b = 2.5$ mm, drifting at a velocity $v_{z0} = 0.5c$ along a $1 \times 1$ cm$^2$ rectangular waveguide. Figure 1 shows the dispersion curves of the first three fundamental slow and fast space-charge modes, calculated with $7 \times 7 = 49$ waveguide modes. The asymptotic behavior of the waves when $X \to \infty$ is of the form $Y = \beta_{z0} X \pm 1$ (e.g., $\omega = v_{z0}\beta \pm \omega_p$) for the fast and slow modes. The plasma-frequency reduction factor for the various transverse plasma modes is illustrated in Fig. 2.

## VI. CONCLUSIONS

In the present paper we developed a coupled-mode method for finding the space-charge waves of a confined beam, which partially fills the waveguide. Instead of solving the Maxwell equations together with the plasma fluid equations and matching boundary conditions at the beam and waveguide surfaces, as is done in usual analyses, the total electromagnetic field is expanded as a linear combination of the empty waveguide TM modes. Using coupled-mode theory, we derived a set of coupled linear equations for the density and the waveguide-mode amplitudes from which a dispersion relation in the form of a matrix eigenvalue problem is obtained. This relation can be solved numerically quite straightforwardly. The solutions of this dispersion relation results in the dispersion curves and the plasma reduction factor parameter of the space-charge waves, which are the natural modes of the beam- (plasma) loaded waveguide.

This method is very useful in problems where the space-charge waves cannot be found analytically because of the complexity of the boundary conditions or because the $e$ beam (plasma) density is not distributed uniformly. Using coupled-mode analysis makes it possible to calculate the plasma-frequency reduction factor in practical asymmetrical schemes, which are found in many electronic devices. The theory is applicable for free-electron lasers which are operating in the collective (Raman) regime.

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