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Symmetry analysis of the Infeld-Rowlands equation

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A fourth-order nonlinear evolution equation in $2+1$ dimensions, arising in the study of soliton stability, is analyzed. Its symmetry group is shown to be infinite dimensional and is used to obtain particular solutions. The equation is shown not to have the Painlevé property.

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I. INTRODUCTION

An important problem in the study of nonlinear phenomena in physics and other sciences, is the extension of results obtained in $1+1$ dimensions of space-time, to higher dimensions. Large classes of "integrable" equations exist in $1+1$ dimensions, having soliton and multisoliton solutions, infinitely many conservation laws, and all the other attributes of integrability [1–4]. Typically such equations arise in the description of weakly nonlinear and weakly dispersive phenomena and the existence of stable solitons is due to a compensation between nonlinearity and dispersion. Mathematically, the same equations occur as compatibility conditions for Lax pairs. These are pairs of linear operators, providing systems of *a priori,* incompatible linear equations [1–4].

Typical integrable soliton equations of this type are the Korteweg–de Vries (KdV) equation, the nonlinear Schrödinger equation, the sine-Gordon equation, the Boussinesq equation, and many others [1–4].

A much larger class of nonlinear partial differential equations with two independent variables also has solitary wave solutions, without being integrable [5]. In some cases the solitary waves are stable with respect to perturbations. The existence of n-soliton solutions with n arbitrary is, on the other hand, a characteristic of integrable systems. Typical examples of nonintegrable equations with solitary wave solutions are the Landau-Ginzburg equation [6] and nonlinear Schrödinger or nonlinear Klein-Gordon equations with various polynomial nonlinearities (for recent studies of exact solutions and their stability, see, e.g., Refs. [7–9]).

Soliton equations in $n+1$ dimensions with $n \geq 2$ are much more difficult to find than in the case $n=1$. Well-known integrable equations involving three independent variables are the Kadomtsev-Petviashvili equation [10], the Davey-Stewartson equation [11], the full three-wave resonant interaction equations [12,13] and a few others. All of these equations have infinite-dimensional Lie point symmetry groups with a specific Kac-Moody-Virasoro structure [14–17].

Soliton equations in higher dimensions are usually generated in two different manners. One is a mathematical one [18–21] starting from a formalism that guarantees integrability. The specific equations are then obtained by making restrictions on the formalism. The other approach is more physical. The starting point is a soliton equation in one space dimension. The stability of solutions with respect to small-amplitude noncollinear perturbations is investigated and this leads to a new nonlinear equation. There is of course no guarantee that this equation will be integrable and will allow for the existence of soliton solutions. Luckily, some physically interesting equations are produced by both approaches, e.g., the above-mentioned Kadomtsev-Petviashvili and Davey-Stewartson equations.

The perturbative approach to solitons in higher dimensions has recently been adopted systematically by Frycz and Infeld [22] and by Infeld and Rowlands [5,23]. In particular, a study [23] of the stability of the Landau-Ginzburg equation has led to a fourth-order nonlinear evolution equation, which after a rescaling, we write as

$$u_t+2u_x u_{xx}+u_{xxxx}+u_{xy}=0. \tag{1.1}$$

We shall call Eq. (1.1) the Infeld-Rowlands equation and abbreviate it as IR.

In Sec. II we find the symmetry group of this equation and present a classification of its one- and two-dimensional subgroups. In Secs. III and IV we use these subgroups to perform symmetry reduction and to obtain some group invariant solutions. Some conclusions are drawn in Sec. V.

II. SYMMETRY GROUP OF THE INFELD-ROWLANDS EQUATION

A. The symmetry group

Methods for calculating Lie point symmetry groups of differential equations are well known and are explained in many books [24,25]. Moreover, the methods can be implemented on a computer or at least in a computer-assisted way [26,27]. Using a MACSYMA package [27], we find that the Infeld-Rowlands equations are invariant under an infinite-dimensional symmetry group G.

A basis for the Lie algebra L of G is given by

$$\begin{gathered}P_0=\partial_t,\quad P_2=\partial_y,\quad D=4t\partial_t+3y\partial_y+x\partial_x-u\partial_u\ ,\\ V(f)=f(y)\partial_x+\left[\frac{x}{2}f'(y)-\frac{tf''(y)}{2}\right]\partial_u\ ,\\ W(h)=h(y)\partial_u\ ,\end{gathered} \tag{2.1}$$

where $f(y)$ and $h(y)$ are arbitrary functions of y (C^∞ functions on some open subset of $\mathbb{R}$, to be more precise).

The structure of the Lie algebra is that of a semidirect sum

$$L=F \triangleright N, \quad [F,F]\subseteq F, \quad [N,N]\subset N, \quad [F,N]\subseteq N, \tag{2.2}$$

where

$$F=\{D,P_0,P_2\}, \quad N=\{V(f),W(h)\} . \tag{2.3}$$

The commutation relations are

$$[P_0,D]=4P_0, \quad [P_2,D]=3P_2 ,$$
$$[V(f_1),V(f_2)]=\tfrac{1}{2}W(f_1f_2'-f_1'f_2) , \tag{2.4a}$$

$$[V(f),W(h)]=0 ,$$
$$[W(h_1),W(h_2)]=0 , \tag{2.4b}$$

$$[D,V(f)]=V(3yf-f), \quad [D,W(h)]=W(3yh'+h) ,$$
$$[P_0,V(f)]=-\tfrac{1}{2}W(f''), \quad [P_0,W(h)]=0 , \tag{2.4c}$$
$$[P_2,V(f)]=V(f'), \quad [P_2,W(h)]=W(h') .$$

The subalgebra $\langle P_0,P_2,D\rangle$ simply corresponds to the invariance of the IR equation (1.1) under translations in time and in the y direction, and under appropriate dilations. The operator $W(h)$ for any chosen $h(y)$ corresponds to a certain gauge transformation:

$$\tilde{u}(\tilde{x},\tilde{y},\tilde{t})=u(x,y,t)+\lambda h(y) , \tag{2.5}$$

i.e., an arbitrary function of y can be added to any solution.

The generator $V(f)$ is more interesting. It can be integrated to yield

$$\tilde{x}=x+\lambda f(y), \quad \tilde{y}=y, \quad \tilde{t}=t ,$$
$$\tilde{u}=u+\frac{\lambda}{2}[xf'(y)-tf''(y)]+\frac{\lambda^2}{4}f(y)f'(y) , \tag{2.6a}$$

i.e., if $u(x,y,t)$ is a solution of Eq. (1.1), then so is

$$\tilde{u}(\tilde{x},\tilde{y},\tilde{t})=u[\tilde{x}-\lambda f(\tilde{y}),\tilde{y},\tilde{t}]+\frac{\lambda}{2}[\tilde{x}f'(\tilde{y})-\tilde{t}f''(\tilde{y})]$$
$$-\frac{\lambda^2}{4}f(\tilde{y})f'(\tilde{y}) . \tag{2.6b}$$

Notice that for $f=1$, transformation (2.6) is a translation in x. For $f(y)=y$ it is a "shear" transformation. In any case, y is invariant, an arbitrary function of y is added to x (and u is appropriately adjusted).

One conclusion that can be drawn directly from the commutation relations in (2.4) is that the Lie algebra L does not contain a Virasoro subalgebra, typical for integrable equations in 2+1 dimensions [14–17].

B. Low-dimensional subalgebras of the symmetry algebra

The main application of the symmetry group of a partial differential equation (PDE) is to perform symmetry reduction. This amounts to requiring that a solution be invariant under some subgroup G_0 of the symmetry group (rather than be transformed into another solution). Invariance leads to a "dimensional reduction:" fewer independent variables. The IR equation involves three independent variables. We wish to reduce that number to two or one. For this it will suffice to make use of one- and two-dimensional subgroups. To do this systematically, we need a classification of one- and two-dimensional subalgebras into conjugacy classes under the action of the symmetry group G. Algorithmic methods for doing this exist. They are explained in Ref. [25], which gives references to the original work. Here we will only present the resulting lists.

We shall present complete lists of the subalgebras, though only some of them will actually be useful.

1. *One-dimensional subalgebras*

$$S_{1,1}=\{P_0\}, \quad S_{1,4}=\{P_0+V(f)\}, \quad S_{1,7}=\{W(h)\} ,$$
$$S_{1,2}=\{P_2\}, \quad S_{1,5}=\{D\} , \tag{2.7}$$
$$S_{1,3}=\{P_0-P_2\}, \quad S_{1,6}=\{V(f)\} .$$

2. *Two-dimensional non-Abelian subalgebras*

$$S_{2,1}=\{D,P_0-\tfrac{1}{6}V(y^{-1})+\alpha W(y^{-5/3}\}, \quad \alpha\neq 0 ,$$
$$S_{2,2}=\{D,P_0+\alpha V(y^{-1})\}, \quad a\neq 0 ,$$
$$S_{2,3}=\{D,P_2\} ,$$
$$S_{2,4}=\{D,P_0\} ,$$
$$S_{2,5}=\{D,V(y^b)\}, \quad b\neq\tfrac{1}{3} , \tag{2.8}$$
$$S_{2,6}=\{D,W(y^b)\}, \quad b\neq-\tfrac{1}{3} ,$$
$$S_{2,7}=\{P_2,V(e^{\lambda y})\} ,$$
$$S_{2,8}=\{P_2,W(e^{\lambda y}\} ,$$
$$S_{2,9}=\{P_0-P_2,V(e^{-\lambda y})-\frac{\lambda^2}{2}W(ye^{-\lambda y})\} ,$$
$$S_{2,10}=\{P_0-P_2,W(e^{-\lambda y})\} .$$

Throughout (2.8) we have $\lambda\neq 0$, $\lambda\in\mathbb{R}$.

3. Two-dimensional Abelian subalgebras

$$S_{2,11}=\{P_2,P_0-\alpha V(1)+\beta W(1)\},\quad \alpha=0,\pm 1\ ,$$
$$S_{2,12}=\{P_2,V(1)+\alpha W(1)\}\ ,$$
$$S_{2,13}=\{P_2,W(1)\}\ ,$$
$$S_{2,14}=\{P_0 V(y)+W(g)\}\ ,$$
$$S_{2,15}=\{P_0,V(1)+W(g)\}\ ,$$
$$S_{2,16}=\{P_0,W(g)\}\ ,$$
$$S_{2,17}=\{P_0-P_2,V(1)+\alpha W(1)\}\ , \tag{2.9}$$
$$S_{2,18}=\{P_0+V(f),W(g)\}\ ,$$
$$S_{2,19}=\{P_0+V(f),V(F)+W(g)\}\quad \text{with } \ddot{F}-\dot{F}f+F\dot{f}=0\ ,$$
$$S_{2,20}=\{V(f),W(g)\}\ ,$$
$$S_{2,21}=\{W(g_1),W(g_2)\},g_1,g_2\ \text{linearly independent,}$$
$$S_{2,22}=\{D,V(y^{1/3})+\alpha W(y^{-1/3})\}\ ,$$
$$S_{2,23}=\{D,W(y^{-1/3})\}\ .$$

Throughout $g=g(y)$, $f=f(y)$ are arbitrary functions.

III. REDUCTIONS BY ONE-DIMENSIONAL SUBGROUPS

We shall run through the list (2.7) of subalgebras of the symmetry algebra and use the corresponding subgroups to reduce Eq. (1.1) to a PDE involving two independent variables. For the subalgebra $\{V(f)\}$ we can solve the PDE explicitly in full generality. For $\{P_0\}$ we reduce to a well-known integrable equation, the KdV equation. For $\{P_0+V(f)\}$ we reduce to a specified forced KdV equation. The algebra $\{W(h)\}$ does not provide a reduction, since it does not act on space-time. The remaining algebras $\{P_2\}$, $\{P_0-P_2\}$, and $\{D\}$ provide reductions to PDE's that we are not able to solve. They will be reduced further to ordinary differential equations (ODE's) in Sec. IV, where we use the appropriate two-dimensional subalgebras.

A. $S_{1,1}=P_0$ and static solutions

The invariant solutions are time independent $u=u(x,y)$ and Eq. (1.1) reduces to

$$2u_x u_{xx}+u_{xxxx}+u_{xy}=0\ . \tag{3.1}$$

Putting

$$u_x=v \tag{3.2}$$

we obtain the Korteweg–de Vries equation

$$v_y+2vv_x+v_{xxx}=0\ , \tag{3.3}$$

in which y plays the role of time. This is the prototype of an integrable equation [1–4] and of course, many solutions are known. These will be static solutions and no time dependence can be introduced by the group transformations corresponding to the algebra (2.1). The invariant solutions in this case are, for instance, the analogs of traveling waves:

$$V=F(\xi),\quad \xi=x+\alpha y \tag{3.4}$$

with F satisfying

$$F_\xi^2=-\tfrac{2}{3}F^3-\alpha F^2+\beta F+\gamma\ , \tag{3.5}$$

where β and γ are integration contants.

We rewrite Eq. (3.5) as

$$F_\xi^2=-\tfrac{2}{3}(F-F_1)(F-F_2)(F-F_3),\quad F_i=\text{const.} \tag{3.6}$$

In particular for

$$F_2=F_3\le F\le F_1\ , \tag{3.7}$$

we obtain "solitons"

$$F=F_2+(F_1-F_2)\left[\cosh\frac{1}{2}\left[\frac{2(F_1-F_2)}{3}\right]^{1/2}(x+\alpha y-\xi_0)\right]^{-2}\ . \tag{3.8}$$

These are time-independent structures that are finite for x and y real and have a maximum along some line $x+\alpha y-\xi_0=0$ in the (x,y) plane. More general solutions that are real, finite, and periodic occur for

$$F_3<F_2\le F\le F_1 \tag{3.9}$$

and these can be given in terms of Jacobi elliptic functions [28] as

$$F=F_3+\frac{(F_1-F_3)(F_2-F_3)}{(F_1-F_3)-(F_1-F_2)\mathrm{sn}^2\left[\frac{1}{2}\left[\frac{2(F_1-F_3)}{3}\right]^{1/2}(x+\alpha y-\xi_0),k\right]}\ ,$$
$$k^2=\frac{F_1-F_2}{F_1-F_3}\ . \tag{3.10}$$

Solutions (3.8) and (3.10) are just examples. Any solution of the KdV equation will in this manner produce solutions of the IR equation. Moreover, more complicated patterns in the (x,y) plane will result as wave crests if we apply the transformation (2.6) to such solutions.

B. $S_{1,6}=V(f)$, $f(y)\neq 0$, and explicit solutions

The reduction formula in this case is

$$u=\frac{x^2}{4}\frac{f'}{f}-\frac{tx}{2}\frac{f''}{f}+F(y,t)\ , \tag{3.11}$$

and Eq. (1.1) reduces to a trivial equation

$$F_t=\frac{t}{2}\frac{f'''}{f}\ . \tag{3.12}$$

We obtain an explicit family of solutions depending on two arbitrary functions of y, namely, $f(y)$ and $h(y)$:

$$u=\frac{x^2}{4}\frac{f'}{f}-\frac{tx}{2}\frac{f''}{f}+\frac{t^2}{4}\frac{f'''}{f}+h(y)\ . \tag{3.13}$$

C. $S_{1,4}=\{P_0+V(f)\}$, $f\neq 0$, and reduction to a forced KdV equation

The reduction formula in this case is

$$u=\frac{tx}{2}f'-\frac{ff'+f''}{4}t^2+F(y,\xi),\quad \xi=x-tf(y)\ , \tag{3.14}$$

where $F(y,\xi)$ satisfies

$$F_{\xi\xi\xi\xi}+2F_{\xi}F_{\xi\xi}-fF_{\xi}+F_{\xi y}+\xi\frac{f'}{2}=0\ . \tag{3.15}$$

Putting

$$F_{\xi}=H\ , \tag{3.16}$$

we obtain a third-order ODE,

$$H_{\xi\xi\xi}+2HH_{\xi}-fH+H_y+\xi\frac{f'}{2}=0\ . \tag{3.17}$$

Equation (3.17) can be transformed into a forced KdV equation of the type that occurs in the study of water waves in shallow water of variable depth [29]. To do this, we first introduce a function $K(y)$, related to $f(y)$ by the equation

$$f=\frac{1}{3}\frac{K''}{K'},\quad K',K''\neq 0\ . \tag{3.18}$$

We then transform from the variables (H,ξ,y) to (W,ρ,η), putting

$$H(\xi,y)=2K'^{2/3}\left[W(\rho,\eta)-\frac{1}{2}\right]-\frac{1}{6}\frac{K''}{K'}\xi\ , \tag{3.19a}$$

$$\rho=2(K')^{1/3}\xi+4K(y)\ , \tag{3.19b}$$

$$\eta=8K(y)\ . \tag{3.19c}$$

The dependent variable $W(\rho,\eta)$ satisfies the forced KdV equation

$$W_{\rho\rho\rho}+WW_{\rho}+W_{\eta}+\tfrac{1}{288}(K'')^2(K')^{-4}\left[\rho-\tfrac{1}{2}\eta\right]=0\ . \tag{3.20}$$

Here η plays the role of time and K is to be viewed as a function of η [via Eq. (3.19c), we have $y=K^{-1}(\eta/8)$]. If Eq. (3.20) were to be taken literally, rather than as an auxiliary equation for obtaining $H(\xi,y)$ and ultimately $u(x,y,t)$ of Eq. (3.14), it would describe, e.g., the propagation of water waves towards a shore. The depth decreases linearly and the slope may be "time" (η) dependent.

For the remaining subalgebras we just give the reduced PDE for completeness.

1. $S_{1,2}=\{P_2\}$

$$u=u(x,t),\quad u_t+2u_xu_{xx}+u_{xxxx}=0\ . \tag{3.21}$$

2. $S_{1,3}=\{P_0-P_2\}$

$$u=u(x,\xi),\quad \xi=y+t, \tag{3.22}$$

$$u_{\xi}+2u_xu_{xx}+u_{xxxx}+u_{x\xi}=0\ .$$

3. $S_{1,5}=\{D\}$

$$u=t^{-t/4}F(\xi,\eta),\quad \xi=xt^{-1/4},\quad \eta=yt^{-3/4}\ , \tag{3.23}$$

$$F_{\xi\xi\xi\xi}+2F_{\xi}F_{\xi\xi}+F_{\xi\eta}-\tfrac{1}{4}(\xi F_{\xi}+3\eta F_{\eta}+F)=0\ . \tag{3.24}$$

IV. REDUCTIONS BY TWO-DIMENSIONAL SUBGROUPS

We have seen that the symmetry algebra has 23 classes of subalgebras [Eqs. (2.8) and (2.9)]. Many of them can be discarded offhand as far as symmetry reduction is concerned. First of all, subalgebras containing $W(g)$ as an element [with $g(y)$ general, or particular] will not provide reductions to ODE's. Subalgebras containing the element P_0 will lead to particular solutions of the KdV equation (and we already know that any solution of the KdV equation yields a solution of the IR equation). Subalgebras containing an element of the type $V(f)$ [or an element conjugate to $V(f)$] will give special cases of solution (3.13).

Taking all the above comments into account, we see that the only subalgebras that will provide reductions to ODE's that would provide alternative solutions are $S_{2,1}$, $S_{2,2}$, $S_{2,3}$, and $S_{2,11}$ of the previous section. We shall run through these four cases below and in each case establish whether the reduced equation has the Painlevé property [30]. We recall here that a nonlinear ODE is said to have the Painlevé property if its general solution has no movable critical points [30–34]. A critical point is a branch point, or an essential singularity, or any singularity, other than a pole. Movable means depending on initial conditions. Equations having this property are much easier to solve than those that do not.

An algorithmic test for the Painlevé property exists [30]. It has been implemented as a MACSYMA program [35]. The test is passed for an ODE of order n if it is possible to expand the general solution in a Laurent series about a generic singular point ξ_0:

$$u(\xi)=\sum_{k=0}^{\infty} a_k(\xi-\xi_0)^{p+k} . \tag{4.1}$$

The expansion must satisfy (i) p is a negative integer; (ii) $(n-1)$ of the coefficients a_k are arbitrary (the corresponding values of k are called resonances); (iii) the "resonance condition," i.e., a certain compatibility condition, is satisfied at each resonance. For a full description of the test see Refs. [30,34,35].

Let us now run through the individual cases.

A. Algebra $S_{2,1}$

Invariance in this case tells us that the solution must have the form

$$u=\frac{xt}{12y^2}+\frac{13t^2}{(12)^2y^3}+\alpha t y^{-5/3}+y^{-1/3}F(\xi) , \quad \xi=\left[x+\frac{t}{6y}\right]y^{-1/3} . \tag{4.2}$$

Putting $F_\xi=W$ we obtain an equation for W:

$$W_{\xi\xi\xi}+2WW_\xi-\tfrac{1}{2}W-\tfrac{1}{3}\xi W_\xi+\alpha+\tfrac{1}{12}\xi=0 . \tag{4.3}$$

This equation fails the Painlevé test. We obtain $p=-2$, $a_0=-6$ in expansion (4.1). There are two resonances, as required for a third-order equation, namely, $k=4$ and $k=6$. The compatibility condition is satisfied for $k=4$, not, however, for $k=6$.

B. Algebra $S_{2,2}$

The situation is quite similar. We have

$$u=-\frac{\alpha}{2}\frac{xt}{y^2}+\frac{\alpha(\alpha-2)}{4y^3}t^2+y^{-1/3}F(\xi) , \quad \xi=\left[x-\frac{\alpha t}{y}\right]y^{-1/3} . \tag{4.4}$$

We again introduce $W=F_\xi$ and obtain the equation

$$W_{\xi\xi\xi}+2WW_\xi-\left[\alpha+\frac{2}{3}\right]W-\frac{1}{3}\xi W_\xi-\frac{\alpha}{2}\xi=0 . \tag{4.5}$$

As far as the Painlevé property is concerned, we again have $p=-2$, $a_0=-6$, and resonances at $k=4$ and 6. The first compatibility condition is satisfied. The condition at $k=6$ is only satisfied for $\alpha=0$. This case was excluded from consideration, since the algebra would be $\{D,P_0\}=S_{2,4}$ and we would obtain similarity solutions of the KdV equation.

C. Algebra $S_{2,3}$

We obtain

$$u=t^{-1/4}F(\xi), \quad \xi=xt^{-1/4} , \tag{4.6}$$

with $F(\xi)$ satisfying

$$F_{\xi\xi\xi}+F_\xi^2-\tfrac{1}{4}\xi F+\gamma=0 , \tag{4.7}$$

where γ is an integration constant. The Painlevé test for Eq. (4.7) yields $p=-1$, $a_0=6$, and resonances at $k=1$ and $k=6$. The resonance condition at $k=1$ is satisfied, at $k=6$ it is not.

Particular solutions of Eq. (4.7) can be obtained by expanding as in Eq. (4.1) and truncating at some value of k. If we truncate at $k=6$ (the second resonance) and choose $\xi_0=0$, we obtain two different cases:

$$\gamma=\frac{3}{2}, \quad F=\frac{6}{\xi} , \tag{4.8}$$

$$\gamma=\frac{7}{3}, \quad F=\frac{6}{\xi}+\frac{1}{36}\xi^3. \tag{4.9}$$

D. Algebra $S_{2,11}$

We put

$$u=\beta t+F(\xi), \quad \xi=x+\alpha t, \quad W=F_\xi , \tag{4.10}$$

and obtain

$$W_{\xi\xi\xi}+2WW_\xi+\alpha W+\beta=0 . \tag{4.11}$$

The Painlevé test is passed successfully if and only if we have $\alpha=0$. This is allowed in the present case as long as we have $\beta\neq 0$ and we concentrate on this value. For $\alpha=0$, Eq. (4.11) can be integrated once to give

$$W_{\xi\xi}=-W^2-\beta\xi-\gamma, \quad \beta\neq 0 . \tag{4.12}$$

Putting

$$W(\xi)=-6^{-3/5}\beta^{2/5}H(\eta) , \quad \eta=\left[\frac{\beta}{6}\right]^{1/5}\xi+\left[\frac{1}{6\beta^4}\right]^{1/5}\gamma , \tag{4.13}$$

we reduce Eq. (4.12) to the standard equation for the first Painlevé transcendent P_I [33,34]:

$$H_{\eta\eta}=6H^2+\eta . \tag{4.14}$$

Thus we obtain a solution of the IR equation in terms of $P_I(\eta)$.

V. CONCLUSIONS

The Infeld-Rowlands equation (1.1) does not seem to belong to the class of integrable equations in three dimensions. Indeed, some of its reductions to ODE's do not have the Painlevé property, as was shown in Sec. IV. It does not pass the Painlevé test for PDE's [34,36] either.

Moreover, while the symmetry algebra of the equation is infinite dimensional (2.1), it does not have the Kac-Moody-Virasoro structure typical for integrable equations [14–17].

The symmetry group was used to obtain particular solutions. Thus, if $v(x,y)$ is any solution of the KdV equation (with y playing the role of time), then

$$u(x,y)=\int^{x} v(x',y)dx' \tag{5.1}$$

is a static solution of the IR equation and all static solutions are obtained in this manner.

A class of explicit solutions, that are second-order polynomials in x and t, and involve two arbitrary functions of y, is given in Eq. (3.13).

Further solutions are obtained from any solution of the forced KdV equation (3.20), using Eqs. (3.14)–(3.19).

Finally, Eq. (4.10)–(4.14) provide solutions in terms of the Painlevé transcendent P_I.

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