# Fractal random processes with iterated logarithmic tails: A generalization of the Shlesinger-Hughes stochastic renormalization approach

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A probabilistic interpretation of the Shlesinger-Hughes stochastic renormalization approach as a cascade of random amplification events is used to describe the dynamics of multiple hierarchical processes. For a positive random variable, a cascade of stochastic amplifications leads to distributions with long tails described by inverse power laws. A "hierarchy of hierarchies" of amplification events is generated by assuming that the mean number of amplification events is also subject to stochastic amplification. The tails of the distributions generated by this procedure are much broader than the ones given by an inverse power law: they are inverse powers of the logarithm of the random variable. By considering a cascade of cascades (similar to a Russian doll), the tails of the corresponding distributions are given by inverse powers of the multiple iterated logarithm of the random variable. This type of asymptotic behavior is universal in the sense that it is independent of the details of the initial probability densities. The results are of interest for the description of very slow processes in frozen systems.

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The Shlesinger-Hughes stochastic renormalization approach (SH) [1] has been applied to a broad class of natural phenomena ranging from solid-state physics and chemistry to biology and even to econometry [2–4]. It may be used to generate probability distributions with long tails belonging to the inverse power-law type. Recently we have suggested a probabilistic interpretation of the SH as a succession of stochastic multiplicative processes [5]. This interpretation allows one to clarify the relationships between SH and the dynamics of hierarchical processes [6]. The aim of this paper is to show how this interpretation may be used to generalize the SH for random distributions with very broad tails having a logarithmic shape. The underlying idea is to build a "hierarchy of hierarchies" of multiplicative random processes.

Let us consider a positive random variable $X$ characterized by a probability density $P(X)dX$. For instance, $X$ may be the absolute value of the displacement vector for a symmetric random walk or the waiting time between two jumps, etc. We assume that $P(X)$ has a short tail, i.e., that there is a cutoff value $X_M$ of $X$ for which

$$P(X>X_m)\sim 0 \ , \tag{1}$$

and that all positive moments of $X$ exist and are finite. By using the SH we can derive a renormalized probability distribution $\tilde{P}(X)dX$ by introducing a hierarchy of amplification events characterized by a certain amplification factor $b>1$:

$$\tilde{P}(X)=\sum_{q=0}^{\infty}\chi_q\int_0^{\infty}\delta(X'b^q-X)P(X')dX' \ , \tag{2}$$

where $\chi_q$ is the probability that a hierarchy is made up of $q$ amplification events; it is given by

$$\chi_q=\lambda^q(1-\lambda) \ , \tag{3}$$

where $1\geq\lambda\geq 1/b$ is the probability that an amplification event occurs. Equations (2) and (3) lead to a scaling equation for $\tilde{P}(X)$:

$$\tilde{P}(X)=(1-\lambda)P(X)+(\lambda/b)\tilde{P}(X/b) \ . \tag{4}$$

The solution of Eq. (4) has an inverse power tail as $X\to\infty$:

$$\tilde{P}(X)\underset{X\to\infty}{\sim}\Xi(\ln X)/X^{1+\rho} \tag{5}$$

where $1\geq\rho>0$ is a fractal exponent given by

$$\rho=\ln(1/\lambda)/\ln b \ , \tag{6}$$

and $\Xi(\ln X)$ is a periodic function of $\ln X$ with period $\ln b$. To avoid the complications related to these logarithmic oscillations in this paper we consider the limit

$$\lambda\nearrow 1, b\searrow 1 \quad \text{with } \rho=\ln(1/\lambda)/\ln b=\text{const} \ . \tag{7}$$

In this limit the logarithmic oscillations disappear but the inverse power tail is still present. The scaling equation (4) reduces to an ordinary differential equation in $P$. Writing in Eq. (4) $b=1+\delta$ and $\lambda=(1+\delta)^{-\rho}$ in the limit $\delta\to 0$ we come to

$$\frac{Xd\tilde{P}}{dX}+(\rho+1)\tilde{P}=\rho P \ . \tag{8}$$

The normalized solution of Eq. (8) is

$$\tilde{P}(X|\rho)=\rho X^{-(1+\rho)}\int_0^X P(Z)Z^{\rho}dZ \ . \tag{9}$$

From Eq. (9) we recover the asymptotic behavior (5) with the difference that now $\Xi$ is a constant,

$$\Xi=\rho\int_0^{\infty}P(X)Z^{\rho}dZ=\rho\langle X^{\rho}\rangle \ . \tag{10}$$

We denote by $\nu$ the mean number of amplification events in a hierarchy. We have

$$\nu=\sum_{0}^{\infty} q\chi_q=\lambda/(1-\lambda) . \quad (11)$$

A "hierarchy of hierarchies" structure may be generated by assuming that the mean number of amplification events is itself random and is subjected to another cascade of amplification processes similar to the one described by Eq. (2). From Eqs. (6) and (11) we have

$$\rho=\ln(1+1/\nu)/\ln b . \quad (12)$$

Since for $\lambda \to 1$ we have $\nu \to \infty$, it follows that in the limit (6) the fractal exponent $\rho$ is given by

$$\rho \approx 1/[\nu \ln b] . \quad (13)$$

From Eq. (13) it follows that a cascade of amplification processes for $\nu$ is equivalent to a cascade of diminution processes for $\rho$. We assume that $\rho$ is initially selected from a probability density $\phi(\rho)d\rho$ with finite moments. By applying the SH we get the following expression for the corresponding renormalized probability density $\tilde{\phi}(\rho)$:

$$\tilde{\phi}(\rho)=\sum_{q=0}^{\infty} \epsilon^q(1-\epsilon)\int_0^1 \delta(\rho' c^q-\rho)\phi(\rho')d\rho' , \quad (14)$$

where $c<1$ is a subunitary scaling factor and $1 \geq \epsilon \geq c$ is the probability that an event in the cascade occurs. In terms of $c$ and $\epsilon$ we define a second fractal dimension

$$H=\ln\epsilon/\ln c , \quad 1 \geq H > 0 . \quad (15)$$

Now we introduce a limit similar to the one given by Eq. (7):

$$c,\epsilon \nearrow 1 , \quad H=\text{const} . \quad (16)$$

In this limit Eq. (14) leads to the differential equation

$$H\phi+\rho d\tilde{\phi}/d\rho+(1-H)\tilde{\phi}=0 , \quad (17)$$

which has the normalized solution:

$$\tilde{\phi}(\rho|H)=H\rho^{H-1}\int_{\rho}^1 \phi(Y)Y^{-H}dY . \quad (18)$$

The final probability density of $X$, $f(X|H)dX$, that is characteristic of a "hierarchy of hierarchies" structure is given by

$$\begin{aligned} f(X|H)&=\int_0^1 \tilde{P}(X|\rho)\tilde{\phi}(\rho|H)d\rho \\ &=X^{-1}H\int_0^X\int_0^1 dZ\, dY P(Z)\phi(Y)Y^{-H}[\ln(X/Z)]^{-(H+1)}\gamma(H+1, Y\ln(X/Z)) , \end{aligned} \quad (19)$$

where

$$\gamma(a,X)=\int_0^X t^{a-1}\exp(-t)dt \quad (20)$$

is the incomplete gamma function. By using Eq. (1) and the asymptotic properties of $\gamma(a,X)$ we get

$$f(X|H) \underset{X \gg X_M,\, X\to\infty}{\sim} AX^{-1}[\ln(X/X_M)]^{-(H+1)} , \quad (21)$$

where

$$A=H\Gamma(1+H)\int_0^1 Y^{-H}\phi(Y)dY=H\Gamma(1+H)\langle \rho^{-H}\rangle , \quad (22)$$

and $\Gamma(a)=\gamma(a,\infty)$ is the complete gamma function.

In order to clarify the nature of the asymptotic behavior of $f(X|H)$ we compute the cumulative distribution

$$F(X|H)=\int_0^X f(X|H)dX . \quad (23)$$

We obtain

$$1-F(X|H) \underset{X\to\infty}{\cong} AH^{-1}[\ln(X/X_M)]^{-H} , \quad (24)$$

an equation that clearly shows the existence of an inverse power law in the logarithm of the random variable $X$. In fact, Eq. (21) can be written as

$$f(X|H)dX \underset{X\to\infty}{\sim} Ad[\ln(X/X_M)]/[\ln(X/X_M)]^{1+H} . \quad (25)$$

The decay of the tails given by Eqs. (21)–(25) is much slower than the one given by the inverse power law (5): not only are the positive moments of $X$ infinite, but also all the positive moments of the logarithm $\ln(X/X_M)$ are infinite. However, as $H>0$, the distribution $f(X|H)dX$ obeys the normalization condition $\int f(X|H)dX=1$.

This procedure can be easily generalized: the "hierarchies of hierarchies" can be lumped into a "hierarchy of hierarchies of hierarchies," etc. We can construct a multiple hierarchy of cascades of amplification events by assuming that the mean number of amplifications corresponding to a previous stage is itself a random variable subject to stochastic amplification; for a process of $n$ such steps we have

$$f_n(X|H)=\int_0^1 \cdots \int \tilde{P}(X|\rho_1)\tilde{\phi}_1(\rho_1|\rho_2)\cdots\tilde{\phi}_n(\rho_n|H) \times d\rho_1 \cdots d\rho_n . \quad (26)$$

Here the renormalized densities $\tilde{\phi}_1,\ldots,\tilde{\phi}_n$ can be derived from Eq. (18)

$$\tilde{\phi}_l(\rho|\rho')=\rho'\rho^{\rho'-1}\int_{\rho}^1 \phi_l(Y)Y^{-\rho'}dY, \quad l=1,\ldots,n , \quad (27)$$

where $\phi_l(Y)$ are the corresponding nonrenormalized distributions. The asymptotic behavior of $f_n(X|H)$ is given by

$$\begin{aligned} f_n(X|H)dX &\cong \frac{\text{const}\times dX}{X\ln(X/X_M)\ln^2(X/X_M)\cdots\ln^{n-1}(X/X_M)[\ln^n(X/X_M)]^{H+1}} \\ &\underset{X\to\infty}{\cong} \frac{\text{const}\times d[\ln^n(X/X_M)]}{[\ln^n(X/X_M)]^{H+1}} , \end{aligned} \quad (28)$$

where $\ln^n a$ is the $n$th iterated logarithm of $a$. For the variable $n$ this procedure generates a family of fractal random processes, for each process the decay of the tail being much slower than in the case of the preceding process. The passage from $n$ to $n+1$, i.e., the lumping of "hierarchies of hierarchies" into a "super hierarchy" corresponds to a logarithmic scale change of the random variable. A remarkable feature of the asymptotic behavior given by Eq. (28) is its universality, that is, its independence of the details of the nonrenormalized probability distributions $P(X), \phi_1(\rho_1), \ldots, \phi_n(\rho_n)$.

The probabilistic interpretation of the SH as a cascade of stochastic amplification processes enlarges the validity range of the method; by considering multiple hierarchical structures of amplification events the tails of the distributions become logarithmic. Although somewhat formal, the procedure used in this paper outlines the existence of a new type of asymptotic behavior for fractal random processes. If the random variable $X$ is the time interval between two events, the method introduced in this paper is of interest in the study of very slow processes occurring for instance in frozen systems far from equilibrium (e.g., in glasses ) or in geology. An ultrametric model with random rates for frozen systems far from equilibrium displays an asymptotic behavior similar to the one given by Eq. (28). Work on this problem is in progress and will be presented elsewhere.

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