BRIEF REPORTS

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Wake-field accelerator in a ferromagnetic waveguide

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A novel high-gradient wake-field accelerator is presented in which the drive-beam current leaves behind a high-gradient wake field, accelerating the witness beam to very high energy. The theoretical analysis is based on Faraday's law, which provides a second-order partial differential equation of the azimuthal magnetic field, under the assumption that $\mu\epsilon \gg 1$. The accelerating field can be more than $\frac{1}{2}$ GV/m in an appropriate choice of system parameters.

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In recent years, there has been strong progress in high-current electron-beam technology. Electron beams with an energy of 10 MeV and a current of 10 kA are easily available in the present technology. In addition, a tremendous improvement has been made in the effective control of these electron beams, including the focus, modulation, and a timely termination of the beam current. Thus the electron beam itself is used as drive current in the wave-field accelerators, where a short and intense bunch of electrons passes through a plasma [1–3] or dielectric waveguide [4–6], leaving behind an intense electromagnetic field. The axial component of this electromagnetic field accelerates charged particles in the witness beam, which follows the drive electron beam. Based on the transverse magnetic (TM) waveguide modes, a preliminary theory [5,6] in a dielectric waveguide accelerator has been developed to estimate the acceleration field, which is the fundamental-radial mode in most cases. However, in reality, the acceleration field is a sum of the whole radial modes, which is a complicated function of various physical parameters, including the geometric configuration, the material properties of the waveguide, and so on. In addition, evolution of the acceleration field in time is again a sum of the every radial-mode evolution. In this regard, I develop a fully self-consistent theory of the wake-field accelerator, which consists of a waveguide with a ferromagnetic material. As will be seen below, the accelerating field is proportional to the square root of the parameter μ/ϵ, where μ and ϵ are the permeability and dielectric constant of the waveguide material. The higher the permeability, the higher the accelerating field.

The theoretical model is based on the induced electric field due to the decay of the field energy stored in an energy storage device. I assume that an electron beam with current $I(t)$ propagates through a hole with radius R_1 in the field-energy storage with radius of R_2. The energy storage device is a waveguide with a ferromagnetic material. Whenever the drive-beam current $I(t)$ decreases, the induced electric field $E_z(r,t)$ appears in the system. The induced axial-electric field E_z is calculated from Faraday's law:

$$\nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial}{\partial t}\mathbf{B} , \tag{1}$$

where **E** and **B** are the electric and magnetic fields, respectively. The ratio of the radial electric field E_r to the azimuthal magnetic field B_θ is on the order of $1/\beta^2\epsilon\mu$ or less. Neglecting E_r in comparison with B_θ for the energy storage material with $\mu\epsilon \gg 1$, the induced electric field E_z is related to the magnetic field B_θ by

$$\frac{\partial}{\partial r}E_z(r,t) = \frac{1}{c}\frac{\partial}{\partial t}B_\theta(r,t) . \tag{2}$$

The azimuthal magnetic field B_θ is obtained from Ampere's law:

$$\nabla\times\mathbf{B} = \frac{\mu\epsilon}{c}\frac{\partial}{\partial t}\mathbf{E} \tag{3}$$

in the Maxwell equation. Making use of Faraday's law in Eq. (1), the curl of Eq. (3) is expressed as

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rB_\theta)\right] - \frac{\mu\epsilon}{c^2}\frac{\partial^2}{\partial t^2}B_\theta = 0 \tag{4}$$

in the storage device defined by the range of r satisfying $R_1 < r < R_2$. In obtaining Eq. (4), I have neglected the term $(\partial^2/\partial z^2)B_\theta = (1/\beta^2 c^2)(\partial^2/\partial t^2)B_\theta$, which is much less than the term proportional to $\mu\epsilon$ in Eq. (4) provided $\mu\epsilon \gg 1$. The solution of Eq. (4) is expressed as

$$B_\theta(r,t) = \int_0^\infty dk\, a_k J_1(kr) q_k(t) , \tag{5}$$

where $J_1(x)$ is the Bessel function of the first kind of order one,

$$q_k(t)=\exp\left[-\frac{2\pi\sigma}{\epsilon}t\right]\cos\left[\frac{kc}{\sqrt{\mu\epsilon}}t\right] \tag{6}$$

is the time function, and σ is the residual conductivity in the material, although it is very small (zero in a practical sense). The time function $q_k(t)$ satisfies the initial and final conditions, $q_k(t=0)=1$ and $q_k(t=\infty)=0$.

I now calculate the magnetic field $B_\theta(r,t)$ driven by the current $I(t)=I(t')U(t-t')$, where $U(x)$ is the Heaviside step function defined by $U(x)=1$ for $x>0$ and 0 otherwise. It is obvious that $B_\theta=0$ for $t<t'$ by the causality. The magnetic field at the time $t>t'$ is expressed as

$$B_\theta(t)=\frac{2\mu I(t')}{cr}U((R_2-r)(r-R_1)) + \int_0^\infty dk\, a_k J_1(kr)q_k(t-t') , \tag{7}$$

where the first term in the right-hand side represents the steady-state solution and the second term represents the time-transient solution. Note that the time-transient solution in Eq. (7) vanishes at the time $t\to\infty$. In obtaining Eq. (7), I have neglected the steady-state solution outside of the energy storage material, assuming that the magnetic permeability of the material is much higher than unity ($\mu\gg1$). This approximation is good because of the following reason: (1) Electrons of the drive beam fill most of the hole with radius of R_1. Assuming a uniform electron density, the magnetic-field energy in the volume characterized by the hole is not important in comparison with that in the energy storage. (2) The induced electric field due to the change of the magnetic field in the range of $0<r<R_1$ occurs instantaneously. In this region, the speed of the electromagnetic wave is the speed of light in vacuum. The witness beam follows the drive beam with a certain time lag. The induced electric field due to the change of the magnetic field in the hole is already gone at the time the witness beam arrives. Meanwhile, we note that the induced electric field due to the change of the magnetic field in the storage material arrives at the time the witness beam comes. Note that the speed of the electromagnetic wave in the material is $c/(\mu\epsilon)^{1/2}$. (3) The magnetic field in the region of $r>R_2$ can be expressed by a combination of Bessel functions and is inversely proportional to r. To make the subsequent field calculation analytically tractable, we neglect the field in the region of $r>R_2$. (4) In addition, the permeability of the energy storage material can be considerably higher than unity. If needed, the corrections associated with the field outside the energy storage material can be incorporated into the theory in a straightforward manner.

Making use of the initial condition $q_k(t-t')=1$ at $t=t'$, I obtain

$$\frac{2\mu I(t')}{cr}U((R_2-r)(r-R_1))+\int_0^\infty dk\, a_k J_1(kr)=0 , \tag{8}$$

from Eq. (7). Multiplying Eq. (8) by $rJ_1(k'r)$ and making use of the orthogonality of the Bessel function

$$\int_0^\infty x\, dx\, J_1(\eta x)J_1(\xi z)=\frac{\delta(\xi-\eta)}{\xi} , \tag{9}$$

I obtain

$$a_k=2\mu\frac{I(t')}{c}[J_0(kR_2)-J_0(kR_1)] , \tag{10}$$

where $J_0(x)$ is the Bessel function of the first kind of order zero. Substituting Eq. (10) into Eq. (7), the magnetic field at the time $t>t'$ is therefore expressed as

$$B_\theta(r,t)=\frac{2\mu}{cr}I(t') + \frac{2\mu I(t')}{c}\int_0^\infty dk[J_o(kR_2)-J_0(kR_1)] \times J_1(kr)q_k(t-t') \tag{11}$$

for $R_1<r<R_2$.

It is necessary to evaluate the magnetic field due to the drive-beam pulse defined by

$$I(t)=I(t')U((t'+\Delta t'-t)(t-t'))$$

with the pulse length $\Delta t'$. Paralleling the derivation of Eq. (11), the magnetic field at the time $t>t'+\Delta t'$ is given by

$$\Delta B_\theta(r,t)=-2\mu\frac{I(t')}{c}\int_0^\infty dk[J_0(kR_2)-J_0(kR_1)] \times J_1(kr)\left[\frac{d}{dt'}q_k\right]\Delta t' , \tag{12}$$

which is the magnetic field contributed by a segment $\Delta t'$ of the drive-beam current $I(t')$. In obtaining Eq. (12), I have assumed that the pulse length $\Delta t'$ is very small. Integrating Eq. (12) over the time t', I can show that the magnetic field $B_\theta(r,t)$ due to a continuous drive beam is expressed as

$$B_\theta(r,t)=-\frac{2\mu}{c}\int_0^\infty dk[J_0(kR_2)-J_0(kR_1)]J_1(kr) \times\int_{-\infty}^t dt' I(t')\frac{\partial}{\partial t'}q_k(t-t') , \tag{13}$$

which determines the magnetic field in the storage material for an arbitrary time profile of the drive-beam current $I(t')$.

The axial electric field E_z is calculated by substituting Eq. (13) into Eq. (2). Neglecting the azimuthal magnetic field outside the energy storage material ($r>R_2$), I approximate the boundary condition of the axial electric field by $E_z(r,t)=0$ at $r=R_2$. In calculation of the accelerating field, I also neglect the contribution from the magnetic field outside the storage material, assuming that the magnetic premeability of the material is higher than unity. Substituting Eq. (13) into Eq. (2) and carrying out partial integrations in time and radial coordinate, I obtain the accelerating field

$$E_z(t)=-\frac{2}{c}\left[\frac{\mu}{\epsilon}\right]^{1/2}\int_{-\infty}^t dt'\left[\frac{dI}{dt'}\right]\int_0^\infty dk[J_0(kR_2)-J_0(kR_1)]^2\sin\left[\frac{c(t-t')}{\sqrt{\mu\epsilon}}k\right] , \tag{14}$$

where the abbreviation $E_z(t)$ represents $E_z(0,t)$. In obtaining Eq. (14), I have neglected the terms proportional to the residual conductivity σ.

For convenience, in the subsequent analysis the normalized times τ_1 and τ_2 are defined by

$$\tau_1=\frac{c(t-t')}{2R_1\sqrt{\mu\epsilon}}\ ,\quad \tau_2=\frac{c(t-t')}{2R_2\sqrt{\mu\epsilon}}\ , \tag{15}$$

where I note that $\tau_1=R_2\tau_2/R_1$. For $R_2 \gg R_1$, the normalized time τ_1 is much longer than the time τ_2. Making use of Eq. (15), I can rewrite Eq. (14) by

$$E_z(t)=-\frac{2}{cR_1}\left[\frac{\mu}{\epsilon}\right]^{1/2}\int_{-\infty}^{t}dt'\left[\frac{dI}{dt'}\right]\int_0^{\infty}dx\,J_0^2(x)\left[\sin(2\tau_1x)+\frac{R_1}{R_2}\sin(2\tau_2x)-\frac{2R_1}{R_2}\frac{J_0(R_1x/R_2)}{J_0(x)}\sin(2\tau_2x)\right]. \tag{16}$$

In the remainder of this article, the analysis is restricted to the case when the inner radius R_1 of the energy storage device is much less than the outer radius R_2, i.e., $R_1 \ll R_2$. In this limit, we note several points from Eqs. (15) and (16). First, the term proportional to sin $(2\tau_1x)$ in the integrand in Eq. (16) dominates. The corrections associated with other terms are of the order $(R_1/R_2)^{1/2}$ or less. Second, the peak values of the integration over the variable x in Eq. (16) occur around the time t satisfying $\tau_1\approx 1$, $\tau_2\approx 1$ and $\tau_2\approx(R_1/R_2)^{1/2}$, which correspond to the contributions from the terms proportional to $\sin(2\tau_1x)$, $(R_1/R_2)\sin(2\tau_2x)$, and $J_0(R_1x/R_2)$, respectively, on the right-hand side of Eq. (16). In the early stage, the term proportional to $\sin(2\tau_1x)$ dominates. In this regard, I keep the term proportional to sin $(2\tau_1x)$ in Eq. (16), neglecting other terms. If needed, the corrections associated with other terms can be calculated in a straightforward manner.

The integration over the variation x is carried out by making use of the integral [7]

$$\pi\int_0^{\infty}dx\,J_0^2(x)\sin(2\tau_1x)=\begin{cases}K(\tau_1), & \tau_1<1\\[4pt] \dfrac{1}{\tau_1}K\left[\dfrac{1}{\tau_1}\right], & \tau_1>1\ ,\end{cases} \tag{17}$$

where $K(x)$ is the elliptical function of the first kind defined by

$$K(x)=\frac{\pi}{2}\left[1+(\tfrac{1}{2})^2x^2+\left[\frac{1\times 3}{2\times 4}\right]^2x^4+\cdots\right]. \tag{18}$$

After carrying out a straightforward calculation, I show that the acceleration field E_z in Eq. (16) is approximated by

$$E_z(t)=-\frac{2}{\pi cR_1}\left[\frac{\mu}{\epsilon}\right]^{1/2}\times\int_{-\infty}^{t}dt'\left[\frac{dI}{dt'}\right]\left[K(\tau_1)U(1-\tau_1)+\frac{1}{\tau_1}K\left[\frac{1}{\tau_1}\right]U(\tau_1-1)\right], \tag{19}$$

where $U(x)$ is the Heaviside step function. Equation (19) can be used to calculate the acceleration-gradient field for a broad range of system parameters, where the drive current changes quickly. Note that the drive-beam current $I(t)$ in Eq. (19) is not specified yet.

In order to investigate the long pulse-driven accelerator, I consider the drive current defined by

$$I(t)=\begin{cases}I_m, & t<0\\[4pt] I_m\left[1-\dfrac{t}{\Delta t}\right], & 0<t<\Delta t\\[4pt] 0\ , & t>\Delta t\ ,\end{cases} \tag{20}$$

where the parameter Δt is the termination time of the drive-beam current. In reality, the drive-beam current $I(t)$ at $t<0$ increases very slowly to I_m at $t=0$. Thus Eq. (20) is a good approximation. Substituting Eq. (20) into Eq. (19) and making use of the definitions in Eq. (15) and

$$\tau=\frac{ct}{2R_1\sqrt{\mu\epsilon}}\ ,\quad \eta=\frac{c\Delta t}{2R_1\sqrt{\mu\epsilon}}\ , \tag{21}$$

the acceleration field can be expressed as

$$E_z(t)=\frac{2I_m}{\pi R_1c}\sqrt{\mu/\epsilon}\,q(\tau)\ , \tag{22}$$

where the function $q(\tau)$ for the drive current in Eq. (20) is defined by

$$q(\tau)=\frac{1}{\eta}\int_0^{\tau}d\tau'\,U(\eta-\tau')\times\left[K(\tau_1)U(1-\tau_1)+\frac{1}{\tau_1}K\left[\frac{1}{\tau_1}\right]U(\tau_1-1)\right] \tag{23}$$

and $\tau'=\tau-\tau_1$.

Figure 1 presents plots of the function $q(\tau)$ versus the normalized time τ obtained from Eq. (23) for $\eta=0.05$ (solid line), 0.1 (bold dashed line), 0.2 (dotted line), and 0.4 (dashed line). Several points are noteworthy in Fig. 1. First, the shorter the normalized termination time, the higher the peak value of the function q. Second, the peak value of the function $q(\tau)$ is about 2.5 even for a relatively slow termination time. This peak value occurs at $\tau\approx 1$. Third, the function $q(\tau)$ is always positive for the choice of the drive current in Eq. (20). Fourth, the value of the function q in the range of τ satisfying $0<\tau<\eta$ increases linearly with time τ. As I note from Eq. (20), the drive current decreases linearly to zero in this range of τ. Be-

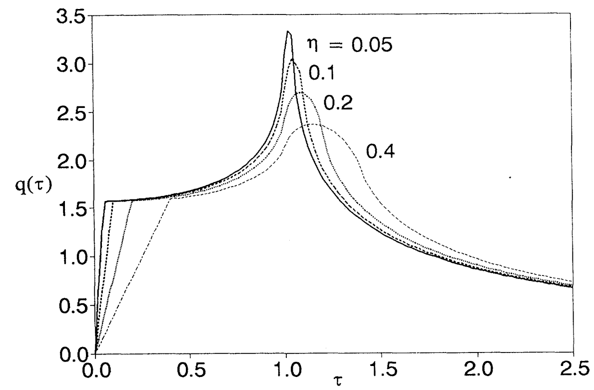

FIG. 1. Plots of the function $q(\tau)$ vs the normalized time τ obtained from Eq. (23) for $\eta=0.05$ (solid line), 0.1 (bold dashed line), 0.2 (dotted line), and 0.4 (dashed line).

cause the q value of this tail portion of the drive beam increases with time, the termination slope stiffens further. This mechanism may decrease the normalized termination time η as time goes by. Finally, I emphasize that the time duration of the high-acceleration field is quite broad. This property is important for a long witness beam. In the limit when the normalized termination time η is much less than unity, i.e., $\eta \ll 1$, Eq. (23) is approximated by

$$q(\tau)=\begin{cases} K\left[\frac{\tau}{2}\right]\frac{\tau}{\eta}\ , & 0<\tau<\eta \\ K\left[\tau-\frac{\eta}{2}\right]\ , & \eta<\tau<1 \\ 1.4+\frac{1}{2}\ln\left[\frac{2}{\eta}\right]\ , & 1<\tau<1+\eta \\ \frac{1}{\tau}\ln\left[\frac{1}{\tau}\right]\ , & \tau>1+\eta\ , \end{cases} \tag{24}$$

which agrees reasonably well with the numerical result in Fig. 1 even for $\eta=0.4$.

As an example, I assume that the current termination parameter is equal to $\eta=0.05$, for which the peak value of the function $q=3.2$ and the accelerating field is given by

$$E_m=\frac{2I_m}{R_1 c}\left[\frac{\mu}{\epsilon}\right]^{1/2}. \tag{25}$$

Assuming that the drive current $I_m=20$ kA, the hole radius $R_1=0.4$ cm, and $\mu/\epsilon=4$, I find from Eq. (25) that the accelerating field is given by $E_m=0.6$ GV/m, which is a very encouraging number. The current termination parameter $\eta=0.05$ corresponds to the real termination time of $\Delta t=2.6$ ps for $\mu\epsilon=4$. As shown in Eq. (20), the rise time of the drive-beam current must be considerably longer than the termination time. The rise time of 26 ps may be enough for the present example. The accelerating field in Eq. (25) for a ferromagnetic waveguide is six times of that in a dielectric waveguide [5] for similar system parameters. I remind the reader that the whole pulse length in the example is less than 1 cm, thereby practically indicating that the driven beam is an intense bunch of electrons. The total charge of the drive-beam current in the example is less than 300 nC. Tailoring the beam pulse as mentioned above is very important to achieve a high accelerating gradient. Obviously, the wake-field accelerator in a ferromagnetic waveguide has a great potential for high gradient acceleration of electrons.

The major technical issue in this accelerator concept is the development of a material that satisfies all the necessary requirements. The material must have a reasonable value of permeability, but have a small conductivity. The permeability and conductivity of a typical ferrite are $\mu=2000$ and $\sigma=100$ S/m, which is six orders of magnitude less than that of copper. Apparently, the permeability and conductivity of the ferrite are too high for this accelerator concept. There is no single element that will satisfy the necessary conditions, but a composite of several different elements may. The conductivity and permeability of heterogeneous mixtures can be determined in terms of the volume fraction of the dispersed components [8]. The time for magnetic field change may be reduced further by the domain structures of the magnetic material [9]. Nevertheless, preparation of the right field-storage material is a very important job and the success of this acceleration concept may depend on it. Another technical issue is instability of the drive beam arising from a phase delay of the fields due to the field storage material. The stability issue of the drive beam is rather a large subject, which is beyond the scope of this Brief Report. However, the growth rate of the instability is proportional to $1/\gamma^2$, where γ is the relativistic mass factor of the drive beam. Growth of the unstable perturbation can be kept under a controllable level by increasing the γ value and by shortening the drive-beam pulse.

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