Reunion and survival of interacting walkers

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The reunion and survival probabilities of p random walkers in d dimensions with a mutual repulsive interaction are formulated via appropriate partition functions of directed polymers. The exponents that describe the decay of these probabilities with length are obtained through renormalization-group theory $O(\epsilon^2)$, where $\epsilon = 2 - d$. The distribution function and the probability of n out of p walkers meeting are also discussed. To first order, the distribution function is a Gaussian one modified by an anomalous exponent of the length of the polymer, N. The procedure is generalized to multicritical many-body interactions. For these multicritical cases, the exponents are obtained to second order in the relevant ϵ. At the upper critical dimension of the interaction, there is a logarithmic correction other than the Gaussian exponent. An interesting consequence is a logarithmic correction for one-dimensional walkers with a three-body repulsive interaction.

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I. INTRODUCTION

A problem that quite often arises, e.g., in various phase transitions, is to find out how the probability of reunion of a set of mutually interacting (generally repulsive) random walkers decays with the length of the walk. We are familiar with such phenomena occuring in nature. One known example is the commensurate-incommensurate (CI) transition [1, 2]. The dislocations present in the medium are responsible for the creation and annihilation of the domain walls [1–5]. This physical picture in terms of formation of loops of walkers helps in visualizing the nature of wetting transitions [1,2], melting of commensurate phases [1, 2], the Ising model [6], etc. Such loops are also relevant in the context of self-organized criticality as in Ref. [7], where they study the critical behavior of directed Abelian two-dimensional (2D) sandpile model and 1D voter model. These repulsive walkers have been called vicious walkers in literature [1,4]. See Refs. [1, 2] for more on one-dimensional interacting walkers.

The particular class of problems we like to address in this paper involves the reunion and survival probabilities for a set of p random walkers. Given that p walkers start at, say, the origin in the d-dimensional space at time zero, we want to know the probability that they will all meet (reunite) at some point $\mathbf{r}$ at length (or time) N, N being very large [8]. For reunion anywhere, an integration over the end point coordinate is required. Another quantity to consider is the survival probability where at time N the walkers can be anywhere in space. This requires independent integrations of the end point coordinates, thereby encompassing the reunion case. Asymptotically for large N, power-law decays are expected. Such power laws generally signify universality in the sense of irrelevance of microscopic details of the walks. Therefore, for these universal asymptotics, one can choose a continuum description—and this is where directed polymers (DP) come out quite handy.

By treating the time or the length of the walk as a special dimension, a d-dimensional random walker can be viewed as a $(d+1)$-dimensional DP. Defined in a general way, directed polymers in $d+1$ dimensions are random walkers directed in a particular direction with entropic fluctuations in the transverse d-dimensional space. This, in turn, means that a walker cannot come back to its previous $d+1$ dimensional positions, signifying a self-avoidance built in by its construction.

The relevance of directed polymers (DP) in the context of many physical phenomena has made them an alive topic of an extensive research in statistical physics. For example, the fluctuating domain walls in the uniaxial CI transition can be identified as directed polymers in two dimensions [1]. The flux lines in high-T_c superconductors [9], polymeric nematics [10], etc. are examples of DP's in three dimensions. DP in a random medium is a topic of interest in the context of surface growth [11] and as a simple model random system [12]. It is known already that several properties of interacting DP's can be studied exactly using renormalization-group (RG) theory [13, 14]. We have shown elsewhere that even a randomly interacting system can also be treated exactly [15].

The reunion probability of p interacting walkers in one dimension [equivalent to (1+1)-dimensional DP] was derived using diffusion theory methods [2], but the whole approach was restricted to one dimension only. In fact, the development of an approach for higher dimensional systems was remaining as an open problem.

Our main intention is to achieve results for arbitrary dimensional systems using the renormalization-group technique. This is not really beyond reach. The reunion probability for a system of two interacting walkers can be calculated exactly in the renormalization-group approach. It is, indeed, possible to resum [16] the whole perturbation series in interaction, instead of renormalizing it. The solution obtained through this exact calculation is, therefore, applicable to any dimension. However, it appeared that for a system of more than two chains such an exact resummation is not feasible, and, therefore, RG approach turns out to be unavoidable. Such

an effort has been made here for a system of more than two chains using ϵ $(= 2 - d)$ expansion. Though it is not possible to proceed exactly, formula up to $O(\epsilon^2)$ could be obtained. Some of these results have already been reported [17]. We provide the details and several new results.

The natural tendency, infused by the success in the interacting-walker problem, is to extend the above problem to polymers interacting with multicritical many-body interaction [18, 19]. We ask the same question of reunion and survival for p chains with m $(< p)$ body interaction. The $m = 2$ case is the interacting walker problem. It turns out that such a problem with many-body interaction, which is relevant both in the polymer context and in the wide regime of condensed-matter physics, possesses all the mathematical and physical aspects of the interacting-walkers case in a much more general way. To obtain results for arbitrary dimensions we again use the renormalization-group technique.

Why RG? For noninteracting walkers, the exponents follow (see below) from the classic single walk (or a DP) result or purely from dimensional analysis. The exact $1 + 1$ dimensional results for interacting walkers show that the interaction do change the exponents (discussed later). The difference, to be called the anomalous exponent, seems to violate dimensional analysis, and a length scale is needed to take care of this. Here, the RG approach comes to our rescue. In this approach, the noninteracting case serves as the starting point to explore the effects of the interaction as a perturbation. The divergences in this perturbation series are then cured through renormalization by introducing a length scale that paves the way for the anomalous exponents.

Once we know how to settle the divergence problem through RG, many other questions can be answered. Specifically, the distribution function for the reunion point, the probability that out of p walkers any n can reunite at time N. It is also possible to study the thermodynamic limit where we want to know the probability of reunion of say two walkers in a finite density of particles. The last problem will, however, be discussed elsewhere.

This paper is arranged as follows. In Sec. II the model for p interacting walkers is posed. For convenience the relevant quantities to be evaluated and the procedure followed for the evaluation are formally presented here. In Sec. III the exponents for reunion probability are derived. Section IV discusses the distribution function for the reunion point, and the survival case is taken up in Sec. V. The general problem of n out of p walkers meeting at a point is discussed in Sec. VI. Section VII contains the description of the model for many walkers with many-body interaction and the solution for the reunion problem. Conclusions can be found in Sec. VIII. Mathematical details are presented in the two appendices.

II. MODEL

Since we are actually interested in the asymptotic behavior (large length scale limit), we adopt the continuum approach. Following the Edwards approach for conventional polymers [20], the Hamiltonian, in the path integral formalism, describing p DP's with mutual repulsive interaction is given by [9]

$$H_p = \tfrac{1}{2}\sum_{i=1}^{p}\int_0^N dz\,\left(\frac{\partial \mathbf{r}_i(z)}{\partial z}\right)^2 + v_0 \sum_{\substack{i,j \\ i>j}}\int_0^N dz\,\delta(\mathbf{r}_{ij}(z)), \tag{2.1}$$

where $\mathbf{r}_i(z)$ is the d-dimensional position vector of the ith chain at the contour length z measured along the chain. This z is the steplength for the random walker or the ordinary polymer formed after projection of the directed one in the transverse d-dimensional space. Here $\mathbf{r}_{ij}(z) = \mathbf{r}_i(z) - \mathbf{r}_j(z)$ and N is the total length of the polymer. The first term, which implies the chain connectivity, contributes the entropic part of the Gaussian chains. The second term causes the interaction of the chains through a mutual equal time δ-function repulsive interaction $(v_0 > 0)$. If we consider this Hamiltonian as that of particles (quantum or random walker [21]), then z plays the role of time. The first term produces the Wiener measure for random walks (or the kinetic energy of the quantum particles), while the second term is the interaction among the particles.

Interaction, as introduced here by a δ-function repulsion, allows intersection of the polymers though at a cost of finite energy. Apparently this cost of finite energy does not support, in the true sense, the required mutual avoidance which can be recovered only in the limit $v_0 \to \infty$. A simple dimensional analysis tells us that v_0 should always occur in the combination $v_0 N^{2-d}$, so that for $d < 2$ this quantity goes to infinity as $N \to \infty$, even for finite v_0. In the RG approach, this scaling limit is taken care of by introducing a renormalized coupling constant that approaches a nontrivial fixed point $(\neq 0)$ value in the large length scale limit. In other words, the RG approach shows that the finite energy cost does not affect the conclusions as long as we are interested in the macroscopic behavior of the system.

The quantity of interest is the weighted number or the partition function for p walkers to be reunited once they start together (i.e., $z = 0$) from some spatial point considered, for simplicity, to be the origin in the d-dimensional space. Formally this can be written as

$$Z_{R,p}(\mathbf{0}) = \int \mathcal{D}\mathbf{r}\; e^{-H_p} \prod_{i=1}^{p}[\delta^d(\mathbf{r}_i(0))\delta^d(\mathbf{r}_i(N))]. \tag{2.2}$$

Here $\int \mathcal{D}\mathbf{r}$ takes care of the sum over all possible paths in this constrained partition function. For simplicity the reunion is assumed to occur at the origin of the d-dimensional space. This partition function is basically the total weight of all possible walks starting from origin and reuniting again at the origin. The reunion at any arbitrary point $\mathbf{r}$, and reunion anywhere are described by the partition functions

$$Z_{R,p}(\mathbf{r}) = \int \mathcal{D}\mathbf{r}\; e^{-H_p} \prod_{i=1}^{p}[\delta^d(\mathbf{r}_i(0))\delta^d(\mathbf{r}_i(N) - \mathbf{r})],$$

and

$$\mathcal{Z}_{R,p} = \int d^d r Z_{R,p}(\mathbf{r}). \tag{2.3}$$

Without the end point constraint imposed by the δ function for $\mathbf{r}_i(N)$, the above equation yields the partition function $Z_{S,p}$ (survival probability). These partition functions, for large N, show power law decays like

$$Z_{g,p} \sim N^{-\psi_{g,p}}, \tag{2.4}$$

where g stands for R and S. These decays define the exponents $\psi_{R,p}$ and $\psi_{S,p}$. For $\mathcal{Z}_{R,p}$, reunion anywhere, the exponent is denoted by $\Psi_{R,p}$.

A. Noninteracting chains

A noninteracting Gaussian chain is described by the normalized partition function ("propagator")

$$G(\mathbf{r} \mid z) = (2\pi z)^{-d/2} \exp(-r^2/2z), \tag{2.5}$$

where $\mathbf{r}$ is the end to end spatial distance at length z along the chain. $Z_{R,p}(\mathbf{r})$, and $Z_{R,p}(\mathbf{0})$ are just products of such p propagators with same $\mathbf{r}$ for all the chains. Since G is normalized,

$$Z_{S,p} = \int \prod_i [d^d r_i G(\mathbf{r}_i \mid N)] = 1, \tag{2.6}$$

giving $\psi_{S,p} = 0$. The "Gaussian" exponents are, therefore,

$$\psi_{S,p} = 0, \ \psi_{R,p} = pd/2, \text{ and } \Psi_{R,p} = (p-1)d/2. \tag{2.7}$$

These exponents, though following from the free propagator Eq. (2.5), can also be obtained from the definitions of the partition functions, Eqs. (2.2) and (2.3), by dimensional analysis. These are therefore the cannonical dimensions of the partition functions. These "Gaussian" numbers are to be compared with the exact $d = 1$ results for interacting walkers [1]

$$\psi_{S,p} = \frac{p(p-1)}{4}, \ \psi_{R,p} = \frac{p^2}{2}, \text{ and } \Psi_{R,p} = \frac{p^2-1}{2}. \tag{2.8}$$

We write $\psi_{R,p} = pd/2 + \eta_p$ with η_p as the anomalous exponent. Since $\psi_{S,p} = 0$ for Gaussian chains, this exponent is the anomalous part by itself.

B. RG analysis

Renormalization group in statistical mechanics is used to understand the long distance behavior of a system in arbitrary dimensions. Since our interest is basically in the $N \to \infty$, the field theoretic renormalization group can be adopted [22].

The partition functions are evaluated by a perturbation expansion, done diagrammatically, in the coupling constant v_0. The coefficients are divergent at $d = 2$. These divergences are identified by evaluating the terms by analytic continuation in d (dimensional regularization). The systematic removal of these divergences requires a multiplicative renormalization constant for the whole series apart from the renormalization of the coupling constant. This multiplicative renormalization constant is the origin of the anomalous exponent η_p, reminiscent of what happens in, say, the ϕ^4 theory.

Let us give the general argument in support of the presence of the anomalous dimension from RG [22], taking $Z_{R,p}(\mathbf{0})$ as an example. We define a dimensionless coupling constant $u_0 = v_0 L^\epsilon$ where $\epsilon = 2 - d$ and L is an arbitrary length scale. The renormalized coupling constant u and the multiplicative renormalization constant $R_{R,p}(u)$ are defined in such a way that the renormalized partition function $Z_{R,p|r}(L, u, N) = R_{R,p}(u) Z_{R,p}(v_0, N)$ is finite and has a well defined $\epsilon \to 0$ limit. (We have, for the time being, suppressed $\mathbf{0}$ but made the dependence on the coupling constant, L, and N explicit.) The price we pay (or the bonus we gain) is the L dependence of $Z_{R,p|r}$.

The renormalization-group equation, that originates from the condition of L independence of the bare theory (i.e., $L\, \partial Z_{R,p}(v_0, N)/\partial L = 0$), is

$$\left(L\frac{\partial}{\partial L} + \beta(u)\frac{\partial}{\partial u} - 2\gamma_{R,p}(u) \right) Z_{R,p|r}(L, u(v_0, L), N) = 0, \tag{2.9}$$

where

$$\beta(u) \equiv L\frac{\partial u}{\partial L}\bigg|_{v_0} \text{ and } 2\gamma_{R,p}(u) = \beta(u)\frac{\partial}{\partial u} \ln R_{R,p}(u). \tag{2.10}$$

The solution of the above equation at the fixed point $u = u^*$, determined by $\beta(u^*) = 0$, has a form

$$Z_{R,p|r}(L, u^*, N) = L^{2\gamma^*} \Phi(N, u^*), \tag{2.11}$$

where $\gamma^* = \gamma_{R,P}(u^*)$. From the dimensional analysis argument and the above solution, the N dependence can be found out as

$$Z_{R,p|r}(L, u^*, N) \sim N^{-\Lambda - \gamma^*}, \tag{2.12}$$

where Λ is the cannonical dimension for the partition function. The factor γ^* (in the exponent of N) is completely an outcome of the renormalization-group analysis and is not predictable by the dimensional analysis argument. This is the anomalous dimension η_p. Let us repeat that an anomalous dimension can occur if and only if the quantity in question requires a multiplicative renormalization constant.

III. AN ENSEMBLE OF INTERACTING WALKERS

For a detailed investigation we first study the case of reunion (at origin) of interacting walkers. The partition function $Z_{R,p}(\mathbf{0})$, Eq. 2.2, is expanded perturbatively in the coupling constant v_0 [13, 14]. The two-chain problem can be solved exactly and is discussed in Ref. [15]. The diagrams up to second order in the perturbative series are shown in Fig. 1. The rules for evaluating the

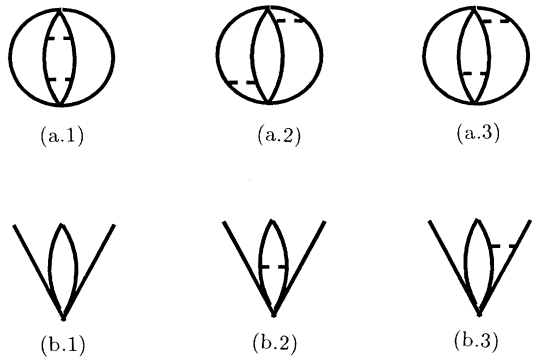

FIG. 1. (a.1)–(a.3) Second-order diagrams for reunion of p chains. For clarity, only four chains are shown. (b.1) Zeroth-order diagram for reunion of n chains with the remaining $p-n$ chains free at $z = N$. (b.2) and (b.3) Two possible first-order diagrams.

diagrams corresponding to the terms in the perturbative series are given below. The evaluation of the diagrams will be performed in detail in Appendix A.

A. Diagram rules

The rules for the diagrams are as follows. (i) Each solid line is identified as a directed polymer represented by the Gaussian propagator $G(\mathbf{r} \mid z)$ of Eq. 2.5. (ii) Each of the dashed lines represents the equal time δ-function interaction and contributes $-v_0$ to the expression. At the point of interaction, the two polymers have the same position and z coordinate. (iii) All the internal coordinates (spatial), i.e., the space coordinates of the points of interaction are to be integrated over. (iv) Integrations over the z coordinates of the interaction points are to be done. (v) Each diagram has a symmetry factor coming purely from combinatorics.

The two integrations of steps (iii) and (iv) are needed because the polymers can interact anywhere in space and at any point along the chain. The ordering of interaction points along the chain is to be preserved in doing the z integrals. Such a time ordering cancels out the factorial one gets from the expansion of the exponential.

All integrations are to be done by analytic continuation in d. A little reflection shows that the integrals are divergent for $d < 2$ (see Appendix A). In the analytically continued form, such divergences show up as poles at $d = 2$. A Laurent series expansion is done in $\epsilon = 2 - d$ to identify the poles.

B. Zero and first-order diagrams

With the above rules we now evaluate the diagrams for $Z_{R,p}(0)$. The first one is the case of no interaction among the walkers and its contribution is $(2\pi N)^{-pd/2}$. This gives the Gaussian exponent.

In the first order in v_0 there are two mutually interacting walkers other than $p-2$ Gaussian walkers. Following the prescription given above, its contribution is

$$-(2\pi N)^{-pd/2} v_0 N^{1-d/2} \binom{p}{2} (4\pi)^{-d/2} \frac{\Gamma^2(1-d/2)}{\Gamma(2-d)}$$
$$= -(2\pi N)^{-pd/2} \frac{u_0}{4\pi} \binom{p}{2} \left(\frac{4}{\epsilon} + 2\ln x + O(\epsilon) \right), \qquad (3.1)$$

where $u_0 = v_0 L^\epsilon$, $x = 4\pi N/L^2$, and as before $\epsilon = 2 - d$. Here the symmetry factor $\binom{p}{2}$ is to take care of all possible combinations of walkers forming the interacting pair.

C. Second-order diagrams

In the second order there are three different kinds of diagrams. The one in which the interaction forms a ladder type configuration [Fig. 1(a.1)] is simply the higher order extension of the first order one. The analytical expression for this is

$$v_0^2 (2\pi N)^{-(pd/2)} \binom{p}{2} (4\pi)^{-d} N^\epsilon \frac{\Gamma^3(\epsilon/2)}{\Gamma(3\epsilon/2)}$$
$$= (2\pi N)^{-pd/2} \left(\frac{u_0}{4\pi} \right)^2 \binom{p}{2} \left(\frac{12}{\epsilon^2} + \frac{12}{\epsilon} \ln x + O(1) \right). \qquad (3.2)$$

There is another diagram in the second order that has two separate mutually interacting pairs [Fig. 1(a.2)]. The contribution of this diagram (in a way square of the first order one) to the perturbation series with the proper symmetry factor is given by

$$v_0^2 (2\pi N)^{-(pd/2)} \tfrac{1}{2} \binom{p}{2} \binom{p-2}{2} N^{2-d} (4\pi)^{-d} \frac{\Gamma^4(1-d/2)}{\Gamma^2(2-d)}$$
$$= (2\pi N)^{-pd/2} \left(\frac{u_0}{4\pi} \right)^2 \binom{p}{2} \binom{p-2}{2} \left(\frac{8}{\epsilon^2} + \frac{8}{\epsilon} \ln x + O(1) \right). \qquad (3.3)$$

Double counting is avoided by dividing the symmetry factor by 2.

The most crucial contribution comes from the diagram that involves three chains connected by the interaction [Fig. 1(a.3)]. Evaluation of such diagrams requires a bit of technicality. The details are relegated to Appendix A. The final expression for this connected diagram is given by

$$6 v_0^2 (2\pi N)^{-pd/2} \binom{p}{3} N^\epsilon (4\pi)^{-d} \left[\frac{\Gamma^3(\hat{\epsilon})}{\Gamma(3\hat{\epsilon})} \, {}_3F_2(\hat{\epsilon}, 1-\hat{\epsilon}, \hat{\epsilon}; 3\hat{\epsilon}, 1-\hat{\epsilon}; 3/4) \right.$$
$$\left. + \left(\frac{3}{4} \right)^{\hat{\epsilon}} \frac{\Gamma(-\hat{\epsilon})}{\Gamma(1-\hat{\epsilon})} \frac{\Gamma^2(\epsilon)}{\Gamma(2\epsilon)} \, {}_3F_2(\epsilon, \epsilon, 1; 2\epsilon, 1+\hat{\epsilon}; 3/4) \right]. \qquad (3.4)$$

where $\hat{\epsilon} = \epsilon/2$, and ${}_3F_2$ is the generalized hypergeometric function [23, 24]. Extracting the poles in ϵ requires a careful handling of the singularities of the hypergeometric functions. The technicalities can be found in Appendix A. The relevant ϵ expansion is

$$(2\pi N)^{-pd/2}\left(\frac{u_0}{4\pi}\right)^2 6\binom{p}{3}\left(\frac{8}{\epsilon^2}-\frac{2}{\epsilon}\ln\frac{3}{4}+\frac{8}{\epsilon}\ln x+O(1)\right). \tag{3.5}$$

D. Partition function

Combining all the terms, we obtain the following perturbative series in terms of the dimensionless coupling constant u_0:

$$\frac{Z_{R,p}}{(2\pi N)^{-pd/2}} = 1 - u_0\binom{p}{2}\left(\frac{1}{\pi\epsilon}+\frac{\ln x}{2\pi}\right) + u_0^2\left(\frac{C}{4\pi^2\epsilon^2}-\frac{3\binom{p}{3}}{4\pi^2\epsilon}\ln(3/4)+\frac{C}{4\pi^2\epsilon}\ln x\right) + \dots, \tag{3.6}$$

where $C = \binom{p}{2}[3+(p-2)(p+1)]$.

The removal of the divergences requires renormalization of the coupling constant used in the bare theory. Such a renormalization had already been done in connection with previous studies of virial coefficients of directed polymers [13, 14]. The second virial coefficient is related to the connected partition function for two chains with the same Hamiltonian as in Eq. (2.1) but with all ends free. Since it is the same system with identical two body interaction, the renormalization of the coupling constant remains the same. We, therefore, straightaway quote the series for the coupling constant from Refs. [13, 14] as

$$u_0 = u(1 + a_1 u + a_2 u^2 + \cdots) \text{ with } a_p = (2\pi\epsilon)^{-p}, \forall p. \tag{3.7}$$

This renormalization of the coupling constant is necessary but not sufficient to remove the divergences of the partition function in Eq. (3.6). The presence of divergence in the very first order term in u_0 indicates that an overall multiplicative renormalization constant, given by the series

$$R_{R,p} = 1 + b_1 u + b_2 u^2 + \cdots, \tag{3.8}$$

has to be introduced. Since the divergence of $Z_{R,p}$ in the first order in u_0 is not touched by the renormalization of u, b_1 can be obtained without much ado by the requirement of the minimal subtraction of the pole. Replacing u_0 by u, we get

$$b_1 = \binom{p}{2}(\pi\epsilon)^{-1}. \tag{3.9}$$

In the next order, u_0 is to be replaced by the series in terms of u, Eq. (3.7), and demand that R be such that the poles are removed minimally. This gives

$$b_2 = \frac{\binom{p}{2}(p^2-p+1)}{4\pi^2\epsilon^2}+\frac{3\binom{p}{3}}{4\pi^2\epsilon}\ln(3/4). \tag{3.10}$$

We are now in a position to evaluate the β and the γ functions of Eq. (2.10). The β function, which physically means the variation of the coupling constant with the macroscopic length L, can be formally evaluated as

$$\beta(u) = u\epsilon\,[1 - u/(2\pi\epsilon)]. \tag{3.11}$$

This β function is known exactly. The flow of the coupling constant with the change in the length scale can be studied by looking at the stability of the fixed point $u^* = 2\pi\epsilon$ in different dimensions [13, 14]. Following the prescription of Sec. III A the anomalous exponent can be computed from the multiplicative renormalization constant. Using Eq. (2.10)

$$\gamma_{R,p}(u) = \binom{p}{2}\frac{u}{2\pi} + 3\binom{p}{3}\ln(3/4)\frac{u^2}{4\pi^2} + O(u^3), \tag{3.12}$$

which at the stable fixed point gives the anomalous dimension up to $O(\epsilon^2)$ as

$$\eta_p \equiv \gamma_{R,p}(u^*) = \binom{p}{2}\epsilon + 3\binom{p}{3}\ln(3/4)\,\epsilon^2 + O(\epsilon^3). \tag{3.13}$$

An important feature is that the three chain connected diagram in Fig. 1(a.3) is only responsible for the $O(\epsilon^2)$ contribution in $\eta_{R,p}$. The other second order processes where two chains talk to each other pairwise are important for renormalizability, but do not contribute to the exponent. For second order, *three chains should collectively be aware of their existences.* We believe that this collective feature will be carried over in higher orders also. The exponents are

$$\psi_{R,p} = pd/2 + \eta_p, \text{ and } \Psi_{R,p} = (p-1)d/2 + \eta_p \tag{3.14}$$

with η_p given by Eq. (3.13).

E. Digression on RG

Beyond the simple algebra, there is still something that is of significance and can also be used as a method for cross checking or self-consistency. Using the β function of Eq. (3.11), and Eqs. (2.10) and (3.8), $\gamma_{R,p}(u)$ can be written formally as

$$2\gamma_{R,p}(u) = \epsilon u[b_1 + (2b_2 - b_1(2\pi\epsilon)^{-1} - b_1^2)u + ..]. \tag{3.15}$$

In general, b_i's are expected to have expansions of the type

$$b_i = \sum_{p=1}^{i}\frac{b_{ip}}{\epsilon^p} \tag{3.16}$$

because of the condition of minimal subtraction of poles. Now, the finiteness condition of $\gamma_{R,p}(u)$ as $\epsilon \to 0$ puts stringent constraint on the coefficients b_{ip}. For example, substituting the expansions for b_1 and b_2 in Eq. (3.15), we require

$$2b_{22} - b_{11}^2 - (2\pi)^{-1}b_{11} = 0 \tag{3.17}$$

for the coefficient of $O(\epsilon^{-2})$ term to vanish. This leaves behind only $O(\epsilon^{-1})$ coefficient that can stay in the $O(u)$ term in the square brackets of Eq. (3.15)— remember the ϵ outside in this equation. In other words, a consequence of the restriction of finiteness of $\gamma_{R,p}(u)$ is that only $O(\epsilon^{-1})$ terms can survive in the coefficient of each order of u in Eq. (3.15). This, furthermore, guarantees that to find out η to $O(\epsilon^2)$ it is sufficient to know only the coefficient of $O(u^2)$ term, and so on for higher orders. We can now turn the table around and use the finiteness criterion to predict or check the next term in $R_{R,p}$. The renormalizability of a model, therefore, means that the leading divergence in ϵ at a particular order in the perturbation series has to be completely determined by the lower order terms in the series. The universality class, as governed by the exponents, is determined only by the $O(\epsilon^{-1})$ residues.

Of course, the above identity is satisfied by b_1 and b_2 of Eqs. (3.9) and (3.10).

IV. DISTRIBUTION FUNCTION

An important quantity is the distribution function, $Z_{R,p}(\mathbf{r})$, for the reunion point of p interacting walkers. This can also be thought of as the propagator, in a combined fashion, for p interacting walkers meeting at $\mathbf{r}$.

Had the walkers been noninteracting, the partition function is just $G^p(\mathbf{r} \mid N)$. The correction up to first order in the coupling constant can be obtained with slight modification of the steps for the evaluation of the partition function $Z_{R,p}(\mathbf{0})$. The only change required is the replacement of the last part of the propagators in the evaluation of a diagram by $G(\mathbf{r}-\mathbf{r}_1 \mid N-z_1)$ which keeps $\mathbf{r}$. It is straightforward to show that the divergence in the first order is identical to that of $Z_{R,p}(\mathbf{0})$. So, to $O(u)$ the renormalization constant $R_{R,p}(u)$ will be determined by the same b_1 as given in Eq. (3.9). Actually, it is expected that the renormalization constant would be the same for $Z_{R,p}(\mathbf{0})$ and $Z_{R,p}(\mathbf{r})$ to all orders. The simplicity of the first order graph is that the $\mathbf{r}$ dependence comes out as a $G(\mathbf{r} \mid N)$ as for the zeroth order term. This is a feature that does not survive in higher order connected diagrams. We have already seen that to first order no renormalization of the coupling constant is required. No harm is, however, done by replacing the coupling constant by u. We skip the details. The renormalized partition function is given by

$$Z_{R,p|r}(\mathbf{r}) = G^p(\mathbf{r} \mid N)\left[1 - \binom{p}{2}\frac{u}{2\pi}\ln\frac{4\pi N}{L^2}\right]. \tag{4.1}$$

Reexponentiating the logrithmic term, we obtain at the fixed point $u = u^* = 2\pi\epsilon$,

$$Z_{R,p|r}(\mathbf{r}) = (2\pi N)^{-pd/2}(4\pi N L^{-2})^{-\binom{p}{2}\epsilon}\exp\left(-\frac{pr^2}{2N}\right), \tag{4.2}$$

a Gaussian function modified by the anomalous exponent. This $O(\epsilon)$ form is exact for two chains ($p = 2$) for all $\epsilon > 0$. From exact results of Ref. [2], we see that this is also exact for all p at $d = 1$.

We have not attempted to go to second order mainly because of the complexity of the diagram of Fig. 1(a.3). It is, however, easy to check that $\mathbf{r}$ dependence is no longer in the simple exponential form. Previous results on the virial coefficients showed that the $\mathbf{r}$ does not require any anomalous dimension, neither does N, essentially because of the absence of any self-interaction. We, therefore, expect a scaling form

$$Z_{R,p|r}(\mathbf{r}) \sim N^{-(pd/2+\eta_p)}F(\mathbf{r}/N^{1/2}), \tag{4.3}$$

where η_p is given by Eq. (3.13). This immediately gives the result for $\Psi_{R,p}$ as quoted in Eq. (3.14).

V. SURVIVAL PROBABILITY

Survival probability as defined in Sec. II is the total weight of all possible configurations of the interacting walkers originating from the origin but free at the other end. Because of this, the diagrammatics also go through necessary changes—in fact, the diagrams are much easier to calculate.

The zeroth order diagram is simply unity because of the normalized propagator. Similarly, the contribution from the nonintersecting chains is also unity in any diagram. The procedure is identical to the reunion case. We just give below the renormalization constant

$$R_{S,p}(u) = 1 + \binom{p}{2}\frac{u}{2\pi\epsilon} + \binom{p}{2}\frac{p^2-p+2}{4}\frac{u^2}{(2\pi\epsilon)^2} + \frac{3}{2}\binom{p}{3}\frac{u^2}{(2\pi)^2\epsilon}\ln\frac{3}{4} + O(u^3). \tag{5.1}$$

The coefficients again satisfy the condition of Eq. (3.17). The crucial point to note is that $R_{R,p} = R_{S,p}^2$ and hence

$$\psi_{S,p} = \eta_{S,p} = \eta_p/2. \tag{5.2}$$

Using the critical exponents $\Psi_{R,p}$ and $\psi_{S,p}$ it was previously found in Ref. [17] that the critical number of interacting walkers (p_c) that are sure to meet is 2 for $d \leq 2$ up to $O(\epsilon^2)$ [25].

The above scaling relation, especially the connection between the two renormalization constants, has actually been proved to $O(u^2)$. We believe that this is true to all orders. One way of justifying this is to cut the diagrams of the reunion case at the middle to produce two diagrams of the survival type. The reunion diagram is then a product of the two "survival" type diagrams with an integration over the glued points. This remains to be established yet. A more significant outcome would be that one can associate $R_{S,p}$ as the renormalization constant for a "vertex" from where p chains emanate.

VI. REUNION PROBABILITY FOR A SUBSET OF WALKERS

So far we have been considering the situation where all the chains meet. A variation on this theme is a case where, as before, p walkers start together at time 0 but we want to know the probability of mating of any 2. Complicacies arise here because, before their reunion, each might have interacted with any of the remaining $p-2$

chains. The difference comes out even in the first order in ϵ and we stop there.

The perturbative expansion in v_0 has the diagrams as shown in Figs. 1(b.1)–(1b.3), in which the third one is the important one. Its contribution is

$$(2\pi N)^{-d} C_2 \frac{u_0}{4\pi} x^{\epsilon/2} \frac{\Gamma(\epsilon/2)}{\Gamma(1+\epsilon/2)} \times {}_2F_1(1-\epsilon/2, \epsilon/2; \epsilon/2+1; 1/2), \quad (6.1)$$

where the combinatorial factor $C_2 = 2\binom{p}{2}(p-2)$. When combined with the other terms, the partition function $Z_{R,p,2}(\mathbf{0})$ is given by

$$Z_{R,p,2}(\mathbf{0}) = (2\pi N)^{-d}\binom{p}{2}\left(1 - \frac{(p-1)u_0}{\pi\epsilon}\right). \quad (6.2)$$

Since the coefficient of the $O(u)$ term of the multiplicative renormalization constant is identically equal to the magnitude of the coefficient of $O(u)$ term in the above series, the former is given by

$$1 + \frac{p-1}{\pi\epsilon}u. \quad (6.3)$$

The anomalous exponent can be evaluated using Eqs. (2.10), (3.11), and Eq. (6.3) and it is

$$\eta = \epsilon(p-1) + O(\epsilon^2). \quad (6.4)$$

This agrees with the result of Eq. (3.13) for $p = 2$. It is possible to go to higher orders, and unlike the $p = 2$ case, the exponent does not stop at $O(\epsilon)$.

An extension of this analysis to meeting of $n(< p)$ walkers is rather trivial. The only change required is the replacement of $(2\pi N)^{-d}$ and C_2 by $(2\pi N)^{-nd/2}$ and $C_n = \binom{p}{n} n(p-n)$, respectively. The renormalization constant turns out to be $1+n(p-1)u/(2\pi\epsilon)+O(u^2)$ yielding

$$\eta = \epsilon n(p-1)/2 + O(\epsilon^2), \quad (6.5)$$

which agrees with Eq. (6.4) for $n = 2$ and with Eq. (3.13) for $n = p$ to $O(\epsilon)$.

VII. REUNION WITH MANY-BODY INTERACTION

This is the case where only m number of chains can have repulsive interaction. There is no interaction at points of encounters of less than m chains. Here also the aim is to evaluate the exponent $\psi_{R,p}$. $\psi_{s,p}$ follows from $\psi_{R,p}$ as discussed in Sec. V. For simplicity, we use the same notation as for the $m = 2$ case. The Hamiltonian is now given by

$$H_p = \tfrac{1}{2}\sum_{i=1}^{p}\int_0^N dz \left(\frac{\partial \mathbf{r}_i(z)}{\partial z}\right)^2 + v_m \sum_{\{i_j\}}\int_0^N dz \prod_{j=1}^{m-1}\delta(\mathbf{r}_{i_j i_{j+1}}(z)), \quad (7.1)$$

where the summation is over all possible m membered sets from p chains. From dimensional analysis using the dimensionless Hamiltonian it is transparent that the coupling constant v_m becomes dimensionless at $d = d_m = 2/(m-1)$ which is the upper critical dimension for this multicritical problem [18, 19]. To derive an $\epsilon_m = 2-(m-1)d \equiv (d_m - d)(m-1)$ expansion for the anomalous exponent we again perform perturbation expansion in the coupling constant v_m. As before dimensional regularization is performed to identify the poles at the upper critical dimension d_m.

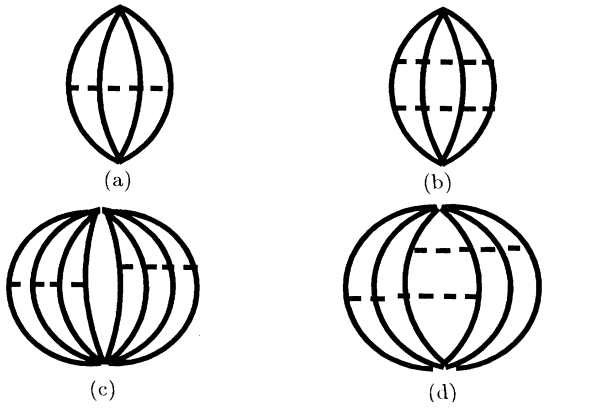

FIG. 2. (a) First-order diagram for m-body interaction. (b) and (c) Two ladder type second-order diagrams. (d) A nonladder diagram involving $m+n$ chains. Here $m = 4, n = 2$. For clarity, noninteracting chains are not shown. The dashed lines represent m-body interactions.

The generalized approach follows identically the procedure for p interacting walkers, i.e., the two-body interaction case. Not much technicality is involved in the evaluation of the first order diagram in v_m [Fig. 2(a)]. In higher orders there are complexities arising out of the connected diagram that appears first in the second order. As shown in Figs. 2(b)–2(d), in the second order, the important diagram involves $m+n$ chains with n going from 1 to $n_{\max} = \min(m-1, p-m)$. At the end we need to do a summation over all possible values of n. Finally, the result can be verified with the known results for $m = 2$. Another elegant check is to use the convergence criteria for the anomalous exponent as a tool and observe whether the magic cancellation, mentioned before Eq. (3.17), occurs here also, as it should.

A. Ladder diagrams

We now proceed to the evaluation of the ladder type graphs [Figs. 2(a)–2(c)] corresponding to the perturbation series. The first order graph with m chains and the two graphs in the second order involving m and $2m$ chains, respectively, do not require any new computation. The basic structure of these graphs is identical to those of Sec. III, except for the replacement of the two interacting chains represented by $G^2(\mathbf{r} \mid z)$ by $G^m(\mathbf{r} \mid z)$ [see Appendix B]. We shall skip the details and state the results straightaway with proper symmetry factors. The contribution from the first order diagram is

$$-\binom{p}{m}\frac{v_m}{4\sigma}(2\pi N)^{-pd/2}(4\sigma N)^{\epsilon_m/2}\frac{\Gamma^2(\epsilon_m/2)}{\Gamma(\epsilon_m)}, \qquad (7.2)$$

where $\sigma = \pi m^{1/(m-1)}/2$, which is π for $m = 2$.

The second order graph, in which two independent groups of m walkers are connected by interaction [Fig. 2(c)], gives

$$\tfrac{1}{2}\binom{p}{m}\binom{p-m}{m}\left(\frac{v_m}{4\sigma}\right)^2(2\pi N)^{-pd/2}(4\sigma N)^{\epsilon_m}\frac{\Gamma^4(\epsilon_m/2)}{\Gamma^2(\epsilon_m)}.$$

The ladder type diagram [Fig. 2(b)] in the second order involving a group of m chains yields

$$\binom{p}{m}\left(\frac{v_m}{4\sigma}\right)^2(2\pi N)^{-pd/2}(4\sigma N)^{\epsilon_m}\frac{\Gamma^3(\epsilon_m/2)}{\Gamma(3\epsilon_m/2)}.$$

B. Nonladder diagrams

There is one more graph in the second order which involves $m+n$ chains. Details of its derivation are given in Appendix B. This graph ends up with identical difficulty related to the convergence of hypergeometric functions as in the case of vicious walkers in Sec. III C and Appendix A. With some special care, this problem can be tackled to extract the leading singularities.

The $O(\epsilon_m^{-2})$ term turns out to be n independent and is equal to $8\epsilon_m^{-2}$. The summation over n is then just a sum over the combinatorial factors as

$$\sum_{n=1}^{n_{\max}}\binom{p}{m+n}\binom{m+n}{n}\binom{m}{n} = \binom{p}{m}^2 - \binom{p}{2m}\binom{2m}{m} - \binom{p}{m}, \qquad (7.3)$$

where the first term on the right-hand side is just the free combinatorics for the two vertices while the last two terms subtract out the two types of second order ladder graphs discussed above. The cancellation of the coefficient of $O(\epsilon_m^{-2})$ in $\gamma_{R,p}(u)$ now turns out to be obvious.

From the previous RG analysis after Eq. (3.15) it is clear that the contributing parts to the exponent are basically the ϵ_m^{-1} terms in the second order diagrams. As we expect, the $O(\epsilon_m^{-1})$ term comes from the connected nonladder diagram. The magnitude of this residue is also the coefficient b_{21} of the renormalization constant. The $O(\epsilon_m^{-1})$ term of the coefficient of u^2 is $-r_{pm}/(4\sigma^2\epsilon_m)$ where

$$r_{pm} = \tfrac{1}{2}m(m+1)\binom{p}{m+1}\left[\gamma + \psi\left(\frac{1}{m-1}\right) + \ln\left(1-\frac{1}{m^2}\right)\right] + \tfrac{1}{2}\sum_{n=2}^{n_{\max}}\binom{p}{m+n}\binom{m+n}{n}\binom{m}{n}\left[\gamma + \psi\left(\frac{n-1}{m-1}\right) - \mathcal{R}_1(n,m) - \mathcal{R}_2(n,m)\right], \qquad (7.4)$$

where γ is the Euler gamma, $\psi(\,)$ is the polygamma function,

$$\mathcal{R}_1(n,m) = \frac{m+n}{m^2}\,{}_3F_2\left(1,1,\frac{m}{m-1};2,2-\frac{n-1}{m-1};1-\frac{n^2}{m^2}\right), \qquad (7.5a)$$

$$\mathcal{R}_2(n,m) = B\left(\frac{1-n}{m-1},\frac{n}{m-1}\right)\left(1-\frac{n^2}{m^2}\right)^{\frac{n-1}{m-1}} \times {}_2F_1\left(\frac{n-1}{m-1},\frac{n}{m-1};1+\frac{n-1}{m-1};1-\frac{n^2}{m^2}\right). \qquad (7.5b)$$

For $m = 2$ we get back the interacting walker results because of the fact that $\psi(1) = -\gamma$.

The renormalization of the coupling constant is, again, known from virial coefficients [18, 19]. Taking u_m as the renormalized coupling constant (defined as $v_m L^{\epsilon_m}$), it is given by a series of the type Eq. (3.7) with $a_p = (2\sigma\epsilon_m)^{-p}$, and the β function is given by Eq. (3.11) with $u/2\pi$ replaced by $u/2\sigma$ and ϵ by ϵ_m. As a matter of fact, once we are satisfied with the cancellation of the $O(\epsilon_m^{-2})$ term, the details of renormalization need not be carried out. The anomalous exponent follows as

$$\eta_p = \binom{p}{m}\epsilon_m + \mathsf{r}_{pm}\,\epsilon_m^2 + O(\epsilon_m^3). \qquad (7.6)$$

For $p = m$, the exponent stops at $O(\epsilon_m)$ [16], and, in fact, like the $p = m = 2$ case, the RG is exact.

C. $d = d_m$

At $d = d_m$, the fixed point diagram and the stability analysis show that the coupling becomes marginally irrelavant. The integration of the β function tells us the length dependence of u, and at $d = 2$, we find

$$u_m(L) = \frac{u_0}{1 + (u_0/2\sigma)\ln(L/L_0)}, \qquad (7.7)$$

where $u_0 = u_m(L_0)$. For large L, u_m decays as $(\ln L/L_0)^{-1}$. The RG equation then produces

$$Z_{R,p} \sim N^{-p/(m-1)}[u_0\ln(N/N_0)]^{-\binom{p}{m}}. \qquad (7.8)$$

Even though the coupling goes to zero in the long length scale regime, the behavior is not that of a noninteracting case. The logarithmic correction, which does not affect the Gaussian power law, is the remanent of the repulsive interaction.

For m=2, the result agrees with Ref. [17], and also with the exact result for $p = m$ [16]. An interesting consequence is the logarithmic correction for p walkers

in 1+1 dimension because $d_m = 1$ for $m = 3$. The RG prediction that follows from the β function and γ is $Z_{R,p} \sim N^{-p/2}(\ln N)^{-\binom{p}{3}}$. This could be checked easily on a lattice numerically.

VIII. SUMMARY

Using the analogy with directed polymers, we have obtained the decay exponents of the reunion and survival probablilities or the partition functions of the interacting DP's, to second order in $\epsilon = 2 - d$ for two-body interaction. This has also been generalized to multicritical many-body interaction for which the expansion parameter is $\epsilon_m = (m-1)(d_m - d)$, where $d_m = 2/(m-1)$. We expect in this case, as explained in Sec. IV, the survival exponent to be half of the reunion one of Eq. (7.6). For the two-body interaction we have also studied the distribution function and the question of reunion of a subset of walkers. These are done to $O(\epsilon)$. For the subset reunion case, the $O(\epsilon)$ correction does not enjoy a simple combinatorial interpretation unlike the case of reunion of all the chains. Our analysis also yields the logarithmic corrections to the exponents at the upper critical dimension d_m. Explicit results are given for two-body interaction in two dimensions and three-body interaction in one dimension.

It would be interesting to get these exponents on fractals. Since our procedure can be extended without much difficulty to higher orders, this gives a rare opportunity of comparing results of ϵ expansion based on the idea of analytic continuation to nonintegral dimensions and on fractals which are well defined nointegral dimensional entities.

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APPENDIX A: EVALUATION OF A FEW DIAGRAMS FOR INTERACTING WALKERS

In this Appendix we evaluate the diagrams shown in Fig. 1 with necessary details. The mathematical expression corresponding to the first order diagram, as per the rules of Sec. III A, is

$$J_1 = -v_0(2\pi N)^{-(p-2)d/2}\int_0^N dz_1 \int dr_1\, G^2(\mathbf{r}_1 \mid N - z_1) \times G^2(\mathbf{r}_1 \mid z_1). \quad \text{(A1)}$$

The symmetry factor $\binom{p}{2}$ which counts all possible pairings is not considered. This expression can be simplified by using the following identity for the Gaussian propagators:

$$G^p(\mathbf{r} \mid z) = (2\pi z)^{-(p-1)d/2} p^{-d/2} G(\mathbf{r} \mid z/p). \quad \text{(A2)}$$

We use the Markovian property for the propagators

$$\int dr\, G(\mathbf{r}_1 - \mathbf{r} \mid z_1) G(\mathbf{r} - \mathbf{r}_2 \mid z_2) = G(\mathbf{r}_1 - \mathbf{r}_2 \mid z_1 + z_2) \quad \text{(A3)}$$

to perform the integration over the space coordinate in J_1. Finally we are left with the task of evaluating the integration over length z as

$$\begin{aligned} J_1 = &- v_0(2\pi N)^{-(p-2)d/2} G(0 \mid N/2)(4\pi)^{-d} \\ &\times \int_0^N dz_1 (N - z_1)^{-d/2} z_1^{-d/2} \\ &= -v_0(2\pi N)^{-pd/2} N^{1-d/2}(4\pi)^{-d/2} \frac{\Gamma^2(1 - d/2)}{\Gamma(2-d)}, \end{aligned} \quad \text{(A4)}$$

as quoted in Eq. (3.1).

Next, we evaluate the ladder type diagram involving only a pair of walkers with two succesive encounters. This can be written as

$$J_2 = v_0^2(2\pi N)^{-(p-2)d/2}\int_0^N dz_1 \int_0^{z_1} dz_2 \int d\mathbf{r}_1 \int d\mathbf{r}_2\, G^2(\mathbf{r}_1 \mid N - z_1) G^2(\mathbf{r}_1 - \mathbf{r}_2 \mid z_1 - z_2) G^2(\mathbf{r}_2 \mid z_2). \quad \text{(A5)}$$

Performing the spatial integration over $\mathbf{r}_1$ and $\mathbf{r}_2$ using Eq. (A2) and the Markovian property we obtain

$$J_2 = v_0^2(2\pi N)^{-(p-2)d/2}(4\pi)^{-3d/2} G(\mathbf{0} \mid N/2) \int_0^N dz_1 \int_0^{z_1} dz_2 (N - z_1)^{-d/2}(z_1 - z_2)^{-d/2} z_2^{-d/2}, \quad \text{(A6)}$$

which finally gives Eq. (3.2).

Now let us turn to the third diagram which connects three chains. This diagram is

$$\begin{aligned} J_3 \equiv v_0^2(2\pi N)^{-pd/2} N^{2-d}(4\pi)^{-d} J = v_0^2 G^{p-3}(0 \mid N) \int_0^N dz_1 \int_0^{z_1} dz_2 \int d\mathbf{r}_1 \int d\mathbf{r}_2 \\ \times G^2(\mathbf{r}_1 \mid N - z_1) G(\mathbf{r}_{12} \mid z_{12}) G(\mathbf{r}_1 \mid z_1) G(\mathbf{r}_2 \mid N - z_2) G^2(\mathbf{r}_2 \mid z_2), \end{aligned} \quad \text{(A7)}$$

where $z_{12} = z_1 - z_2$. After completion of the spatial integration using the identities of Eqs. (A2) and (A3), we are left with integrations over z_1 and z_2, a bit complicated because of the connected nature of the graph. We obtain

$$J = \int_0^1 dz_1 (1-z_1)^{-d/2} z_1^{1-d} \int_0^1 dz_2 \; z_2^{-d/2} [1-(3z_1+1)z_2/4]^{-d/2}. \tag{A8}$$

The integration over z_2 is in the form of the Euler representation of the hypergeometric function so that

$$J = \frac{\Gamma(1-d/2)}{\Gamma(2-d/2)} \int_0^1 dz_1 (1-z_1)^{-d/2} z_1^{1-d} {}_2F_1\left(\frac{d}{2}, 1-\frac{d}{2}; 2-\frac{d}{2}; \frac{3z_1+1}{4}\right). \tag{A9}$$

Before proceeding further one should be cautious. Eventhough it looks tempting to substitute the series expansion for the hypergeometric function and perform integration term by term, the problem at the upper limit of the integration should be noted. The series expansion for the hypergeometric function is not valid when the variable becomes unity. This problem can be bypassed by analytic continuation. Exploiting the following transformation formula for the hypergeometric function [24]:

$$\begin{aligned} {}_2F_1(a,b;c;z) = {} & \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1(a,b;a+b-c+1;1-z) \\ & + (1-z)^{c-a-b} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} {}_2F_1(c-a,c-b;c-a-b+1;1-z), \end{aligned} \tag{A10}$$

which renders the argument always less than unity through out the domain of integration, in Eq. (A9), the remaining z_1 integrations can be carried out. The final expression is

$$J = \frac{\Gamma^3(\epsilon/2)}{\Gamma(3\epsilon/2)} {}_3F_2\left(\frac{\epsilon}{2}, 1-\frac{\epsilon}{2}, \frac{\epsilon}{2}; \frac{3\epsilon}{2}, 1-\frac{\epsilon}{2}; \frac{3}{4}\right) + \left(\frac{3}{4}\right)^{\epsilon/2} \frac{\Gamma(-\epsilon/2)}{\Gamma(1-\epsilon/2)} \frac{\Gamma^2(\epsilon)}{\Gamma(2\epsilon)} {}_3F_2\left(\epsilon, \epsilon, 1; 2\epsilon, 1+\frac{\epsilon}{2}; \frac{3}{4}\right), \tag{A11}$$

where $\epsilon = 2-d$ and ${}_3F_2$ is the generalized hypergeometric function [23]. The reward, after all this, is the immediate retrieval of the $O(\epsilon^{-2})$ term. The subleading term is also not much of a problem now if the limiting forms

$${}_3F_2\left(\frac{\epsilon}{2}, 1-\frac{\epsilon}{2}, \frac{\epsilon}{2}; \frac{3\epsilon}{2}, 1-\frac{\epsilon}{2}; \frac{3}{4}\right) = 1 + \frac{\epsilon}{6}\ln 4 + O(\epsilon^2) \tag{A12a}$$

and

$${}_3F_2\left(\epsilon, \epsilon, 1; 2\epsilon, 1+\frac{\epsilon}{2}; \frac{3}{4}\right) = 1 + \frac{\epsilon}{2}\ln 4 + O(\epsilon^2), \tag{A12b}$$

obtained from the series expansion, are used. The final result is given in Eq. (3.6).

APPENDIX B: CONNECTED DIAGRAM OF FIG. 2

This graph amounts to the following expression:

$$\begin{aligned} & v_m^2 G^{(p-m-n)d/2}(0 \mid N) \int_0^N dz_1 \int_0^{z_1} dz_2 \int d\mathbf{r}_1 \int d\mathbf{r}_2 [G^m(\mathbf{r}_1 \mid N-z_1) \\ & \qquad \times G^{m-n}(\mathbf{r}_{12} \mid z_{12}) G^n(\mathbf{r}_2 \mid N-z_2) G^n(\mathbf{r}_1 \mid z_1) G^m(\mathbf{r}_2 \mid z_2)], \end{aligned} \tag{B1}$$

where $\mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2$ and $z_{12} = z_1 - z_2$. The identity in Eq. (A2) reduces the powers of the propagators to unity. We furthermore use the following relation:

$$G(\mathbf{r} \mid a) G(\mathbf{r} \mid b) = [2\pi(a+b)]^{-d/2} G\left(\mathbf{r} \mid \frac{ab}{a+b}\right), \tag{B2}$$

which leads to the following form for the integrand in Eq. (B1):

$$\begin{aligned} & (2\pi)^{-d(3m+n-3)/2} (m-n)^{-d/2} [(N-z_1)z_2]^{-d(m-1)/2} [(N-z_2)z_1]^{-d(n-1)/2} \\ & \qquad \times (z_{12})^{-d(m-n-1)/2} (\mu\nu)^{d/2} G\left(\mathbf{r}_1 \mid (N-z_1)z_1\mu\right) G\left(\mathbf{r}_{12} \mid \frac{z_{12}}{m-n}\right) G(\mathbf{r}_2 \mid (N-z_2)z_2\nu\Big), \end{aligned} \tag{B3}$$

where

$$\mu^{-1} = Nn - z_1 n + m z_1,$$

and

$$\nu^{-1} = mN - mz_2 + nz_2.$$

Use of the Markovian property to perform the integration over the spatial coordinates $\mathbf{r}_1$ and $\mathbf{r}_2$ with rescaling of z_1, z_2 by N, produces

$$(2\pi N)^{-pd/2}\frac{v_m^2}{16\sigma^2}(4\sigma N)^{\epsilon_m} I, \tag{B4}$$

where $\sigma = \pi m^{1/(m-1)}/2$ and

$$I = \int_0^1 dz_1(1-z_1)^{-d(m-1)/2} z_1^{1-(m-1)d} \int_0^1 dz_2(1-z_2)^{-d(m-n-1)/2} z_2^{-d(m-1)/2} \times (1-z_2z_1)^{-d(n-1)/2}[1 - z_2(n^2 + m^2z_1 - n^2z_1)/m^2]^{-d/2}. \tag{B5}$$

The integration over z_2 can be performed easily using the standard Euler type formula to get [23]

$$I = A\int_0^1 dz_1(1-z_1)^{-d(m-1)/2} z_1^{1-(m-1)d} F_1\left(\frac{\epsilon_m}{2}, \bar{n}, \frac{d}{2}; \epsilon_m + n\frac{d}{2}; z_1, \hat{z}_1\right), \tag{B6}$$

where F_1 is the first Appell function (multiple hypergeometric function) of two variables [24, 26],

$$A = B[\epsilon_m/2, (\epsilon_m + nd)/2],\ \bar{n} = (n-1)d/2,$$

and

$$\hat{z}_1 = \frac{n^2}{m^2} + z_1\left(1 - \frac{n^2}{m^2}\right),$$

$B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ being the standard beta function. The Appell function has the following series expansion:

$$F_1(a, b_1, b_2; c; x_1, x_2) = \sum_{s,t=0}^{\infty} \frac{(a, s+t)(b_1, s)(b_2, t)}{(c, s+t)s!t!} x_1^s x_2^t \tag{B7}$$

for $|x_1|, |x_2| < 1$. The shorthand notation $(a,b) = \Gamma(a+b)/\Gamma(a)$ is used. This integration requires special attention since at the upper limit both the variables of the multiple hypergeometric function become unity. Using a standard transformation rule for hypergeometric function of double variables [26], namely

$$\begin{aligned} F_1(a, b_1, b_2; c; x_1, x_2) = {} & \frac{\Gamma(c)\Gamma(a + b_2 - c)}{\Gamma(a)\Gamma(b_2)}(1-x_1)^{-b_1}(1-x_2)^{c-a-b_2} \\ & \times F_1\left(c-a, b_1, c-b_1-b_2; c-a-b_2+1; \frac{1-x_2}{1-x_1}, 1-x_2\right) \\ & + \frac{\Gamma(c)\Gamma(c-a-b_2)}{\Gamma(c-a)\Gamma(c-b_2)} F_2(a, b_1, b_2; c-b_2, 1-c+a+b_2; x_1, 1-x_2). \end{aligned} \tag{B8}$$

F_2 being the second Appell function [24, 26], we obtain $I = I_1 + I_2$, where

$$I_1 = A_1\int_0^1 dz_1(1-z_1)^{\epsilon_m - 1} z_1^{\epsilon_m - 1} F_1\left(\frac{\epsilon_m + nd}{2}, \bar{n}, \epsilon_m; \frac{\epsilon_m}{2} + \bar{n} + 1; 1 - \frac{n^2}{m^2}, 1 - \hat{z}_1\right), \tag{B9a}$$

and

$$I_2 = A_2\int_0^1 dz_1(1-z_1)^{\epsilon_m/2 - 1} z_1^{\epsilon_m - 1} F_2(\epsilon_m/2, \bar{n}, d/2; \epsilon_m + \bar{n}, 1 - \epsilon_m/2 - \bar{n}; z_1, 1 - \hat{z}_1) \tag{B9b}$$

where

$$A_1 = \left(1 - \frac{n^2}{m^2}\right)^{\epsilon_m/2 + \bar{n}} B[(\epsilon_m + nd)/2, -\epsilon_m/2 - \bar{n}] \tag{B9c}$$

and

$$A_2 = B(\epsilon_m/2 + \bar{n}, \epsilon_m/2). \tag{B9d}$$

The first term, I_1, is free from the above mentioned problem at the upper limit of the hypergeometric function, whereas the second term should be treated in a different procedure because of the first variable in the hypergeometric

function. The integration over z_1 in I_1 gives [27]

$$I_1 = A_1 \frac{\Gamma^2(\epsilon_m)}{\Gamma(2\epsilon_m)} F^{1:2;1}_{1:1;0}\left(\begin{matrix}\epsilon_m/2+nd/2 & :\epsilon_m,\epsilon_m;\ \bar{n};\\ \epsilon_m/2+\bar{n}+1 & :\ \ 2\epsilon_m\ ;\ \ ;\end{matrix} 1-\frac{n^2}{m^2}; 1-\frac{n^2}{m^2}\right) \tag{B10}$$

where

$$F^{1:2;1}_{1:1;0}\left(\begin{matrix}a & :b_1,b_2;\ c;\\ f & :\ \ \ g\ ;\ \ ;\end{matrix} x;y\right) = \sum_{s,t=0}^{\infty} \frac{(a,s+t)(b_1,s)(b_2,s)(c,t)}{(f,s+t)(g,s)} \frac{x^s}{s!}\frac{y^t}{t!} \tag{B11}$$

is the Kampé de Fériet function [26, 27]. The series expansion for I_1 is

$$\begin{aligned} I_1 = &\left(1-\frac{n^2}{m^2}\right)^{\epsilon_m/2+\bar{n}} \frac{\Gamma(-\epsilon_m/2-\bar{n})\Gamma(\epsilon_m/2+\bar{n}+1)}{\Gamma(\bar{n})\Gamma(d/2)} \\ &\times\sum_{s,t}\frac{\Gamma(\epsilon_m/2+nd/2+s+t)\Gamma^2(\epsilon_m+s)\Gamma(\bar{n}+t)}{\Gamma(\epsilon_m/2+\bar{n}+s+t+1)\Gamma(2\epsilon_m+s)}\frac{1}{s!t!}\left(1-\frac{n^2}{m^2}\right)^{s+t}. \end{aligned} \tag{B12}$$

This straightaway reveals the $O(\epsilon_m^{-2})$ singularity from the $s=t=0$ term, *but only for* $n=1$. All n's, however, contribute to $O({\epsilon_m}^{-1})$. The relevant expansion is

$$\begin{aligned} I_1 &= -\frac{4}{\epsilon_m^2} - \frac{2}{\epsilon_m}\left[\gamma+\psi\left(\frac{1}{m-1}\right)+\ln\left(1-\frac{1}{m^2}\right)+\mathcal{R}_1(1,m)\right]+O(1)\ \text{(for } n=1) \\ &= \frac{2}{\epsilon_m}\mathcal{R}_2(n,m)+O(1)\quad(\text{for } n>1), \end{aligned} \tag{B13}$$

where $\mathcal{R}_1(n,m)$ and $\mathcal{R}_2(n,m)$ are defined in Eqs. (7.5a) and (7.5b).

In I_2, the difficulty is with the first variable z_1 which can be separated out into a hypergeometric function as

$$F_2(\epsilon_m/2,\bar{n},d/2,\epsilon_m+\bar{n},1-\epsilon_m/2-\bar{n},z_1,1-\hat{z}_1) = \sum_s \frac{(d/2,s)(\epsilon_m/2,s)}{(1-\epsilon_m/2-\bar{n},s)}\frac{(1-\hat{z}_1)^s}{s!}\,{}_2F_1(\epsilon_m/2+s,\bar{n};\epsilon_m+\bar{n};z_1). \tag{B14}$$

The next step is to divide the range of integration into two equal parts, $[0,1/2]$ and $[1/2,1]$. These two parts will be denoted by I_{21} and I_{22}.

For I_{21}, one gets rid of the trouble in performing the series expansion in the hypergeometric function of variable z_1 and can integrate term by term. This procedure leads to a sum of incomplete beta functions of variables ϵ_m+t and $\epsilon_m/2+s$ where t is the summation index used for the expansion of the hypergeometric function. Using the connection between the incomplete beta function and hypergeometric function, we finally obtain

$$I_{21} = A_2\sum_{s,t}\frac{2^{-\epsilon_m-t}1}{\epsilon_m+t}\,{}_2F_1(\epsilon_m+t,1-\epsilon_m/2-s;\epsilon_m+t+1;1/2)\frac{(d/2,s)(\epsilon_m/2,s+t)(\bar{n},t)}{(1+\epsilon_m/2-n_\epsilon,s)(n_\epsilon,t)}\left(1-\frac{n^2}{m^2}\right)^s\frac{1}{s!t!}, \tag{B15}$$

where $n_\epsilon=\epsilon_m+\bar{n}$.

As before, $n=1$ and $n>1$ cases are to be treated separately . The $O(\epsilon_m^{-2})$ term comes from $s=t=0$. The expansions are

$$I_{21} = \frac{4}{\epsilon_m^2}+\frac{2}{\epsilon_m}\mathcal{R}_1(1,m)+O(1)\quad(\text{for } n=1) \tag{B16a}$$

$$= \frac{2}{\epsilon_m^2}-\frac{1}{\epsilon_m}\left[\gamma+\psi\left(\frac{n-1}{m-1}\right)-\mathcal{R}_1(n,m)\right]+O(1)\quad(\text{for } n>1). \tag{B16b}$$

For I_{22} involving $\int_{1/2}^1$, we use the transformation rule for the hypergeometric equation that was used in the context of p interacting walkers to avoid identical problem with the series expansion. The transformation formula leads to two hypergeometric functions of variable $(1-z_1)$ each of which is regular throughout the domain of integration. After this substitution one can verify that

$$I_{22} = I_{221}+I_{222}, \tag{B17}$$

where

$$I_{221} = \sum_s \mathcal{A}_s \int_{1/2}^{1} (1-z_1)^{s_+-1} z_1^{\epsilon_m-1} \frac{\Gamma(s_-)\Gamma(n_\epsilon)}{\Gamma(\epsilon_m)\Gamma(\bar{n}+s_-)} {}_2F_1(s_+, \bar{n}; 1-s_-; 1-z_1), \tag{B18a}$$

$$I_{222} = \sum_s \mathcal{A}_s \int_{1/2}^{1} (1-z_1)^{\epsilon_m-1} z_1^{\epsilon_m-1} \frac{\Gamma(n_\epsilon)\Gamma(-s_-)}{\Gamma(s_+)\Gamma(\bar{n})} {}_2F_1(\bar{n}+s_-, \epsilon_m; 1+s_-; 1-z_1), \tag{B18b}$$

where

$$\mathcal{A}_s = A_2 \frac{(d/2, s)(\epsilon_m/2, s)}{(1+\epsilon_m/2-n_\epsilon, s)} \frac{1}{s!} \left(1-\frac{n^2}{m^2}\right)^s, \text{ and } s_\pm = \epsilon_m/2 \pm s. \tag{B18c}$$

At this stage use of series expansion is perfectly alright. The final result again comes out in terms of incomplete beta function which after converting into hypergeometric function yields for the right-hand side of Eq. (B18c),

$$I_{221} = \sum_{s,q} \mathcal{A}_s \frac{\Gamma(s_-)\Gamma(n_\epsilon)}{\Gamma(\epsilon_m)\Gamma(n_\epsilon - s_+)} \frac{(s_+, q)(\bar{n}, q)}{(1-s_-, q)} \frac{2^{-(s_++q)}}{(q+s_+)} \frac{\mathcal{F}(q+s_+)}{q!}, \tag{B19a}$$

$$I_{222} = \sum_{s,q} \mathcal{A}_s \frac{\Gamma(n_\epsilon)\Gamma(-s_-)}{\Gamma(s_+)\Gamma(\bar{n})} \frac{(n_\epsilon - s_+, q)(\epsilon_m, q)}{(1+s_-, q)} \frac{2^{-\epsilon_m-q}}{(\epsilon_m+q)} \frac{\mathcal{F}(\epsilon_m+q)}{q!}, \tag{B19b}$$

where

$$\mathcal{F}(x) = {}_2F_1(x, 1-\epsilon_m; 1+x, 1/2).$$

The useful observation here is that for leading singularity, i.e., $O(\epsilon_m^{-2})$, the first terms [$= 1$] in the expansions of the hypergeometric functions are the only relevant ones. To search for the lower order singularity, i.e., for $O(\epsilon_m^{-1})$ terms, one has to consider separately the higher order terms in the expansion. We quote the series retaining terms up to $O(\epsilon_m^{-1})$,

$$I_{221} = \frac{8}{\epsilon_m^2} + O(1) \quad (\text{for } n \geq 1), \tag{B20a}$$

$$I_{222} = O(1) \quad (\text{for } n = 1) \tag{B20b}$$

$$= -\frac{2}{\epsilon_m^2} - \frac{1}{\epsilon_m}\left[\gamma + \psi\left(\frac{n-1}{m-1}\right) - \mathcal{R}_1(n,m)\right] + O(1) \quad (\text{for } n > 1). \tag{B20c}$$

Combining all the terms we obtain

$$I = \frac{8}{\epsilon_m^2} - \frac{2}{\epsilon_m}\left[\gamma + \psi\left(\frac{1}{m-1}\right) + \ln\left(1-\frac{1}{m^2}\right)\right] + O(1) \quad (\text{for } n = 1) \tag{B21a}$$

$$= \frac{8}{\epsilon_m^2} - \frac{2}{\epsilon_m}\left[\gamma + \psi\left(\frac{n-1}{m-1}\right) - \mathcal{R}_1(n,m) - \mathcal{R}_2(n,m)\right] + O(1) \quad (\text{for } n > 1). \tag{B21b}$$

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