COMMENTS

Comments are short papers which criticize or correct papers of other authors previously published in the **Physical Review**. *Each Comment should state clearly to which paper it refers and must be accompanied by a brief abstract. The same publication schedule as for regular articles is followed, and page proofs are sent to authors.*

Comment on "Unified expression for Fermi and Bose distributions"

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The incorrectness of the recently proposed expressions of the Dirac δ function relating to both Fermi and Bose distributions by Chen [Phys. Rev. A **46**, 3538 (1992)] is pointed out. The corresponding correct theorems are provided.

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Very recently Chen proposed three expressions of the Dirac δ function relating to both Fermi and Bose distributions [1]. Unfortunately, close inspection reveals that all of these expressions are incorrect. For the convenience of discussions, we rewrite these equations as follows.

Equation (3) of Ref. [1] relates to the Fermi distribution, which is

$$A(x,y)=\sum_{m=0}^{\infty}\frac{(-1)^m\pi^{2m}}{(2m+1)!}\frac{\partial^{2m+1}}{\partial y^{2m+1}}(1+e^{x-y})^{-1}=\delta(x-y)\ . \tag{1}$$

Equation (15) of Ref. [1], relevant to Bose distribution, is written as

$$B(x,y)=\sum_{m=0}^{\infty}\frac{(-1)^m\pi^{2m}}{(2m+1)!}\frac{\partial^{2m+1}}{\partial y^{2m+1}}(1-e^{x-y})^{-1}=\delta(x-y)\ . \tag{2}$$

Equation (16) of Ref. [1], which was claimed as a general expression, has the form

$$C(x,y)=\sum_{m=0}^{\infty}\frac{(-1)^m\pi^{2m}}{(2m+1)!}\frac{\partial^{2m+1}}{\partial y^{2m+1}}(1+\lambda e^{x-y})^{-1}=\delta(x+\ln\lambda-y)\ . \tag{3}$$

Obviously, when $\lambda=-1$, one cannot reduce Eq. (3) to Eq. (2). This is not surprising because Eq. (2) is incorrect. In fact, as we will show in the following, Eqs. (1) and (3) are also incorrect. Let us first prove the incorrectness of Eq. (1).

The left-hand side of Eq. (1) is equivalent to the expression

$$A(x,y)=\frac{1}{\pi}\sin\left[\pi\frac{\partial}{\partial y}\right](1+e^{x-y})^{-1}=\frac{1}{2i\pi}\{e^{i\pi(\partial/\partial y)}-e^{-i\pi(\partial/\partial y)}\}(1+e^{x-y})^{-1}\ . \tag{4}$$

By using the property of displacement operator [2]

$$e^{a(\partial/\partial y)}F(x,y)=F(x,y+a)\ , \tag{5}$$

Eq. (4) reads

$$\begin{aligned}A(x,y)&=\frac{1}{2i\pi}\{(1+e^{x-y-i\pi})^{-1}-(1+e^{x-y+i\pi})^{-1}\}\\&=\frac{1}{2i\pi}\{(1-e^{x-y})^{-1}-(1-e^{x-y})^{-1}\}\\&\equiv 0\ .\end{aligned} \tag{6}$$

Considering the well-known definition of the Dirac δ function [3,4]

$$\delta(x)=\begin{cases}0\ , & x\neq 0\\ \infty\ , & x=0\ ,\end{cases} \tag{7}$$

and

$$\int_{-\infty}^{\infty}\delta(x)dx=1\ , \tag{8}$$

it is obvious that

$$A(x,y)\neq\delta(x-y)\ . \tag{9}$$

By performing the same procedure on Eqs. (2) and (3) as the above, one will find that both Eqs. (2) and (3) are incorrect, too. In fact, the left-hand-side expressions in Eqs. (1)–(3) cannot play the role of δ function because their definitions, as we will state in the following, are illegal. Therefore applying them to practical problems will lead to mistakes. However, we find that expressions of the Dirac δ function really can be generated by Fermi and Bose distributions, which we present as follows.

The expression of the Dirac δ function corresponding to the Fermi distribution is

$$f(x,y)\equiv\lim_{\eta\to 0^+}\frac{1}{\pi}\sin\left[(\pi-\eta)\frac{\partial}{\partial y}\right](e^{x-y}+1)^{-1}=\delta(x-y)\ , \tag{10}$$

where $x,y,\eta\in(-\infty,\infty)$.

Proof:

(I) $x \neq y$:

$$f(x,y)=\lim_{\eta\to 0^+}\frac{1}{2i\pi}[e^{i(\pi-\eta)(\partial/\partial y)}-e^{-i(\pi-\eta)(\partial/\partial y)}](1+e^{x-y})^{-1}$$
$$=\lim_{\eta\to 0^+}\frac{1}{2i\pi}[(1+e^{x-y-i\pi+i\eta})^{-1}-(1+e^{x-y+i\pi-i\eta})^{-1}]$$
$$=\lim_{\eta\to 0^+}\frac{1}{2i\pi}[(1-e^{x-y+i\eta})^{-1}-(1-e^{x-y-i\eta})^{-1}]$$
$$=0\ . \tag{11}$$

(II) $x=y$:

$$f(x,x)=\lim_{\eta\to 0^+}\frac{1}{2i\pi}[(1-e^{i\eta})^{-1}-(1-e^{-i\eta})^{-1}]$$
$$=\lim_{\eta\to 0^+}\frac{1}{2\pi}\cot\frac{\eta}{2}$$
$$=\infty\ . \tag{12}$$

(III) $y\in(-\infty,\infty)$:

$$\int_{-\infty}^{\infty}f(x,y)dy=\int_{-\infty}^{\infty}\lim_{\eta\to 0^+}\frac{1}{2i\pi}[(1-e^{x-y+i\eta})^{-1}-(1-e^{x-y-i\eta})^{-1}]dy$$
$$=\lim_{\eta\to 0^+}\frac{\sin\eta}{\pi}\int_{-\infty}^{\infty}\frac{dy}{e^{y-x}+e^{-(y-x)}-2\cos\eta}$$
$$=\lim_{\eta\to 0^+}\frac{\sin\eta}{\pi}\int_{-\infty}^{\infty}\frac{d\zeta}{e^{\zeta}+e^{-\zeta}-2\cos\eta}$$
$$=\lim_{\eta\to 0^+}\frac{2\sin\eta}{\pi}\int_{0}^{\infty}\frac{d\zeta}{e^{\zeta}+e^{-\zeta}-2\cos\eta}\ . \tag{13}$$

By setting $z=e^{\zeta}$, Eq. (13) becomes

$$\int_{-\infty}^{\infty}f(x,y)dy=\lim_{\eta\to 0^+}\frac{2\sin\eta}{\pi}\int_{1}^{\infty}\frac{dz}{z^2-2z\cos\eta+1}$$
$$=\lim_{\eta\to 0^+}\frac{2\sin\eta}{\pi}\frac{1}{\sin\eta}\tan^{-1}\left[\frac{z-\cos\eta}{\sin\eta}\right]\Bigg|_{z=0}^{z=\infty}$$
$$=\lim_{n\to 0^+}\frac{2}{\pi}\left[\frac{\pi}{2}-\tan^{-1}\left[\tan\frac{\eta}{2}\right]\right]$$
$$=\lim_{\eta\to 0^+}\frac{2}{\pi}\left[\frac{\pi}{2}-\frac{\eta}{2}\right]=1\ . \tag{14}$$

According to the requirements of the δ function given by Eqs. (7) and (8),

$$f(x,y)=\delta(x-y) \tag{15}$$

is proved.

The situation for Bose systems is subtle because the Bose distribution is essentially a singular function. Therefore we must modify its definition at first. Considering the fact $(e^{x-y}-1)^{-1}=\lim_{\epsilon\to 0}(e^{x-y+i\epsilon}-1)^{-1}$, we propose that the expression of the Dirac δ function for the Bose distribution with variable chemical potentials is

$$g(x,y)\equiv\lim_{\eta\to 0^+}\lim_{\epsilon\to 0}\frac{1}{\pi}\sin\left[\eta\frac{\partial}{\partial y}\right](e^{x-y+i\epsilon}-1)^{-1}$$
$$=\delta(x-y)\ , \tag{16}$$

where $x,y,\eta,\epsilon\in(-\infty,\infty)$, and the order to take the limits $\epsilon\to 0$ and $\eta\to 0^+$ is not interchangeable. This formula can also be proved by performing a similar procedure as used above.

The theorems (10) and (16) can be extended to more general situations, which are

$$\lim_{\eta\to 0^+}\frac{1}{\pi}\sin\left[(\pi-\eta)\frac{\partial}{\partial y}\right](\lambda e^{x-y}+1)^{-1}$$
$$=\delta(x-y+\ln\lambda)\ ,\quad \lambda>0\ , \tag{17}$$

$$\lim_{\eta\to 0^+}\lim_{\epsilon\to 0}\frac{1}{\pi}\sin\left[\eta\frac{\partial}{\partial y}\right](\lambda e^{x-y+i\epsilon}-1)^{-1}$$
$$=\delta(x-y+\ln\lambda)\ ,\quad \lambda>0\ . \tag{18}$$

In summary, we have shown that Chen's recently proposed expressions of the Dirac δ function generated by Fermi and Bose distributions are incorrect. Correct theorems are proposed in this Comment and proved rigorously. The expressions obtained by us might have potential applications to different physical problems.

To conclude we would like to stress the following.

(i) The Dirac δ function used in physics, as pointed out by many authors [5,6], can be considered as a simplified symbol representing a complicated limiting process which guarantees the conditions (7) and (8). Due to the character of this ideal, or generalized function, the representation of $\delta(x)$ can be obtained as the result of a passage to limit from different well-behaved functions of x which involve an auxiliary quantity as parameter governing the limiting process. Therefore any such expression of $\delta(x)$ should explicitly or inexplicitly indicate the limiting process it represents. Obviously, the expressions suggested in this Comment satisfy this requirement. It is at this point, in exception of other reasons, that the expressions proposed in Ref. [1] failed.

(ii) It is well known that different representations of $\delta(x)$ suit the needs of different problems in physics. The expressions addressed in this Comment, we believe, might bring convenience to solving certain physical problems. Considering the similarity between Eqs. (1) and (10), one may think that from the application point of view, the ill-defined expression (1) should still be used to solve some practical problems. But, since the limit process explicitly indicated in expression (10) is not involved in Eq. (1), direct application of it is risky.

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