Fokker-Planck solution for the spherical symmetry of the electron distribution function of a fully ionized plasma

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The linearized electron-electron collision operator for the velocity-dependent coefficient of index $n=0$ in the Legendre expansion in the pitch angle of the electron distribution function of a fully ionized plasma is restated through a redefinition of the dependent variable as an exact differential. Two specific Coulomb loss terms for particles and energy are brought to evidence in the process of transformation to the differential form which are associated with the conserving properties of the collision operator. The inversion of the differential operator can be achieved by simple quadratures. The method is applied to determine the isotropic component of the electron distribution function in velocity space for a number of problems of interest in fusion research.

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I. INTRODUCTION

Among the methods that are available to solve the steady-state Fokker-Planck equation for the distribution function of the particles in a plasma the one of expansion in Legendre polynomials appears to have some definite advantages. For most of the problems of physical interest the plasma is immersed in a magnetic field about which it is azimuthally symmetric, it being enough to consider the local dependence of the distribution function on two coordinates in velocity space, namely, the speed of the particles and the angle between the velocity vector and the direction of the magnetic field, usually called the pitch angle. If the forces driving the plasma out of thermodynamical equilibrium are weak, two approximations can also be made with considerable simplification to the original problem. First, the ion distribution can be assumed to be known, and described by a function of the impulsive type as the limit of an infinitely narrow Maxwellian. Second, the electron distribution function f_e can be taken to be a Maxwellian F_{Me} plus a small deviation $\hat{f}_e$, which can be treated as a perturbation. The consequence of this latter assumption is twofold. On the one hand, it permits us to linearize the electron collision integral $C_{ee}(f_e,f_e)$, in the sense that the self-interaction of the perturbation $C_{ee}(\hat{f}_e,\hat{f}_e)$ is neglected and only the reciprocal effects of the perturbation and the electron background distribution function, $C_{ee}(\hat{f}_e,F_{Me})$ and $C_{ee}(F_{Me},\hat{f}_e)$, are taken into account. On the other hand, the driving term for collisions, which is in general itself dependent on the form of the distribution function, can be approximated by its zeroth-order approximation, and becomes a known function $S(\mathbf{v})$ in velocity space. These are schematically the terms of statement of the problem which we shall take as a reference frame in this paper.

The expansion of the electron distribution function in a series of Legendre polynomials of the pitch angle,

$$f_e(\mathbf{v})=F_{Me}(v)+\sum_{n=0}^{\infty}\hat{f}_{e_n}(v)P_n(\cos\theta) , \tag{1.1}$$

reduces the problem to one of only one variable, the speed. The advantage of the use of the Legendre polynomials as base functions lies in that they are eigenfunctions of the linearized collision operator and thus that the expansion brings the original equation to unfolding into an infinite set of uncoupled equations for the speed-dependent coefficients $\hat{f}_{e_n}(v)$. In other words, this is precisely the method by which the *order* of the problem, which is unusually high in physics problems, can be brought to appear in its strict minimum. Furthermore, and as a consequence, boundary conditions which concern the conservation properties of the collision operator and the related macroscopic physical contents of the solution are to be applied to only the two first coefficients $\hat{f}_{e_0}(v)$ and $\hat{f}_{e_1}(v)$; the remaining coefficients of the expansion are devoid of globally determined meaning and for them the boundary conditions take the more usual form of analytical restrictions.

The equations that govern the coefficients $\hat{f}_{e_n}(v)$ are of the integro-differential type. By the simple device of taking as the dependent variable one of the integrals over $\hat{f}_{e_n}(v)$ already present in the equation rather than the unknown function itself [1], they can all be restated as purely differential equations. In regard to their mathematical structure, they can be classed into three groups.

(1) The equation for the coefficient $\hat{f}_{e_0}(v)$. From a practical point of view, a third-order differential equation has to be solved, but the general solution for $\hat{f}_{e_0}(v)$ is constructed from four independent solutions, one of them being a constant corresponding to the trivial solution of the differential equation. Of the four arbitrary multiplying constants, two are fixed by boundary conditions which are to be applied along with the process of derivation of the solution and which represent constraints that must by necessity be obeyed if a solution to the perturbation problem is to be found at all; the other two, however, remain mathematically undetermined as the equation (either in the differential or in the original integro-differential form, but anyway containing the specific driv-

ing term for collisions) together with its boundary conditions keeps on being satisfied for any values they may take. Physically, values can be assigned to these free constants by imposing global boundary conditions on the solution (not the equation) which are specifications of the number of particles and the energy associated with $\hat{f}_{e_0}(v)$.

The Coulomb collision integral conserves particles and energy, that is, it is annihilated by the operations of taking the zeroth- and second-order moments in velocity space, whatever be the function f to which the collision operator is applied:

$$\int C_{ee}(f,f)d\mathbf{v}=0 \ , \tag{1.2}$$

$$\int C_{ee}(f,f)\tfrac{1}{2}m_e v^2 d\mathbf{v}=0 \ . \tag{1.3}$$

This means that a precondition for the existence of a solution is that these same two moments of the non-Coulomb terms present in the kinetic equation must also vanish. It turns out, however, that very often there is interest in finding the response of the plasma to a driving force which plays the role of an energy source; in the greatest generality we should consider the possibility that it would also be a particle source. The remedy, in these cases, is to add one or more *ad hoc* terms to balance the particle and energy input from the given external source in order to guarantee the existence of a solution.

The differential formulation which we shall develop in this paper sheds light on the question of the conserving properties of the collision operator. Two specific loss terms for particles and energy, which do not depend on the shape of the solution, appear naturally as belonging to the structure of the Coulomb collision operator and come multiplied by two arbitrary constants which can be adjusted in the course of the derivation of the solution to meet the proper balance with the source. Given their origin, we shall refer to them as the Coulomb loss terms. One of the two plays the role of a drain for both particles and energy, but the other is just an energy sink, which shows itself to be the very same steady-state time rate of change of a Maxwellian with a variable temperature of frequent use [2]. The problem of conservation and balance should then be better looked at as one of boundary conditions, and two new arbitrary constants must be added to the original set which, so enlarged, comprises now a total of six.

If the compensating terms for particles and energy are taken differently from the Coulomb ones, the arbitrary constants with which these come associated must be chosen to be zero. There is, however, a practical advantage in the use of terms of the Coulomb form in that they do not demand a particular solution of their own, and thus do not obscure the effects of an external source which one intends to study; their only role is to specify the time rate at which particles and energy must be removed from the system and make sure that a steady-state solution does exist.

(2) The equation for the coefficient $\hat{f}_1(v)$. This is essentially the Spitzer-Härm problem. It is a fourth-order differential equation, one which admits two simple integrals and that can thus be reduced to a second-order differential equation [1]. The global physical condition to be preserved is the momentum conservation of the electrons upon mutual collisions in the steady state. A loss mechanism is provided by the collisions of the electrons with the ions, which can be made to occur at the same rate as the momentum input from the external source by adjusting a normalization constant appearing in the differential equation. The total momentum and the energy current carried by the electrons then stem from the solution.

(3) The equations for $\hat{f}_{e_n}(v)$, $n=2,3,4,\ldots$. These can be shown to be differential equations of the sixth order. They all have in common the structure of their singularities and thus represent from a mathematical viewpoint one and the same problem. No global constraint comes about from physical considerations and the boundary conditions appear as local restrictions on continuity and finiteness.

The main purpose of this paper is to solve the equation of the first group, namely, the equation for $\hat{f}_{e_0}(v)$.

II. EXPRESSION OF THE COLLISION TERM IN TERMS OF EXACT DIFFERENTIALS

It is convenient to write $\hat{f}_{e_0}(v)$ as the product of a function $a(v)$, which we shall refer to as the factorized distribution function, and a Maxwellian: $\hat{f}_{e_0}(v)=a(v)F_{Me}(v)$. The linearized electron-electron collision operator can then be written

$$C_{ee}^{(0)}(f_e,f_e)=C_{ee}^{(0)}(F_{Me},\hat{f}_{e_0})+C_{ee}^{(0)}(\hat{f}_{e_0},F_{Me}) \ , \tag{2.1}$$

where

$$\frac{1}{\Gamma_{ee}}C_{ee}^{(0)}(F_{Me},\hat{f}_{e_0})=\frac{n_eF_{Me}(x)}{v_{Te}^3}\left\{\frac{\Lambda'(x)}{x^2}a(x)+\frac{4}{3x}J_4(x)-\frac{2}{x}J_2(x)+\left[\frac{4}{3}x^2-2\right][J_1(\infty)-J_1(x)]\right\} , \tag{2.2}$$

$$\frac{1}{\Gamma_{ee}}C_{ee}^{(0)}(\hat{f}_{e_0},F_{Me})=\frac{n_eF_{Me}(x)}{v_{Te}^3}\left\{\frac{\Lambda(x)}{2x^3}a''(x)+\frac{1}{2x^3}\left[\Lambda'(x)-\left[2x+\frac{1}{x}\right]\Lambda(x)\right]a'(x)\right\} . \tag{2.3}$$

In these expressions, the free variable is $x = v/v_{Te}$, where v is the electron speed and v_{Te} is the electron thermal velocity, defined by $v_{Te} = (2T_e/m_e)^{1/2}$, T_e being the electron temperature and m_e the electron mass; $F_{Me}(x)$ is the Maxwellian distribution:

$$F_{Me}(x) = \frac{n_e}{\pi^{3/2} v_{Te}^3} e^{-x^2} , \tag{2.4}$$

n_e being the electron density; the functions $J_1(x)$, $J_2(x)$, and $J_4(x)$ are defined by

$$J_n(x) = \frac{4}{\sqrt{\pi}} \int_0^x a(y) y^n e^{-y^2} dy \quad (n = 1,2,4) , \tag{2.5}$$

$\Lambda(x)$ is the function

$$\Lambda(x) = \frac{4}{\sqrt{\pi}} \int_0^x y^2 e^{-y^2} dy , \tag{2.6}$$

and, writing e for the electron charge, Γ_{ee} is the quantity defined by $\Gamma_{ee} = 4\pi(\ln\lambda)e^4/m_e^2$, which we shall treat, in accordance with the common practice, as a constant, ignoring the (weak) dependence of the Coulomb logarithm $\ln\lambda$ on the velocity [3].

The function $\Lambda(x)$, which connects to the error function $\phi(x) = (2/\sqrt{\pi}) \int_0^x \exp(-y^2) dy$ through

$$\Lambda(x) = \phi(x) - x\phi'(x) , \tag{2.6'}$$

where the prime represents the derivative with respect to the argument, plays an important role in the theory of Coulomb collisions in a plasma. It is an instance of the functions defined by the integrals:

$$\Lambda_n(x) = \frac{4}{\sqrt{\pi}} \int_0^x y^n e^{-y^2} dy , \tag{2.7}$$

which will appear frequently in the subsequent analysis. Properties of the function $\Lambda(x)$ and of the functions $\Lambda_n(x)$ are given in Appendixes A and B, respectively.

We shall take Eq. (2.1) as representing the full electron collision term, thus neglecting the contribution coming from the interaction of the electrons with the ions. This can be justified as follows. If the plasma is not far from thermodynamical equilibrium, the width of the ion distribution is typically of the order $\alpha = (m_e/m_i)^{1/2}$, m_i being the ion mass, times smaller than the spread of the electron distribution, and just a very small fraction of the electrons with normalized velocities in the range $|\Delta x| \propto \alpha$ would suffer the effect of the details of the structure of the ion distribution. These few are the only electrons that could exchange energy with the ions and may be ignored without introducing appreciable error into the results: the ion distribution function can thus be represented as an infinitely thin, isotropic Maxwellian, like a Dirac δ function in velocity space. While giving a term of electron-ion collisions of the same order of magnitude as the term of electron-electron collisions in the equations $n = 1,2,3,\ldots$, this approximation suppresses the ions altogether from the kinetic equation for the electron distribution function of index $n = 0$. From the point of view of energy balance no real damage results from this, since the processes between electrons and ions, in any case, would not be able to drain out the energy from the electron population at the rate which is introduced by external sources.

We shall assume that the unknown function $a(x)$ remains finite at the origin and that it grows less rapidly than e^{x^2} as x goes to infinity, so that moments of all orders can be obtained from the distribution function. Since the Maxwellian distribution must annihilate the electron-electron collision term, and thus $a(x) = 1$ must be a solution, in seeking to lower the order of the equation we reexpress all the integrals $J_n(x)$ in terms of $a'(x)$. This can be achieved by recourse to partial integration, most simply by first noting that the integrals can be written as

$$J_n(x) = \int_0^x a(y) \Lambda_n'(y) ay , \tag{2.5'}$$

and then using the expressions for the $\Lambda_n(x)$'s given in Appendix B. We obtain

$$J_1(x) = \frac{2}{\sqrt{\pi}} a(0) - \frac{2}{\sqrt{\pi}} a(x) e^{-x^2} + \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} a'(y) dy , \tag{2.8}$$

$$J_2(x) = a(x)\Lambda(x) - \int_0^x \Lambda(y) a'(y) dy , \tag{2.9}$$

$$J_4(x) = \tfrac{3}{2} a(x)\Lambda(x) - \tfrac{1}{2} a(x) x \Lambda'(x) - \tfrac{3}{2} \int_0^x \Lambda(y) a'(y) dy + \tfrac{1}{2} \int_0^x y \Lambda'(y) a'(y) dy . \tag{2.10}$$

By substituting these expressions into Eq. (2.2), the terms in $a(0)$ and $a(x)$ cancel out, and we are left with

$$\frac{\pi^{3/2} v_{Te}^3}{n_0 \nu_0} x^2 C_{ee}^{(0)}(F_{Me}, \hat{f}_{e_0}) = \left[\frac{2}{3} x^2 - 1 \right] \Lambda'(x) \left[\int_0^\infty e^{-y^2} a'(y) dy - \int_0^x e^{-y^2} a'(y) dy \right] + \frac{2}{3} \frac{\Lambda'(x)}{x} \int_0^x y^3 e^{-y^2} a'(y) dy , \tag{2.11}$$

where we have introduced the electron collision frequency

$$\nu_0 = \frac{n_e \Gamma_{ee}}{v_{Te}^3} . \tag{2.12}$$

Upon multiplication by x^2, Eq. (2.3) can be brought to the form of an exact differential:

$$\frac{\pi^{3/2}v_{Te}^3}{n_e\nu_0}x^2C_{ee}^{(0)}(\hat{f}_{e_0},F_{Me})=\frac{d}{dx}\left[\frac{\Lambda(x)}{2x}e^{-x^2}a'(x)\right] . \tag{2.13}$$

We thus obtain for the total collision term an integro-differential expression in which only $a'(x)$ appears:

$$\frac{\pi^{3/2}v_{Te}^3}{n_e\nu_0}x^2C_{ee}^{(0)}(f_e,f_e)=\frac{d}{dx}\left[\frac{\Lambda(x)}{2x}e^{-x^2}a'(x)\right]+\left[\frac{2}{3}x^2-1\right]\Lambda'(x)\left[\int_0^\infty e^{-y^2}a'(y)dy-\int_0^x e^{-y^2}a'(y)dy\right]$$
$$+\frac{2}{3}\frac{\Lambda'(x)}{x}\int_0^x y^3e^{-y^2}a'(y)dy . \tag{2.14}$$

We proceed by introducing the quantity

$$K(x)=\frac{4}{\sqrt{\pi}}\int_0^x e^{-y^2}a'(y)dy+C_1 , \tag{2.15}$$

where C_1 is an arbitrary constant, the derivative of which is the function

$$K'(x)=\frac{4}{\sqrt{\pi}}e^{-x^2}a'(x) . \tag{2.16}$$

With the replacement

$$\int_0^x y^3e^{-y^2}a'(y)dy=\frac{\sqrt{\pi}}{4}x^3K(x)-\frac{3\sqrt{\pi}}{4}\int_0^x y^2K(y)dy \tag{2.17}$$

for the last integral term in Eq. (2.14), the expression for the collision operator takes the form

$$\frac{4\pi v_{Te}^3}{n_e\nu_0}x^2C_{ee}^{(0)}(f_e,f_e)=\frac{d}{dx}\left[\frac{\Lambda(x)}{2x}K'(x)\right]+\Lambda'(x)K(x)$$
$$-2\frac{\Lambda'(x)}{x}\int_0^x y^2K(y)dy$$
$$+K(\infty)\left[\frac{2}{3}x^2-1\right]\Lambda'(x) . \tag{2.18}$$

We next define the new dependent variable

$$F(x)=\int_0^x y^2K(y)dy+C , \tag{2.19}$$

where C is another arbitrary constant, the derivative of which is related to $K(x)$ through

$$K(x)=\frac{F'(x)}{x^2} . \tag{2.20}$$

The second and third terms on the right-hand side of Eq. (2.18) can be combined into a single differential term:

$$\frac{\Lambda'(x)F'(x)}{x^2}-2\frac{\Lambda'(x)F(x)}{x}=\frac{d}{dx}\left[\frac{\Lambda'(x)}{x^2}F(x)\right] , \tag{2.21}$$

and we rewrite Eq. (2.18) as

$$\frac{8\pi v_{Te}^3}{n_e\nu_0}C_{ee}^{(0)}(f_e,f_e)=\frac{1}{x^2}\frac{dU}{dx}+\frac{16}{\sqrt{\pi}}C\frac{e^{-x^2}}{x}$$
$$+\frac{8}{\sqrt{\pi}}K(\infty)\left[\frac{2}{3}x^2-1\right]e^{-x^2} , \tag{2.22}$$

where

$$U(x)=\frac{\Lambda(x)}{x}K'(x)+2\frac{\Lambda'(x)}{x^2}F(x) . \tag{2.23}$$

With the help of the properties of the function $\Lambda(x)$ given in Appendix A and from the connection between $K(x)$ and $F(x)$ stated in Eq. (2.20) we may show that $U(x)$ can be written as

$$U(x)=\frac{1}{x}\frac{dW}{dx} , \tag{2.24}$$

where

$$W(x)=\Lambda(x)K(x)-\frac{\Lambda'(x)}{x^2}F(x) . \tag{2.25}$$

Similarly we find that $W(x)$ in its turn can also be expressed in terms of an exact differential:

$$W(x)=\frac{\Lambda^2(x)}{x^2}\frac{d}{dx}\left[\frac{F(x)}{\Lambda(x)}\right] . \tag{2.26}$$

This brings the expression for the collision term to the following final compact form:

$$\frac{8\pi v_{Te}^3}{n_e\nu_0}C_{ee}^{(0)}(f,f)=\frac{1}{x^2}\frac{d}{dx}\left\{\frac{1}{x}\frac{d}{dx}\left[\frac{\Lambda^2(x)}{x^2}\frac{d}{dx}\left[\frac{F(x)}{\Lambda(x)}\right]\right]\right\}$$
$$+\frac{16}{\sqrt{\pi}}C\frac{e^{-x^2}}{x}$$
$$+\frac{8}{\sqrt{\pi}}K(\infty)\left[\frac{2}{3}x^2-1\right]e^{-x^2} . \tag{2.27}$$

We note first that, in addition to a differential operator, Eq. (2.27) contains two algebraic terms, not depending on the unknown function, which come associated with two constants $K(\infty)$ and $C\equiv F(0)$. The constant $C_1\equiv K(0)$ introduced at the step of defining $K(x)$ does not contribute with any term to the transformed operator and is indeed superfluous, the boundary condition on $K(x)$ being already implied by $K(\infty)$. These two terms should be considered as spontaneous balancing terms for particles and energy which are built into the structure of the original collisional integral and brought to evidence by the process of reduction to the differential form. Because of their origin we shall refer to them as the Coulomb loss terms. If the overall external driving term (which may or may not include loss mechanisms) does not introduce particles and energy into the system, they do not have any role to play, and the constants $K(\infty)$ and C must be chosen to be zero; if, however, the driving term acts as a source or sink of any or both particles and energy, by properly choosing the multiplying constants they can be made to balance the input and save the conserving properties of the collision operator.

It is useful to give a physical meaning to these spontaneous sources. The collisional effects of an isotropic, monoenergetic electron beam of speed v_b interacting with the plasma electrons can be shown to be described by a source $S(v)$ of the form

$$\frac{\pi^{3/2}v_{Te}^3}{n_b\nu_0}S(v)=\frac{\delta(x-\xi)}{x^2}e^{-x^2}+\begin{cases}\dfrac{2}{\xi}\left[\dfrac{2}{3}x^2-1\right]e^{-x^2} & (x<\xi)\\ 2\left[\dfrac{2}{3}\xi^2-1\right]\dfrac{e^{-x^2}}{x} & (x>\xi)\,,\end{cases}\tag{2.28}$$

where n_b is the space particle density in the beam and $\xi=v_b/v_{Te}$. Except for the Dirac δ function, the effect of which is to guarantee that particles are not created by collisions, this source reproduces the form of the Coulombic loss terms we have just identified in the differential collision operator. In particular, in the limit of zero speed electron beam we have

$$\frac{\pi^{3/2}v_{Te}^3}{n_b\nu_0}S(v)=\frac{\delta(x)}{x^2}-2\frac{e^{-x^2}}{x}\quad(0\le x<\infty)\,,\tag{2.28a}$$

and in the limit $v_b \gg v_{Te}$:

$$\frac{\pi^{3/2}v_{Te}^3}{n_b\nu_0}S(v)=\frac{2}{\xi}\left[\frac{2}{3}x^2-1\right]e^{-x^2}\quad(0\le x<\infty)\,.\tag{2.28b}$$

The term of the form e^{-x^2}/x in the collision operator thus describes how electrons of zero speed are created by collisions in the plasma; and since energy is continually redistributed among the remaining, fewer particles, it is accompanied by some net enhancement in the energy density. That is, it represents a source for both particles and energy. On the other hand, the term $-(\frac{2}{3}x^2-1)e^{-x^2}$ corresponds just to a removal of energy by the suprathermal particles, the number of which is vanishingly small.

This latter term is frequently used to counterbalance the energy input from a source which does not introduce particles into the system ([2,4,5] among others). It can also be interpreted as the time rate of change of the zeroth-order electron distribution function caused by a steady-state removal of its energy:

$$-\frac{\partial F_{Me}}{\partial t}=-\frac{\partial F_{Me}}{\partial T_e}\frac{dT_e}{dt}=-\frac{3}{2}\frac{1}{\tau_E}\left[\frac{2}{3}x^2-1\right]F_{Me}\,,\tag{2.29}$$

where $\tau_E=T_e(dT_e/dt)^{-1}$ is the characteristic energy loss time.

Since the form of the Coulomb loss terms does not depend on the unknown distribution function, they may be treated on the same footing as the external sources, and we write for the true collision operator, instead of Eqs. (2.22) and (2.27), the purely differential expressions

$$\frac{8\pi v_{Te}^3}{n_e\nu_0}C_{ee}^{(0)}(f_e,f_e)=\frac{1}{x^2}\frac{dU}{dx}\,,\tag{2.22$'$}$$

$$\frac{8\pi v_{Te}^3}{n_e\nu_0}C_{ee}^{(0)}(f_e,f_e)=\frac{1}{x^2}\frac{d}{dx}\left\{\frac{1}{x}\frac{d}{dx}\left[\frac{\Lambda^2(x)}{x^2}\frac{d}{dx}\left[\frac{F(x)}{\Lambda(x)}\right]\right]\right\}.\tag{2.27$'$}$$

Successive integrations of this "telescope" third-order differential operator will uncover $F(x)$ for any given driving term and introduce three arbitrary constants; a chain of operations to follow will lead first to $K(x)$ through Eq. (2.20) and then to $a'(x)$ through Eq. (2.16); finally a last-step integration will yield the factorized distribution function $a(x)$ and introduces a fourth arbitrary constant.

It remains to discuss the boundary conditions to be applied to the collision operator. Conservation of the total number of particles implies that

$$\int C_{ee}^{(0)}(f_e,f_e)d\mathbf{v}=4\pi v_{Te}^3\int_0^\infty C_{ee}^{(0)}(f_e,f_e)x^2dx=0\,,\tag{2.30}$$

which, by Eq. (2.22′), is the same as

$$U(x)\Big|_0^\infty=0\,.\tag{2.31}$$

Similarly, the condition that the total energy is conserved upon mutual collisions of the electrons translates as

$$\int C_{ee}^{(0)}(f_e,f_e)\tfrac{1}{2}m_ev^2d\mathbf{v}=2\pi m_ev_{Te}^5\int_0^\infty C_{ee}^{(0)}(f_e,f_e)x^4dx=0\,,\tag{2.32}$$

which, using again Eq. (2.22′), gives by partial integration

$$\int_0^\infty\frac{dU}{dx}x^2dx=x^2U(x)\Big|_0^\infty-2\int_0^\infty xU(x)dx=0\,.\tag{2.33}$$

By substituting $U(x)$ as given by Eq. (2.24) into the integral on the right-hand side, we get

$$x^2U(x)-2W(x)\Big|_0^\infty=0\ . \qquad (2.34)$$

These conditions must be compatible with the balance of particles and energy in the kinetic equation. Writing this symbolically as

$$L(x)+S(x)=C_{ee}^{(0)}(f_e,f_e)\ , \qquad (2.35)$$

where $S(x)$ represents the given external source and $L(x)$ the term of losses, and taking moments of the zeroth and second order, we have

$$\int_0^\infty L(x)x^2dx+\int_0^\infty S(x)x^2dx=\int_0^\infty C_{ee}^{(0)}(f_e,f_e)x^2dx=0\ , \qquad (2.36)$$

$$\int_0^\infty L(x)x^4dx+\int_0^\infty S(x)x^4dx=\int_0^\infty C_{ee}^{(0)}(f_e,f_e)x^4dx=0\ . \qquad (2.37)$$

These constraints fix the rates of removal of particles and energy by $L(x)$, for example by specifying two parameters τ_N and τ_E with the meaning of characteristic loss times; if the Coulombic loss terms are employed, the characteristic times can be simply related to the constants C and $K(\infty)$ by which they come multiplied in Eq. (2.27). Since Eqs. (2.36) and (2.37) do not require a knowledge of the solution, they are easier to apply than Eqs. (2.31) and (2.34). These latter ones, however, retain their usefulness, helping to simplify the derivation of the solution as will be shown subsequently (Sec. IV). We also postpone to the applications the discussion on the global boundary conditions to be applied solely to the solution.

III. THE SOLUTIONS FOR THE COLLISION OPERATOR

A. The general solution

We solve first the homogeneous equation

$$\frac{8\pi v_{Te}^3}{n_e\nu_0}C_{ee}^{(0)}(f_e,f_e)=\frac{1}{x^2}\frac{d}{dx}\left\{\frac{1}{x}\frac{d}{dx}\left[\frac{\Lambda^2(x)}{x^2}\frac{d}{dx}\left[\frac{F(x)}{\Lambda(x)}\right]\right]\right\}=0\ . \qquad (3.1)$$

By repeated integrations we obtain

$$F(x)=A_1\Lambda(x)\int_\xi^x\frac{y^4dy}{\Lambda^2(y)}+A_2\Lambda(x)\int_\xi^x\frac{y^2dy}{\Lambda^2(y)}+A_3\Lambda(x)\ , \qquad (3.2)$$

where ξ is some reference value of the variable and A_1,A_2,A_3 are arbitrary constants. From this we may construct the following four basic independent solutions for $F(x)$:

$$\begin{aligned}&F_1(x)=0\ ,\\&F_2(x)=-\Lambda(x)\ ,\\&F_3(x)=\Lambda(x)\left[\frac{4}{\sqrt{\pi}}\int_\xi^x\frac{y^4dy}{\Lambda^2(y)}-\frac{4}{\sqrt{\pi}}\int_\xi^x\frac{y^2dy}{\Lambda^2(y)}\right],\\&F_4(x)=\Lambda(x)\frac{4}{\sqrt{\pi}}\int_\xi^x\frac{y^2dy}{\Lambda^2(y)}\ .\end{aligned} \qquad (3.3)$$

We have explicitly included the trivial solution in the set since there will be associated with it a nontrivial solution for the distribution function. The solution labeled $F_3(x)$ is a combination of the two solutions that come multiplied by A_1 and A_2 in Eq. (3.2). The reason we prefer to deal with $F_3(x)$ rather than with the single-integral function $(4/\sqrt{\pi})\int_\xi^x y^4dy/\Lambda^2(y)$ is that the factorized distribution function generated by this latter solution is singular both at the origin and at infinity, while the one generated by $F_3(x)$ exhibits just the singularity at infinity, the other being suppressed by a similar singular behavior at the origin coming from the added $F_4(x)$.

The functions $K(x)$ can be evaluated according to Eq. (2.20) and are

$$\begin{aligned}&K_1(x)=0\ ,\\&K_2(x)=-\frac{4}{\sqrt{\pi}}e^{-x^2}\ ,\\&K_3(x)=\frac{4}{\sqrt{\pi}}e^{-x^2}\left[\frac{4}{\sqrt{\pi}}\int_\xi^x\frac{y^4dy}{\Lambda^2(y)}-\frac{4}{\sqrt{\pi}}\int_\xi^x\frac{y^2dy}{\Lambda^2(y)}\right]+\frac{4}{\sqrt{\pi}}\frac{x^2-1}{\Lambda(x)}\ ,\\&K_4(x)=\frac{16}{\pi}e^{-x^2}\int_\xi^x\frac{y^2dy}{\Lambda^2(y)}+\frac{4}{\sqrt{\pi}}\frac{1}{\Lambda(x)}\ .\end{aligned} \qquad (3.4)$$

By use of Eq. (2.16) the $a'(x)$'s are obtained as

$$\begin{aligned}&a_1'(x)=0\ ,\\&a_2'(x)=2x\ ,\\&a_3'(x)=-2x\left[\frac{4}{\sqrt{\pi}}\int_\xi^x\frac{y^4dy}{\Lambda^2(y)}-\frac{4}{\sqrt{\pi}}\int_\xi^x\frac{y^2dy}{\Lambda^2(y)}\right]+\frac{2xe^{x^2}}{\Lambda(x)}\ ,\\&a_4'(x)=-\frac{8}{\sqrt{\pi}}x\int_\xi^x\frac{y^2dy}{\Lambda^2(y)}\ ,\end{aligned} \qquad (3.5)$$

from which the four independent solutions for the factorized distribution function follow by integration:

$$
\begin{aligned}
a_1(x)&=1\ ,\\
a_2(x)&=x^2\ ,\\
a_3(x)&=\frac{4}{\sqrt{\pi}}\int_\xi^x \frac{y^6dy}{\Lambda^2(y)}\\
&\quad-(1+x^2)\left[\frac{4}{\sqrt{\pi}}\int_\xi^x\frac{y^4dy}{\Lambda^2(y)}-\frac{4}{\sqrt{\pi}}\int_\xi^x\frac{y^2dy}{\Lambda^2(y)}\right]\\
&\quad+\frac{e^{x^2}}{\Lambda(x)}-\frac{e^{\xi^2}}{\Lambda(\xi)}\ ,\\
a_4(x)&=\frac{4}{\sqrt{\pi}}\int_\xi^x\frac{y^4dy}{\Lambda^2(y)}-\frac{4}{\sqrt{\pi}}x^2\int_\xi^x\frac{y^2dy}{\Lambda^2(y)}\ .
\end{aligned}
\tag{3.6}
$$

In the limit $x\to 0$, we find that

$$a_3(x)=\frac{3\sqrt{\pi}}{2}x+O(x^3)\ , \tag{3.7}$$

$$a_4(x)=-\frac{3\sqrt{\pi}}{2}\frac{1}{x}+O(x)\ , \tag{3.8}$$

except for constants, and in the limit $x\to\infty$, that

$$a_3(x)\sim e^{x^2}+O(x^7)\ , \tag{3.9}$$

$$a_4(x)\sim-\frac{8}{15\sqrt{\pi}}x^5+O(e^{-x^2})\ . \tag{3.10}$$

The function represented by $a_3(x)$ is thus finite at the origin but for large values of the variable grows faster than allowed by the physical requirements on the solution, while $a_4(x)$, in contradistinction, remains exponentially bounded at infinity but diverges at the origin.

It is convenient to handle expressions for the solutions in which the singularities, both at the origin and at infinity, appear neatly isolated from the integrals. This can be accomplished with the help of the functions

$$f_2(\xi;x)=\frac{4}{\sqrt{\pi}}\int_\xi^x y^2\left[\frac{1}{\Lambda^2(y)}-\frac{9\pi}{16}\frac{1}{y^6}-\frac{27\pi}{40}\frac{1}{y^4}-1\right]dy\ , \tag{3.11}$$

$$f_4(\xi;x)=\frac{4}{\sqrt{\pi}}\int_\xi^x y^4\left[\frac{1}{\Lambda^2(y)}-\frac{9\pi}{16}\frac{1}{y^6}-1\right]dy\ , \tag{3.12}$$

$$f_6(\xi;x)=\frac{4}{\sqrt{\pi}}\int_\xi^x y^6\left[\frac{1}{\Lambda^2(y)}-1\right]dy\ , \tag{3.13}$$

which remain finite in the whole domain of the variables ξ and x, and in terms of which the singular solutions for the factorized distribution function can be written as

$$
\begin{aligned}
a_3(\xi;x)&=f_6(\xi;x)+(1+x^2)[f_2(\xi;x)-f_4(\xi;x)]+\frac{e^{x^2}}{\Lambda(x)}-\frac{e^{\xi^2}}{\Lambda(\xi)}+\frac{4}{7\sqrt{\pi}}(x^7-\xi^7)\\
&\quad+(1+x^2)\left[-\frac{4}{5\sqrt{\pi}}(x^5-\xi^5)+\frac{4}{3\sqrt{\pi}}(x^3-\xi^3)-\frac{9\sqrt{\pi}}{20}\left[\frac{1}{x}-\frac{1}{\xi}\right]-\frac{3\sqrt{\pi}}{4}\left[\frac{1}{x^3}-\frac{1}{\xi^3}\right]\right]\ ,\\
a_4(\xi;x)&=f_4(\xi;x)-x^2f_2(\xi;x)+\frac{4}{5\sqrt{\pi}}(x^5-\xi^5)+x^2\left[\frac{3\sqrt{\pi}}{4}\left[\frac{1}{x^3}-\frac{1}{\xi^3}\right]-\frac{4}{3\sqrt{\pi}}(x^3-\xi^3)\right]\\
&\quad+\frac{9\sqrt{\pi}}{2}\left[\frac{3}{5}x^2-\frac{1}{2}\right]\left[\frac{1}{x}-\frac{1}{\xi}\right]\ .
\end{aligned}
\tag{3.6'}
$$

Expansions and tables for $f_2(0;x)$, $f_4(0;x)$, and $f_6(0;x)$ are provided in Appendix C.

We calculate now the number of particles and the energy carried by the solutions we have just found. The fractional number of particles and the fractional energy in the electron distribution function are defined, respectively, by

$$\frac{\Delta n}{n_e}=\frac{1}{n_e}\int\hat{f}_e(\mathbf{v})d\mathbf{v}=\frac{4}{\sqrt{\pi}}\int_0^\infty a(x)x^2e^{-x^2}dx\ , \tag{3.14}$$

$$
\begin{aligned}
\frac{\Delta E}{n_eT_e}&=\frac{1}{n_eT_e}\int\frac{1}{2}m_ev^2\hat{f}_e(\mathbf{v})d\mathbf{v}\\
&=\frac{4}{\sqrt{\pi}}\int_0^\infty a(x)x^4e^{-x^2}dx\ .
\end{aligned}
\tag{3.15}
$$

In carrying out the integrations over the solutions $a_3(\xi;x)$ and $a_4(\xi;x)$ we respectively replace the upper and the lower limit by ξ, anticipating the domains of the variable x in which they will contribute to the solution of a physical problem. The results for the four independent solutions then are

$$
\begin{aligned}
&\frac{\Delta n_1}{n_e}=1,\quad \frac{\Delta E_1}{n_eT_e}=\frac{3}{2}\ ,\\
&\frac{\Delta n_2}{n_e}=\frac{3}{2},\quad \frac{\Delta E_2}{n_eT_e}=\frac{15}{4}\ ,\\
&\frac{\Delta n_3}{n_e}=\frac{5}{2}-e^{\xi^2}+\frac{2}{\sqrt{\pi}}(\xi^2-1)\frac{\xi^3}{\Lambda(\xi)}-\frac{4}{7\sqrt{\pi}}\xi^7-g_6(0;\xi)\ ,\\
&\frac{\Delta E_3}{n_eT_e}=\frac{21}{4}-\frac{3}{2}e^{\xi^2}+\frac{5}{\sqrt{\pi}}(\xi^2-1)\frac{\xi^3}{\Lambda(\xi)}-\frac{6}{7\sqrt{\pi}}\xi^7\\
&\qquad-\frac{3}{2}g_6(0;\xi)\ ,
\end{aligned}
\tag{3.16}
$$

$$\frac{\Delta n_4}{n_e} = f_4(\xi;\infty) - \frac{3}{2}f_2(\xi;\infty) - g_4(\xi;\infty) - \frac{9\sqrt{\pi}}{8}\frac{1}{\xi^3} - \frac{9\sqrt{\pi}}{5}\frac{1}{\xi} - \frac{2}{\sqrt{\pi}}\xi^3\left[\frac{1}{\Lambda(\xi)}-1\right],$$

$$\frac{\Delta E_4}{n_e T_e} = \frac{3}{2}f_4(\xi;\infty) - \frac{15}{4}f_2(\xi;\infty) - \frac{3}{2}g_4(\xi;\infty) - \frac{45\sqrt{\pi}}{16}\frac{1}{\xi^3} - \frac{27\sqrt{\pi}}{4}\frac{1}{\xi} - \frac{5}{\sqrt{\pi}}\xi^3\left[\frac{1}{\Lambda(\xi)}-1\right],$$

where

$$g_4(\xi;x) = \frac{4}{\sqrt{\pi}}\int_\xi^x y^4\left[\frac{1}{\Lambda(y)}-1\right]dy, \qquad (3.17)$$

$$g_6(\xi;x) = \frac{4}{\sqrt{\pi}}\int_\xi^x y^6\left[\frac{1}{\Lambda(y)}-1\right]dy. \qquad (3.18)$$

The partial integrations that are needed to arrive at these expressions are facilitated by use of the properties of the functions $\Lambda_n(x)$ listed in Appendix B. The small $-x$ expansion and a numerical table for the newly defined function $g_4(0;x)$ can be found in Appendix C.

B. The particular solution

It may be of interest to have a closed form representation of the solution of the Fokker-Planck equation for a general driving term:

$$C_{ee}^{(0)}(f,f) = S(x).$$

We shall assume that $S(x)$ vanishes for $0 \le x < \xi$, that it is no more singular than a Dirac δ function at the point $x=\xi$, and that it vanishes exponentially for large x: $S(x) \sim e^{-x^2}$ for $x \to \infty$. These conditions correspond to those which we are bound to encounter in all situations of physical interest. The point $x=\xi$ may coincide with the origin.

Using the representation of Eq. (2.22′) for the collision term, the equation to be solved then reads

$$\frac{1}{x^2}\frac{dU}{dx} = \frac{8\pi v_{Te}^3}{n_e \nu_0}S(x). \qquad (3.19)$$

The solution is

$$U(x) = \frac{8\pi v_{Te}^3}{n_e \nu_0}N(x), \qquad (3.20)$$

where

$$N(x) = \int_\xi^x y^2 S(y)\,dy \qquad (3.21)$$

is a function which gives a measure of the rate at which electrons with normalized speeds less than and equal to x are introduced by the source into the plasma.

We next find $W(x)$, which, from Eq. (2.24), is determined by

$$\frac{1}{x}\frac{dW}{dx} = \frac{8\pi v_{Te}^3}{n_e \nu_0}N(x). \qquad (3.22)$$

The solution is

$$W(x) = \frac{4\pi v_{Te}^3}{n_e \nu_0}P(x), \qquad (3.23)$$

where

$$P(x) = x^2 N(x) - 2M(x), \qquad (3.24)$$

and $M(x)$, defined by

$$M(x) = \tfrac{1}{2}\int_\xi^x y^4 S(y)\,dy, \qquad (3.25)$$

represents the power absorbed from the source by particles with normalized speeds less than or equal to x. Note that

$$P'(x) = 2xN(x). \qquad (3.26)$$

To find $F(x)$ we solve Eq. (2.26), which is

$$\frac{\Lambda^2(x)}{x^2}\frac{d}{dx}\left[\frac{F(x)}{\Lambda(x)}\right] = \frac{4\pi v_{Te}^2}{n_e \nu_0}P(x) \qquad (3.27)$$

and yields

$$F(x) = \frac{4\pi v_{Te}^3}{n_e \nu_0}\Lambda(x)\int_\xi^x \frac{y^2 P(y)}{\Lambda^2(y)}dy. \qquad (3.28)$$

By following the steps of Eqs. (2.20) and (2.16), the associated factorized distribution function can easily be found to be

$$a(x) = 0 \quad (x<\xi),$$
$$\frac{n_e \nu_0}{4\pi v_{Te}^3}a(x) = \frac{\sqrt{\pi}}{4}\frac{e^{x^2}P(x)}{\Lambda(x)} - \frac{\sqrt{\pi}}{2}\int_\xi^x \frac{y e^{y^2}P(y)}{\Lambda(y)}dy + (1-x^2)\int_\xi^x \frac{y^2 P(y)}{\Lambda^2(y)}dy + \int_\xi^x \frac{y^4 P(y)}{\Lambda^2(y)}dy \quad (x>\xi). \qquad (3.29)$$

From this formal representation of the solution we see that the source function $P(x)$ is to be interpreted as an effective driving force for disturbances to equilibrium; the definition [Eq. (3.24)] gives it as a measure of the (time rate of) excess of energy of the particles introduced into the system by the source within a sphere of normalized radius x in velocity space if they all had the energy of the particles lying on the surface with respect to the true energy content deposited by the source into the internal volume. The fractional number of particles and the fractional energy can be evaluated as before and are given by

$$\frac{n_e \nu_0}{4\pi v_{Te}^3}\frac{\Delta n}{n_e} = -\frac{\sqrt{\pi}}{2}\int_\xi^\infty\left[\frac{1}{\Lambda(x)}-1\right]x e^{x^2}P(x)\,dx + \int_\xi^\infty\left[\frac{1}{\Lambda(x)}-1\right]\frac{x^4 P(x)}{\Lambda(x)}dx - \frac{1}{2}\int_\xi^\infty \frac{x^2 P(x)}{\Lambda^2(x)}dx - \int_\xi^\infty \frac{x^4 N(x)}{\Lambda(x)}dx, \qquad (3.30)$$

$$\frac{n_e \nu_0}{4\pi v_{Te}^3}\frac{\Delta E}{n_e T_e}=\frac{3}{2}\left[\frac{n_e \nu_0}{4\pi v_{Te}^3}\frac{\Delta n}{n_e}\right]-\frac{3}{2}\int_\xi^\infty \frac{x^2 P(x)}{\Lambda^2(x)}dx \ . \tag{3.31}$$

C. The solutions for the "Coulombic balancing terms"

For sources of the form

$$S_N(x)=\frac{16}{\sqrt{\pi}}\frac{e^{-x^2}}{x} \ , \tag{3.32}$$

$$S_E(x)=\frac{8}{\sqrt{\pi}}\left[\frac{2}{3}x^2-1\right]e^{-x^2} \ , \tag{3.33}$$

which appears in Eqs. (2.22) and (2.27), we obtain, respectively,

$$F_N(x)=-1, \quad K_N(x)=0 \ , \tag{3.34}$$

$$F_E(x)=-\tfrac{1}{3}x^3, \quad K_E(x)=-1 \ , \tag{3.35}$$

and, since in both cases $K'(x)=0$, the particular solutions for the factorized distribution function coincide with the constant solution $a_1(x)$ of Sec. III A. It should be noted, however, that neither of the pairs of functions $F(x)$ and $K(x)$ above appears in the set belonging to the homogeneous differential equation, and in particular that $F_N(0)\neq 0$ and $K_E(\infty)\neq 0$. This indicates that their only effect is to give room for boundary conditions on the equation that ensure that the conserving properties of the collision operator can be preserved in any case.

IV. EXAMPLES OF APPLICATION

A. The ion-beam problem

Besides its practical importance, this problem has a didactic interest in that, because of the form of the source term, the solution can be expressed just as a combination of the solutions of the homogeneous Fokker-Planck differential equation. The perturbative treatment of the equation requires that the fast ion density n_b is much lower than the electron density in the bulk plasma; consistent with this, we shall assume that, to first order in n_b/n_e, the collisional interaction between the electrons in the plasma and the injected ions can be approximated [6] by

$$C_{eb}(f_e,f_b)\simeq C_{eb}(F_{Me},f_b) \ . \tag{4.1}$$

Here f_b is the distribution function of the fast ions, which, for a monoenergetic beam, takes the form

$$f_b(\mathbf{v})=\frac{n_b}{2\pi}\frac{\delta(v-v_b)}{v^2}K(\theta) \ , \tag{4.2}$$

where v_b is the speed and $K(\theta)$ is an arbitrary function which describes the angular distribution of the ions in the beam. This leads to the following form for the isotropic term in the expansion of the collision integral in zonal harmonics of the pitch angle:

$$C_{eb}^{(0)}(F_{Me},f_b)=\frac{rZ_b^2K_0\nu_0 n_b}{\pi^{3/2}v_{Te}^3}S(x) \ , \tag{4.3}$$

where

$$S(x)=\begin{cases}\dfrac{2}{\xi}\left[\dfrac{2}{3}x^2-1\right]e^{-x^2} & (0\le x<\xi)\\[2mm] \dfrac{4}{3}\xi^2\dfrac{e^{-x^2}}{x} & (\xi<x<\infty)\ ,\end{cases} \tag{4.4}$$

$\xi=v_b/v_{Te}$, Z_b is the ionic charge in the beam, K_0 is the space average of $K(\theta)$, r is the ratio of Coulomb logarithms: $r=\ln\lambda_{eb}/\ln\lambda_{ee}$, and a term of the order of the ratio of the electron mass to the fast ion mass has been neglected.

Coulomb collisions only being considered, we may abstain from including a source term for the particle density in the Fokker-Planck equation; energy, however, is effectively transferred from the injected ions to the electrons in the bulk plasma by collisional interaction, and balance requires an energy loss term to be included in the equation, which we shall take to be of the form of Eq. (3.33). The steady-state Fokker-Planck equation then reads

$$C'S_E(x)=C_{ee}^{(0)}(f_e,f_e)+C_{eb}^{(0)}(F_{Me},f_b) \ , \tag{4.5}$$

where C' is some constant, or, using the representation of Eq. (2.22′) for the collision operator:

$$\frac{1}{x^2}\frac{dU}{dx}=-\frac{8}{\sqrt{\pi}}rZ_b^2K_0\left[\frac{n_b}{n_e}\right]S(x)+\frac{8}{\sqrt{\pi}}C\left[\frac{2}{3}x^2-1\right]e^{-x^2} \ . \tag{4.6}$$

The new constant $C=(8\pi v_{Te}^3/n_e\nu_0)C'$ may be determined from the requirement that energy is balanced and is

$$C=2rZ_b^2K_0\left[\frac{n_b}{n_e}\right]\frac{\Lambda(\xi)}{\xi} \ . \tag{4.7}$$

Now, the driving terms for collisions in Eq. (4.6) are exactly of the form of the Coulombic sources discussed in Sec. III, and therefore the solution for the factorized distribution function can be written down immediately as a combination of the solutions of the homogeneous equation. Dividing the domain of the independent variable into two regions I and II separated by $x=\xi$, we have

$$\begin{aligned}a_{\rm I}(x)&=A_1+A_2x^2+A_3a_3(\xi;x) \quad (0\le x<\xi)\ ,\\ a_{\rm II}(x)&=A_1'+A_2'x^2+A_4a_4(\xi;x) \quad (\xi<x<\infty)\ ,\end{aligned} \tag{4.8}$$

where the A's are constants and we have omitted in each region the solution which would introduce a physically unacceptable singularity. The associated solution for $F(x)$, from Eqs. (3.3), (3.34), and (3.35), is

$$F_{\rm I}(x)=B_1+\frac{B_2}{3}x^3-A_2\Lambda(x)+A_3F_3(x) \ , \tag{4.9}$$

$$F_{\rm II}(x)=B_1'+\frac{B_2'}{3}x^3-A_2'\Lambda(x)+A_4F_4(x)\ ,$$

where the B's are constants. The auxiliary function $W(x)$, evaluated from Eq. (2.26), is then

$$\begin{aligned} W_{\rm I}(x)&=-\frac{4}{\sqrt{\pi}}B_1e^{-x^2}+B_2\Lambda(x)-\frac{4}{3\sqrt{\pi}}B_2x^3e^{-x^2}\\ &\quad+\frac{4}{\sqrt{\pi}}A_3(x^2-1)\ ,\\ W_{\rm II}(x)&=-\frac{4}{\sqrt{\pi}}B_1'e^{-x^2}+B_2'\Lambda(x)-\frac{4}{3\sqrt{\pi}}B_2'x^3e^{-x^2}\\ &\quad+\frac{4}{\sqrt{\pi}}A_4\ , \end{aligned} \tag{4.10}$$

from which follows, by Eq. (2.24), the expression for $U(x)$:

$$\begin{aligned} U_{\rm I}(x)&=\frac{8}{\sqrt{\pi}}B_1e^{-x^2}+\frac{8}{3\sqrt{\pi}}B_2x^3e^{-x^2}+\frac{8}{\sqrt{\pi}}A_3\ ,\\ U_{\rm II}(x)&=\frac{8}{\sqrt{\pi}}B_1'e^{-x^2}+\frac{8}{3\sqrt{\pi}}B_2'x^3e^{-x^2}\ . \end{aligned} \tag{4.11}$$

We may now determine the arbitrary constants. Taking the derivative of $U(x)$ and then substituting it into Eq. (4.6), we find

$$\begin{aligned} B_1&=0\ ,\\ B_2&=2rZ_b^2K_0\left[\frac{n_b}{n_e}\right]\frac{1}{\xi}[1-\Lambda(\xi)]\ ,\\ B_1'&=\frac{2}{3}rZ_b^2K_0\left[\frac{n_b}{n_e}\right]\xi^2\ ,\\ B_2'&=-2rZ_b^2K_0\left[\frac{n_b}{n_e}\right]\frac{\Lambda(\xi)}{\xi}\ . \end{aligned} \tag{4.12}$$

We next apply the conditions of conservation of particles and energy by the collision operator, namely, Eqs. (2.31) and (2.34). We obtain

$$\begin{aligned} A_3&=0\ ,\\ A_4&=\frac{\sqrt{\pi}}{2}rK_0Z_b^2\left[\frac{n_b}{n_e}\right]\frac{\Lambda(\xi)}{\xi}\ . \end{aligned} \tag{4.13}$$

Since $S(x)$ exhibits a jump discontinuity at $x=\xi$, $U(x)$ must be continuous: $U_{\rm II}(\xi)-U_{\rm I}(\xi)=0$, a condition automatically fulfilled for the values found for the constants. Similarly, it can be verified that the requirement of continuity upon $W(x)$ is already satisfied. Also $a'(x)$ and $a(x)$ must be continuous across $x=\xi$, and we get

$$\begin{aligned} A_1'&=A_1\ ,\\ A_2'&=A_2\ . \end{aligned} \tag{4.14}$$

With this, the solution for the factorized distribution function can be written as

$$\begin{aligned} a_{\rm I}(x)&=A_1+A_2x^2\quad(0\le x<\xi)\ ,\\ a_{\rm II}(x)&=A_1+A_2x^2+\frac{\sqrt{\pi}}{2}rZ_b^2K_0\left[\frac{n_b}{n_e}\right]\frac{\Lambda(\xi)}{\xi}a_4(\xi;x)\\ &\qquad(\xi<x<\infty)\ . \end{aligned} \tag{4.15}$$

There remain undetermined the constants A_1 and A_2, which are fixed neither by analytical requirements on the solution nor by physical constraints on the equation: the same as for the equilibrium Maxwellian distribution, which also depends on two free constants, here they specify arbitrarily the contents of particle and energy to be carried by the nonequilibrium distribution function. To ascribe definite values to A_1 and A_2 we follow the usual prescription of perturbation theory [7] by which only the leading term is to contain the macroscopic information of the solution represented by the whole perturbation series. The number of particles and the energy associated with $a(x)$ are given by

$$\frac{\Delta n}{n_e}=A_1\left[\frac{\Delta n_1}{n_e}\right]+A_2\left[\frac{\Delta n_2}{n_e}\right]+\frac{\sqrt{\pi}}{2}rK_0Z_b^2\left[\frac{n_b}{n_e}\right]\frac{\Lambda(\xi)}{\xi}\left[\frac{\Delta n_4}{n_e}\right]\ , \tag{4.16}$$

$$\frac{\Delta E}{n_eT_e}=A_1\left[\frac{\Delta E_1}{n_eT_e}\right]+A_2\left[\frac{\Delta E_2}{n_eT_e}\right]+\frac{\sqrt{\pi}}{2}rK_0Z_b^2\left[\frac{n_b}{n_e}\right]\frac{\Lambda(\xi)}{\xi}\left[\frac{\Delta E_4}{n_eT_e}\right]\ . \tag{4.17}$$

Making use of Eqs. (3.16) and imposing $\Delta n/n_e=0$, $\Delta E/n_eT_e=0$, we obtain a system of two algebraic equations which is solved by

$$\begin{aligned} A_1&=\frac{\sqrt{\pi}}{2}rK_0Z_b^2\left[\frac{n_b}{n_e}\right]\left[-f_4(\xi;\infty)+g_4(\xi;\infty)-\frac{9\sqrt{\pi}}{4}\frac{1}{\xi}\right]\frac{\Lambda(\xi)}{\xi}\ ,\\ A_2&=\frac{\sqrt{\pi}}{2}rK_0Z_b^2\left[\frac{n_b}{n_e}\right]\\ &\quad\times\left\{f_2(\xi;\infty)+\frac{3\sqrt{\pi}}{4}\frac{1}{\xi^3}+\frac{27\sqrt{\pi}}{10}\frac{1}{\xi}+\frac{4}{3\sqrt{\pi}}\xi^3\left[\frac{1}{\Lambda(\xi)}-1\right]\right\}\frac{\Lambda(\xi)}{\xi}\ . \end{aligned} \tag{4.18}$$

A graphical representation of the solution for a few particular values of ξ is given in Fig. 1. We observe that for $\xi=0$ the beam ions become indistinguishable from the background ions in the bulk plasma, and we expect accordingly $a(x)$ to vanish identically. This is indeed what our solution shows, but that would not be the case had we taken $\Delta n/n_e\neq0$, $\Delta E/n_eT_e\neq0$.

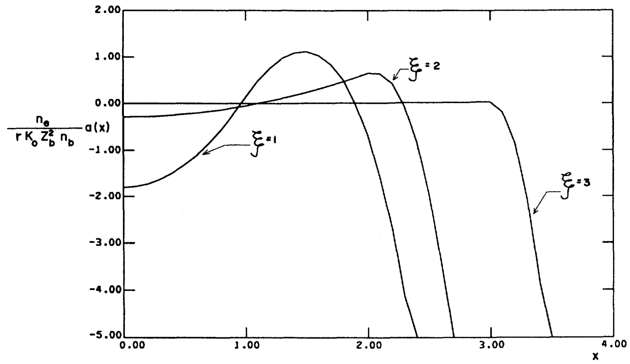

FIG. 1. Solution for the factorized electron distribution function for the ion-beam problem. The parameter ξ is the ratio of the ion-beam speed to the electron thermal speed.

B. The electron cyclotron wave problem

As a second example of solution, we determine the steady-state distribution function of the electrons in a plasma under the influx of energy of an electron cyclotron wave. For the velocity space diffusion caused by the wave we adopt the expression deduced by Kennel and Engelmann [8]:

$$\left[\frac{\partial f_e}{\partial t}\right]_w=\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}\left[D_c v_\perp^{2l-1}\delta\left[\frac{\omega-l\omega_c}{k_\parallel}-v_\parallel\right]\frac{\partial F_{Me}}{\partial v_\perp}\right], \tag{4.19}$$

where $v_\perp$ and $v_\parallel$ are the electron perpendicular and parallel velocities, D_c is a constant in velocity space and is proportional to the wave amplitude squared, ω and ω_c represent the wave frequency and the electron gyrofrequency, respectively, $k_\parallel$ is the wave vector along the magnetic field, and l is an integer that denotes the electron cyclotron harmonic. Consistent with the linearization of the collision operator, we have replaced the electron distribution function f_e on the right-hand side by its zeroth-order approximation F_{Me}, an approximation which is justified for small wave amplitudes. The expansion of (4.19) into a series of Legendre polynomials in the variable $\cos\theta=v_\parallel/v$ gives for the isotropic term in velocity space

$$\left[\frac{\partial f}{\partial t}\right]_w^{(0)}=-\frac{n_e}{\pi^{3/2}v_{Te}^3}\frac{1}{\tau_D}\times\begin{cases}0 & (x\le\xi)\,,\\ \dfrac{1}{x^2}\dfrac{\partial}{\partial x}[(x^2-\xi^2)^l e^{-x^2}] & (x>\xi)\,,\end{cases} \tag{4.20}$$

where $x=v/v_{Te}$ as before, ξ now defines the normalized phase velocity of the wave,

$$\xi=\frac{\omega-l\omega_c}{k_\parallel v_{Te}}\,,$$

and $\tau_D=v_{Te}^{5-2l}/D_c$ is the characteristic diffusion time. The wave introduces energy into the system but not particles, and we thus make the quasilinear term to appear accompanied by a compensating term like the one of Eq. (3.33). The Fokker-Planck equation then reads

$$\frac{1}{\tau_E}\left[x^2-\frac{3}{2}\right]F_{Me}(x)=\left[\frac{\partial f}{\partial t}\right]_w^{(0)}+C_{ee}^{(0)}(f_e,f_e)\,. \tag{4.21}$$

We shall restrict ourselves to the case $l=1$. Energy balance gives the relation between the diffusion time and the energy loss time:

$$\frac{\tau_D}{\tau_E}=\frac{8}{3\sqrt{\pi}}e^{-\xi^2}\,. \tag{4.22}$$

We start by solving for the function $U(x)$, using the representation of the collision operator stated in Eq. (2.22′). Since the wave diffusion term exhibits a jump discontinuity at the wave phase velocity, as in the ion-beam problem we divide the integration domain into two regions separated by the coordinate $x=\xi$. We write then

$$\begin{aligned}&\frac{\sqrt{\pi}}{8}\nu_0\tau_D\frac{1}{x^2}\frac{dU}{dx}=\frac{\tau_D}{\tau_E}\left[x^2-\frac{3}{2}\right]e^{-x^2}\quad(0\le x<\xi)\,,\\ &\frac{\sqrt{\pi}}{8}\nu_0\tau_D\frac{1}{x^2}\frac{dU}{dx}=\frac{\tau_D}{\tau_E}\left[x^2-\frac{3}{2}\right]e^{-x^2}+\frac{1}{x^2}\frac{\partial}{\partial x}[(x^2-\xi^2)e^{-x^2}]\quad(\xi<x<\infty)\,.\end{aligned} \tag{4.23}$$

The solution is

$$\begin{aligned}\nu_0\tau_D U_{\rm I}(x)&=-\frac{\tau_D}{\tau_E}x\Lambda'(x)+A_1\,,\\ \nu_0\tau_D U_{\rm II}(x)&=-\frac{\tau_D}{\tau_E}x\Lambda'(x)+\frac{8}{\sqrt{\pi}}(x^2-\xi^2)e^{-x^2}+A_2\,,\end{aligned} \tag{4.24}$$

where A_1 and A_2 are arbitrary constants. In order to make $U(x)$ a continuous function across $x=\xi$, as it must be, we choose $A_1=A_2\equiv A$, a condition from which the conservation of particles follows as a consequence: $U_{\rm II}(\infty)-U_{\rm I}(0)=0$.

We next find $W(x)$, which stems from Eq. (2.24), and is given by

$$\begin{aligned}\nu_0\tau_D W_{\rm I}(x)&=-\frac{\tau_D}{\tau_E}\left[\frac{3}{2}\Lambda(x)-\frac{1}{2}x\Lambda'(x)\right]+\frac{A}{2}x^2+B\,,\\ \nu_0\tau_D W_{\rm II}(x)&=-\frac{\tau_D}{\tau_E}\left[\frac{3}{2}\Lambda(x)-\frac{1}{2}x\Lambda'(x)\right]+\frac{4}{\sqrt{\pi}}(e^{-\xi^2}-e^{-x^2})-\frac{4}{\sqrt{\pi}}(x^2-\xi^2)e^{-x^2}+\frac{A}{2}x^2+B\,,\end{aligned} \tag{4.25}$$

where the newly introduced integration constants were taken to be equal so as to make $W(x)$ continuous across $x=\xi$. With this, the condition of energy conservation by the collision operator: $x^2U(x)-2W(x)|_0^\infty=0$ furnishes no information besides the relation already found between the diffusion time and the energy loss time.

We proceed by solving Eq. (2.26) for $F(x)$ and find

$$\begin{aligned}\nu_0\tau_D F_{\rm I}(x)&=-\frac{1}{2}\frac{\tau_D}{\tau_E}x^3+\frac{\sqrt{\pi}}{8}AF_3(x)\\&\quad+\frac{\sqrt{\pi}}{4}\left[\frac{A}{2}+B\right]F_4(x)-C_1\Lambda(x)\ ,\\ \nu_0\tau_D F_{\rm II}(x)&=-\frac{1}{2}\frac{\tau_D}{\tau_E}x^3+\frac{\sqrt{\pi}}{8}AF_3(x)\\&\quad+\left[\frac{\sqrt{\pi}}{4}\left[\frac{A}{2}+B\right]+e^{-\xi^2}\right]F_4(x)+1-\xi^2\\&\quad+x^2-2\Lambda(x)\int_\xi^x\frac{y\,dy}{\Lambda(y)}-C_2\Lambda(x)\ ,\end{aligned}\tag{4.26}$$

where C_1 and C_2 are arbitrary constants. Since the functions $F_4(x)$ and $F_3(x)$ give rise to physically unacceptable singularities for the distribution function at the origin and at infinity, respectively, we must have $(A/2)+B=0$ from the expression for $F_{\rm I}(x)$ and $A=0$ from the expression for $F_{\rm II}(x)$. Imposing in addition the requirement of continuity of $F(x)$ across $x=\xi$ we have finally

$$\begin{aligned}\nu_0\tau_D F_{\rm I}(x)&=-\frac{1}{2}\frac{\tau_D}{\tau_E}x^3-C\Lambda(x)\ ,\\ \nu_0\tau_D F_{\rm II}(x)&=-\frac{1}{2}\frac{\tau_D}{\tau_E}x^3+1-\xi^2+x^2\\&\quad-\left[C+\frac{1}{\Lambda(\xi)}\right]\Lambda(x)+e^{-\xi^2}F_4(x)\\&\quad-2\Lambda(x)\int_\xi^x\frac{y\,dy}{\Lambda(y)}\ ,\end{aligned}\tag{4.27}$$

where C is a constant.

The solution for the function $K(x)$ follows from Eq. (2.20) and is

$$\begin{aligned}\nu_0\tau_D K_{\rm I}(x)&=-\frac{3}{2}\frac{\tau_D}{\tau_E}-\frac{4}{\sqrt{\pi}}Ce^{-x^2}\ ,\\ \nu_0\tau_D K_{\rm II}(x)&=-\frac{3}{2}\frac{\tau_D}{\tau_E}+\frac{4}{\sqrt{\pi}}\frac{e^{-\xi^2}}{\Lambda(x)}\\&\quad-\frac{4}{\sqrt{\pi}}\left[C+\frac{1}{\Lambda(\xi)}\right]e^{-x^2}\\&\quad+\frac{16}{\pi}e^{-\xi^2}e^{-x^2}\int_\xi^x\frac{y^2dy}{\Lambda^2(y)}\\&\quad-\frac{8}{\sqrt{\pi}}e^{-x^2}\int_\xi^x\frac{y\,dy}{\Lambda(y)}\ ,\end{aligned}\tag{4.28}$$

from which, using Eq. (2.16), we obtain $a'(x)$ as

$$\begin{aligned}\nu_0\tau_D a_{\rm I}'(x)&=2Cx\ ,\\ \nu_0\tau_D a_{\rm II}'(x)&=2\left[C+\frac{1}{\Lambda(\xi)}\right]x-2\frac{x}{\Lambda(x)}\\&\quad-\frac{8}{\sqrt{\pi}}e^{-\xi^2}x\int_\xi^x\frac{y^2dy}{\Lambda^2(y)}+4x\int_\xi^x\frac{y\,dy}{\Lambda(y)}\ .\end{aligned}\tag{4.29}$$

Note that $a'(x)$ is continuous. So must be the factorized distribution function, which we find to be

$$\begin{aligned}\nu_0\tau_D a_{\rm I}(x)&=D+Cx^2\quad(0\le x\le\xi)\ ,\\ \nu_0\tau_D a_{\rm II}(x)&=D+Cx^2+\hat{a}_p(x)\quad(\xi\le x<\infty)\ ,\end{aligned}\tag{4.30}$$

where

$$\begin{aligned}\hat{a}_p(x)&=\frac{\sqrt{\pi}}{2}[(x^2-1)g_1(\xi;x)-g_3(\xi;x)]+e^{-\xi^2}a_4(\xi;x)\\&\quad+(x^2-\xi^2)\left[\frac{1}{2}(x^2-\xi^2)-1+\frac{1}{\Lambda(\xi)}\right]\\&\quad-\frac{3\sqrt{\pi}}{2}\left[x-\frac{x^2}{\xi}+\frac{1}{\xi}-\frac{1}{x}\right]\end{aligned}\tag{4.31}$$

is the particular solution normalized to $\nu_0\tau_D$, and $g_1(\xi;x)$ and $g_3(\xi;x)$, defined by

$$g_1(\xi;x)=\frac{4}{\sqrt{\pi}}\int_\xi^x y\left[\frac{1}{\Lambda(y)}-\frac{3\sqrt{\pi}}{4}\frac{1}{y^3}-1\right]dy\ ,\tag{4.32}$$

$$g_3(\xi;x)=\frac{4}{\sqrt{\pi}}\int_\xi^x y^3\left[\frac{1}{\Lambda(y)}-1\right]dy\ ,\tag{4.33}$$

are regular functions in the whole domain of variation of ξ and x. Expansions and numerical tables for $g_1(0;x)$ and $g_3(0;x)$ are provided in Appendix C.

It remains to specify values for the constants C and D, which again we shall do by demanding that the perturbation to the distribution function be void of particles and energy. The result is

$$\begin{aligned}D&=\frac{\sqrt{\pi}}{2}[g_1(\xi;\infty)+g_3(\xi;\infty)]-h(\xi;\infty)\\&\quad+e^{-\xi^2}\left[-f_4(\xi;\infty)+g_4(\xi;\infty)-\frac{9\sqrt{\pi}}{4}\frac{1}{\xi}\right]\\&\quad+\frac{3\sqrt{\pi}}{2}\frac{1}{\xi}+\xi^2\left[\frac{1}{\Lambda(\xi)}-1\right]\ ,\\ C&=-\frac{\sqrt{\pi}}{2}g_1(\xi;\infty)\\&\quad+e^{-\xi^2}\left[f_2(\xi;\infty)+\frac{3\sqrt{\pi}}{4}\frac{1}{\xi^3}+\frac{27\sqrt{\pi}}{10}\frac{1}{\xi}-\frac{4}{3\sqrt{\pi}}\xi^3\right]\\&\quad-\frac{3\sqrt{\pi}}{2}\frac{1}{\xi}-\left[\frac{1}{\Lambda(\xi)}-1\right]\ ,\end{aligned}\tag{4.34}$$

where

$$h(\xi;x)=\int_\xi^x\frac{y^2\Lambda'(y)}{\Lambda(y)}dy\tag{4.35}$$

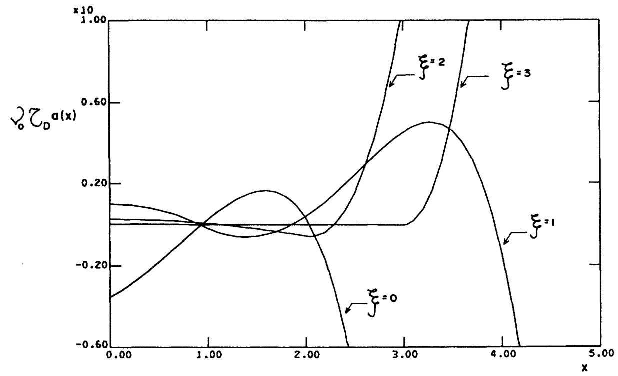

FIG. 2. Solution for the factorized electron distribution function for the electron cyclotron wave problem. The parameter ξ is the ratio of the wave phase speed to the electron thermal speed.

and is tabulated in Appendix C. Graphs of the factorized distribution function are given in Fig. 2.

We take this as a case study for the question of the global boundary conditions on the equation and the conserving properties of the operator. We note, from Eq. (4.27), that $F_{\rm I}(0)=0$, and from Eqs. (4.28) and (4.22), that $K_{\rm II}(\infty)=0$, indicating that the solution indeed meets the conditions of overall balance of particles and energy. Had we omitted the energy loss term in Eq. (4.21), under the same requirements of continuity of $U(x)$ and $W(x)$ of necessity, we would have found, instead of Eq. (2.34), that

$$x^2U(x)-2W(x)|_0^\infty=-\frac{8}{\sqrt{\pi}}\frac{e^{-\xi^2}}{\nu_0\tau_D},$$

apparently confirming that $C_{ee}^{(0)}(f,f)$, as given by Eq. (2.22′), acts as an energy source. The terms depending on τ_D/τ_E would not have appeared during the derivation of the solution and, in particular, they would be missing in Eqs. (4.28) for $K(x)$, which would pass to furnish with the change a non-null result, namely, $\nu_0\tau_D K_{\rm II}(\infty)=4/\sqrt{\pi}e^{-\xi^2}$. But this means that the collision term that has actually been used is not the one of Eq. (2.22′), but rather the one of Eq. (2.27), that is, containing an inhomogeneous term that shows itself to be the precise reproduction of the energy loss term which had been omitted: in one way or another, whether or not we have written it explicitly, it creeps into our equations to ensure that energy is conserved by the collision operator. There is no solution that does not imply an energy balance between sources in the Fokker-Planck equation; and, as can be seen from the present example, that solution is not modified in the least by the "suppression" of the energy loss term, since the constant terms depending on τ_D/τ_E in Eq. (4.28) would be eliminated anyway when taking the derivative of $K(x)$ in the operation of finding $a'(x)$.

C. Diffusion by electron Landau damping of radio frequency waves

As a final example we consider a quasilinear diffusion term of the form proposed by Fisch and Karney [9]:

$$\left[\frac{\partial f_e}{\partial t}\right]_w=\frac{\partial}{\partial v_\parallel}\left[D_L\delta\left[\frac{\omega}{k_\parallel}-v_\parallel\right]\frac{\partial F_{Me}}{\partial v_\parallel}\right], \tag{4.36}$$

where D_L is a constant and f_e on the right-hand side has again been replaced by F_{Me}. Expansion in zonal harmonics gives the source term

$$S(x)=-\left[\frac{\partial f_e}{\partial t}\right]_w^{(0)}=\begin{cases}0 \quad (0<x<\xi)\\ \dfrac{n_e}{\pi^{3/2}v_{Te}^3}\dfrac{\xi}{\tau_L}\left[\dfrac{\delta(x-\xi)}{x}-\dfrac{2\xi}{x}\right]e^{-x^2} \quad (\xi<x<\infty),\end{cases} \tag{4.37}$$

where $\xi=\omega/k_\parallel v_{Te}$ and $\tau_L=v_{Te}^3/D_L$ is the electron diffusion time in velocity space. To counterbalance the energy that is fed by this source into the system, we assume a sink of the same Coulombic form that we have been using throughout. The characteristic removal time τ_E must be chosen in accordance with

$$\frac{\tau_L}{\tau_E}=\frac{8}{3\sqrt{\pi}}\xi^2e^{-\xi^2}. \tag{4.38}$$

The source in this case is a combination of the Coulombic balancing terms, and the solution may proceed as in the ion-beam example, the only difference being that, because of the Dirac δ function, $U(x)$ and $a'(x)$ are no longer continuous, but exhibit a finite jump at $x=\xi$. We merely quote the result for the factorized distribution function:

$$\begin{aligned}\nu_0\tau_D a_{\rm I}(x)&=A_1+A_2x^2 \quad (x<\xi),\\ \nu_0\tau_D a_{\rm II}(x)&=A_1+A_2x^2+\frac{\xi^2}{\Lambda(\xi)}(x^2-\xi^2)\\ &\quad+\xi^2e^{-\xi^2}a_4(\xi;x) \quad (x>\xi),\end{aligned} \tag{4.39}$$

where

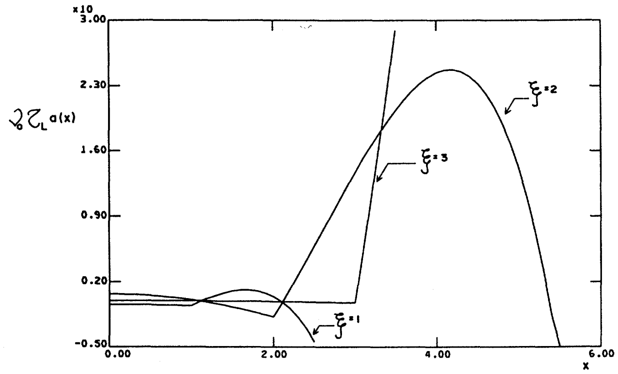

FIG. 3. Solution for the factorized electron distribution function for the problem of electron Landau damping of radio frequency waves. The parameter ξ is the ratio of the wave phase speed to the electron thermal speed.

$$\begin{aligned} A_1 &= \xi^2 e^{-\xi^2}[g_4(\xi;\infty)-f_4(\xi;\infty)] \\ &\quad +\xi^4\left[\frac{1}{\Lambda(\xi)}-1\right]-\frac{9\sqrt{\pi}}{4}\xi e^{-\xi^2}, \\ A_2 &= \xi^2 e^{-\xi^2}f_2(\xi;\infty)-\xi^2\left[\frac{1}{\Lambda(\xi)}-1\right] \\ &\quad +e^{-\xi^2}\left[\frac{3\sqrt{\pi}}{4}\frac{1}{\xi}+\frac{27\sqrt{\pi}}{10}\xi\right]-\frac{4}{3\sqrt{\pi}}\xi^5 e^{-\xi^2}. \end{aligned} \tag{4.40}$$

For $\xi \to 0$, the constants A_1 and A_2 and the complete solution all vanish. A pictorial representation of the solution for a few values of ξ is given in Fig. 3.

V. SUMMARY

The main result of this paper is substantiated in Eq. (2.27′), which expresses the collision term for an isotropic distribution function in velocity space as an exact third-order differential operator. The general solution for the factorized distribution function can be constructed from a combination of the four independent functions described by Eqs. (3.6). The particular solution for any given driving term in the kinetic equation finds its representation in Eq. (3.29). The conditions of particle and energy conservation upon mutual collisions of the electrons translate as Eqs. (2.31) and (2.34), respectively. The Maxwellian distribution function is a solution in the equilibrium of the isolated system and in the presence of sources of the form of Eqs. (3.32) and (3.33), corresponding to the rate at which electrons are generated with zero and with infinite speed, respectively, by collisions in the bulk Maxwellian. These Coulombic sources can be used to balance particles and energy introduced by arbitrary sources and guarantee the existence of a steady-state solution to the kinetic equation.

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APPENDIX A: THE FUNCTION $\Lambda(x)$

The function $\Lambda(x)$ is defined by

$$\Lambda(x)=\frac{4}{\sqrt{\pi}}\int_0^x y^2 e^{-y^2}dy \tag{A1}$$

and is related to the error function

$$\phi(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2}dy$$

through

$$\Lambda(x)=\phi(x)-x\phi'(x). \tag{A2}$$

The first two derivatives of $\Lambda(x)$ are

$$\Lambda'(x)=\frac{4}{\sqrt{\pi}}x^2 e^{-x^2}, \tag{A3}$$

$$\Lambda''(x)=\left[\frac{2}{x}-2x\right]\Lambda'(x). \tag{A4}$$

Particular values taken by $\Lambda(x)$ are

$$\Lambda(0)=0, \tag{A5}$$

$$\Lambda(\infty)=1. \tag{A6}$$

The following expansions for $\Lambda(x)$ are convergent for $0\le x<\infty$:

$$\Lambda(x)=\frac{4}{\sqrt{\pi}}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+3}}{(2n+3)n!}, \tag{A7}$$

$$\Lambda(x)=\frac{4}{\sqrt{\pi}}e^{-x^2}\sum_{n=0}^{\infty}\frac{2^n}{(2n+3)!!}x^{2n+3}, \tag{A8}$$

$$\Lambda(x)=1-\frac{e^{-x^2}}{\sqrt{\pi}}\left[2x+\frac{1}{x+}\,\frac{\frac{1}{2}}{x+}\,\frac{1}{x+}\,\frac{\frac{3}{2}}{x+}\,\frac{2}{x+}\cdots\frac{n}{x+}\,\frac{n+\frac{1}{2}}{x+}\cdots\right]. \tag{A9}$$

For large x, the asymptotic expansion of $\Lambda(x)$ is

$$\Lambda(x)\sim 1-\frac{2}{\sqrt{\pi}}xe^{-x^2}\times\left[1+\frac{1}{2x^2}+\sum_{n=1}^{\infty}(-1)^n\frac{(2n-1)!!}{2^n x^{2n+2}}\right]. \tag{A10}$$

For small values of x the following power series expansions can be used:

$$\Lambda(x)=\frac{4}{3\sqrt{\pi}}x^3-\frac{4}{5\sqrt{\pi}}x^5+\frac{2}{7\sqrt{\pi}}x^7-\frac{2}{27\sqrt{\pi}}x^9+\frac{1}{66\sqrt{\pi}}x^{11}-\frac{1}{390\sqrt{\pi}}x^{13}+\cdots, \tag{A11}$$

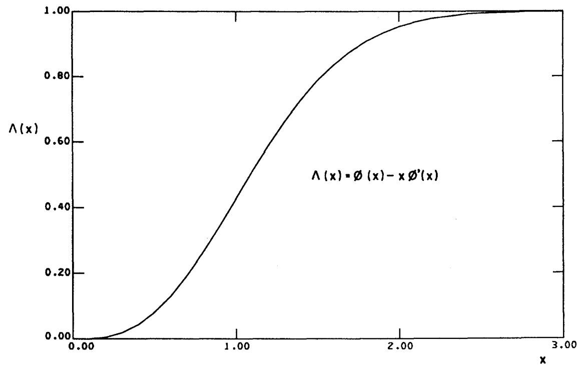

FIG. 4. The function $\Lambda(x)$.

$$\Lambda^2(x)=\frac{16}{9\pi}x^6-\frac{32}{15\pi}x^8+\frac{736}{525\pi}x^{10}-\frac{1856}{2835\pi}x^{12}+\frac{17\,504}{72\,765\pi}x^{14}-\frac{1984}{27\,027\pi}x^{16}+\cdots, \quad (A12)$$

$$\frac{1}{\Lambda(x)}=\frac{3\sqrt{\pi}}{4}\frac{1}{x^3}+\frac{9\sqrt{\pi}}{20}\frac{1}{x}+\frac{153\sqrt{\pi}}{1400}x+\frac{227\sqrt{\pi}}{21\,000}x^3-\frac{4909\sqrt{\pi}}{10\,780\,000}x^5-\frac{132\,751\sqrt{\pi}}{700\,700\,000}x^7+\cdots, \quad (A13)$$

$$\frac{1}{\Lambda^2(x)}=\frac{9\pi}{16}\frac{1}{x^6}+\frac{27\pi}{40}\frac{1}{x^4}+\frac{513\pi}{1400}\frac{1}{x^2}+\frac{401\pi}{3500}+\frac{11\,313\pi}{539\,000}x^2+\frac{146\,151\pi}{87\,587\,500}x^4+\cdots, \quad (A14)$$

$$\frac{\Lambda'(x)}{\Lambda(x)}=\frac{3}{x}-\frac{6}{5}x+\frac{24}{175}x^3+\frac{16}{2625}x^5-\frac{1504}{1\,010\,625}x^7-\frac{11\,456}{65\,690\,625}x^9+\cdots, \quad (A15)$$

$$\frac{e^{x^2}}{\Lambda(x)}=\frac{3\sqrt{\pi}}{4}\frac{1}{x^3}+\frac{6\sqrt{\pi}}{5}\frac{1}{x}+\frac{327\sqrt{\pi}}{350}x+\frac{1234\sqrt{\pi}}{2625}x^3+\cdots. \quad (A16)$$

A graphical representation of $\Lambda(x)$ is given in Fig. 4.

APPENDIX B: THE FUNCTIONS $\Lambda_n(x)$

The functions $\Lambda_n(x)$ are defined by

$$\Lambda_n(x)=\frac{4}{\sqrt{\pi}}\int_0^x y^n e^{-y^2}dy \quad (n=0,1,2,\ldots). \quad (B1)$$

Writing $\Lambda(x)$ for $\Lambda_2(x)$, they can be evaluated recursively by means of the expression

$$\Lambda_n(x)=-\frac{1}{2}x^{n-3}\Lambda'(x)+\left[\frac{n-1}{2}\right]\Lambda_{n-2}(x) \quad (n=2,3,4,5,\ldots), \quad (B2)$$

knowing that

$$\Lambda_0(x)\equiv 2\phi(x)=2\Lambda(x)+\frac{\Lambda'(x)}{x}, \quad (B3)$$

$$\Lambda_1(x)=\frac{2}{\sqrt{\pi}}-\frac{\Lambda'(x)}{2x^2}. \quad (B4)$$

The explicit forms of the ones of use in this paper are

$$\Lambda_3(x)=\frac{2}{\sqrt{\pi}}-\frac{1}{2}\Lambda'(x)-\frac{\Lambda'(x)}{2x^2}, \quad (B5)$$

$$\Lambda_4(x)=\tfrac{3}{2}\Lambda(x)-\tfrac{1}{2}x\Lambda'(x), \quad (B6)$$

$$\Lambda_5(x)=\frac{4}{\sqrt{\pi}}-\frac{1}{2}x^2\Lambda'(x)-\Lambda'(x)-\frac{\Lambda'(x)}{x^2}, \quad (B7)$$

$$\Lambda_6(x)=\tfrac{15}{4}\Lambda(x)-\tfrac{1}{2}x^3\Lambda'(x)-\tfrac{5}{4}x\Lambda'(x), \quad (B8)$$

$$\Lambda_7(x)=\frac{12}{\sqrt{\pi}}-\frac{1}{2}x^4\Lambda'(x)-\frac{3}{2}x^2\Lambda'(x)-3\Lambda'(x)-3\frac{\Lambda'(x)}{x^2}, \quad (B9)$$

$$\Lambda_8(x)=\tfrac{105}{8}\Lambda(x)-\tfrac{1}{2}x^5\Lambda'(x)-\tfrac{7}{4}x^3\Lambda'(x)-\tfrac{35}{8}x\Lambda'(x). \quad (B10)$$

Useful relations among the derivatives are

$$\Lambda_n''(x)=(n-2x^2)\Lambda_{n-1}'(x)=\left[\frac{n}{x}-2x\right]\Lambda_n'(x). \quad (B11)$$

APPENDIX C: EXPANSIONS AND TABLES FOR FUNCTIONS DEFINED BY INTEGRALS

For the functions introduced in the main text the following expansions are valid for small values of x:

$$f_2(0;x)=\frac{4}{\sqrt{\pi}}\int_0^x y^2\left[\frac{1}{\Lambda^2(y)}-\frac{9\pi}{16}\frac{1}{y^6}-\frac{27\pi}{40}\frac{1}{y^4}-1\right]dy$$
$$=\frac{513\sqrt{\pi}}{350}x+\frac{401\sqrt{\pi}}{2625}x^3+\frac{11\,313\sqrt{\pi}}{673\,750}x^5+\frac{146\,151\sqrt{\pi}}{153\,278\,125}x^7+\cdots-\frac{4}{3\sqrt{\pi}}x^3, \quad (C1)$$

$$f_4(0;x)=\frac{4}{\sqrt{\pi}}\int_0^x y^4\left[\frac{1}{\Lambda^2(y)}-\frac{9\pi}{16}\frac{1}{y^6}-1\right]dy$$
$$=\frac{27\sqrt{\pi}}{10}x+\frac{171\sqrt{\pi}}{350}x^3+\frac{401\sqrt{\pi}}{4375}x^5+\frac{11\,313\sqrt{\pi}}{943\,250}x^7+\cdots-\frac{4}{5\sqrt{\pi}}x^5, \quad (C2)$$

$$f_6(0;x)=\frac{4}{\sqrt{\pi}}\int_0^x y^6\left[\frac{1}{\Lambda^2(y)}-1\right]dy$$
$$=\frac{9\sqrt{\pi}}{4}x+\frac{9\sqrt{\pi}}{10}x^3+\frac{513\sqrt{\pi}}{1750}x^5+\frac{401\sqrt{\pi}}{6125}x^7+\cdots-\frac{4}{7\sqrt{\pi}}x^7, \quad (C3)$$

TABLE I. Values of the functions $f_6(0;x)$, $g_3(0;x)$, $g_4(0;x)$, and $h(0;x)$.

x	$f_6(0;x)$	$g_3(0;x)$	$g_4(0;x)$	$h(0;x)$
0.0	0.000 00	0.000 00	0.000 00	0.000 00
0.2	0.810 53	0.603 93	0.060 58	0.059 52
0.4	1.702 29	1.224 86	0.247 20	0.232 41
0.6	2.772 17	1.863 45	0.566 71	0.502 20
0.8	4.136 45	2.506 02	1.016 40	0.843 22
1.0	5.913 96	3.129 15	1.576 67	1.223 46
1.2	8.187 71	3.705 04	2.209 13	1.608 42
1.4	10.954 99	4.207 51	2.860 93	1.965 95
1.6	14.088 85	4.617 74	3.474 63	2.271 10
1.8	17.339 62	4.928 46	4.001 21	2.509 88
2.0	20.393 21	5.145 08	4.411 33	2.680 47
2.2	22.969 65	5.283 17	4.700 18	2.791 28
2.4	24.912 42	5.363 29	4.883 65	2.856 53
2.6	26.218 78	5.405 47	4.988 63	2.891 29
2.8	27.002 27	5.425 62	5.042 76	2.908 05
3.0	27.422 17	5.434 34	5.067 93	2.915 36
3.2	27.623 80	5.437 78	5.078 52	2.918 26
3.4	27.710 80	5.439 00	5.082 56	2.919 30
3.6	27.744 64	5.439 41	5.083 95	2.919 65
3.8	27.756 52	5.439 52	5.084 39	2.919 75
4.0	27.760 30	5.439 56	5.084 52	2.919 78
4.2	27.761 39	5.439 56	5.084 55	2.919 78
4.4	27.761 68	5.439 57	5.084 56	2.919 78
4.6	27.761 75	5.439 57	5.084 56	2.919 79
∞	27.761 76	5.439 57	5.084 56	2.919 79

$$g_1(0;x)=\frac{4}{\sqrt{\pi}}\int_0^x y\left[\frac{1}{\Lambda(y)}-\frac{3\sqrt{\pi}}{4}\frac{1}{y^3}-1\right]dy$$
$$=\frac{9}{5}x+\frac{51}{350}x^3+\frac{227}{26\,250}x^5-\frac{4909}{18\,865\,000}x^7-\cdots-\frac{2}{\sqrt{\pi}}x^2\,, \tag{C4}$$

$$g_3(0;x)=\frac{4}{\sqrt{\pi}}\int_0^x y^3\left[\frac{1}{\Lambda(y)}-1\right]dy$$
$$=3x+\frac{3}{5}x^3+\frac{153}{1750}x^5+\frac{227}{36\,750}x^7-\cdots-\frac{x^4}{\sqrt{\pi}}\,, \tag{C5}$$

$$g_4(0;x)=\frac{4}{\sqrt{\pi}}\int_0^x y^4\left[\frac{1}{\Lambda(y)}-1\right]dy$$
$$=\frac{3}{2}x^2+\frac{9}{20}x^4+\frac{51}{700}x^6+\frac{227}{42\,000}x^8-\cdots-\frac{4}{5\sqrt{\pi}}x^5\,, \tag{C6}$$

$$h(0;x)=\int_0^x\frac{y^2\Lambda'(y)}{\Lambda(y)}dy$$
$$=\frac{3}{2}x^2-\frac{3}{10}x^4+\frac{4}{175}x^6+\frac{2}{2625}x^8-\cdots\,. \tag{C7}$$

For large values of x, Tables I and II can be used. Because $f_2(0;x)$, $f_4(0;x)$, and $g_1(0;x)$ converge too slowly to their limiting values as $x\to\infty$, numerical values are instead tabulated for the closely related functions of quicker convergence:

$$\tilde{f}_2(x)=\frac{4}{\sqrt{\pi}}\int_x^\infty y^2\left[\frac{1}{\Lambda^2(y)}-1\right]dy\,, \tag{C8}$$

TABLE II. Values of the functions $\tilde{f}_2(x)$, $\tilde{f}_4(x)$, and $\tilde{g}_1(x)$.

x	$\tilde{f}_2(x)$	$\tilde{f}_4(x)$	$\tilde{g}_1(x)$
0.5	19.074 28	15.138 36	4.382 48
0.6	12.781 31	13.264 65	3.312 95
0.7	9.161 73	11.751 48	2.546 08
0.8	6.844 52	10.457 71	1.973 66
0.9	5.243 45	9.307 25	1.535 01
1.0	4.075 26	8.257 39	1.193 23
1.2	2.500 58	6.375 63	0.711 78
1.4	1.520 67	4.735 40	0.411 40
1.6	0.894 73	3.338 97	0.227 28
1.8	0.500 83	2.209 99	0.118 72
2.0	0.263 39	1.360 05	0.058 12
2.2	0.128 95	0.772 40	0.026 49
2.4	0.058 40	0.402 63	0.011 18
2.6	0.024 37	0.192 01	0.004 36
2.8	0.009 35	0.083 63	0.001 57
3.0	0.003 29	0.033 25	0.000 52
3.2	0.001 07	0.012 07	0.000 15
3.4	0.000 32	0.004 00	0.000 04
3.6	0.000 09	0.001 21	0.000 01
3.8	0.000 02	0.000 33	0.000 00
4.0	0.000 01	0.000 08	0.000 00
4.2	0.000 00	0.000 01	0.000 00
∞	0.000 00	0.000 00	0.000 00

$$\tilde{f}_4(x)=\frac{4}{\sqrt{\pi}}\int_x^\infty y^4\left[\frac{1}{\Lambda^2(y)}-1\right]dy\ , \tag{C9}$$

$$\tilde{g}_1(x)=\frac{4}{\sqrt{\pi}}\int_x^\infty y\left[\frac{1}{\Lambda(y)}-1\right]dy\ . \tag{C10}$$

It is useful to know that

$$\begin{aligned} f_2(0;\infty)&=0.108\,16\ ,\\ f_4(0;\infty)&=9.654\,52\ ,\\ g_1(0;\infty)&=-0.981\,13\ . \end{aligned} \tag{C11}$$

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[3] Note, however, that this approximation is of no effect in what concerns the mathematical transformations of this section. Also, note that it is not needed to determine analytically the solver of the collision operator: the dependence of the Coulomb logarithm on the speed could easily be taken into account by incorporating it into the definition of the source $S(x)$ in Sec. III B.

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