Multidimensional modulation of Alfvén waves

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A multidimensional version of the derivative nonlinear Schrödinger equation is derived for long-wavelength periodic Alfvén waves propagating in a plasma considered in the two-fluid approximation. It is shown that the coupling with magnetosonic waves takes the form of a mean drift which drastically affects the modulational stability of circularly polarized Alfvén waves.

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I. INTRODUCTION

Alfvén waves are thought to play an important role in astro- and geophysical plasmas such as the interstellar medium or planetary magnetospheres. When the plasma is considered in the two-fluid approximation, these waves are dispersive due to the Hall effect. When electron inertia is neglected and electric quasineutrality assumed, the dynamical equations read in a nondimensional form

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{1.1}$$

$$\rho(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) = -\frac{\beta}{\gamma} \nabla \rho^\gamma + (\nabla \times \mathbf{b}) \times \mathbf{b}, \tag{1.2}$$

$$\partial_t \mathbf{b} + \nabla \times (\mathbf{u} \times \mathbf{b}) = -\frac{1}{R_i} \nabla \times [\tfrac{1}{\rho}(\nabla \times \mathbf{b}) \times \mathbf{b}], \tag{1.3}$$

$$\nabla \cdot \mathbf{b} = 0. \tag{1.4}$$

Here the mean density ρ_0 of the plasma and the amplitude of the uniform ambient magnetic field B_0 are taken as unities. Velocities are measured in terms of the Alfvén velocity $c_A = B_0/\sqrt{\mu_0 \rho_0}$ where μ_0 is the void permeability. The quantity R_i denotes the nondimensional gyromagnetic radius of the ions (in an arbitrary unit). The parameter $\beta = c_S^2/c_A^2$ is the square ratio of the velocities of sonic and Alfvén waves. As usual, γ denotes the polytropic gas constant. Equations (1.1)–(1.4) admit exact solutions in the form of finite amplitude circularly polarized Alfvén waves [1],

$$\begin{gathered} b_x = B_0, \quad u_y = -b_y, \quad u_z = -b_z, \\ b_y + i b_z = B_\perp e^{i(kx - \omega t)} , \end{gathered} \tag{1.5}$$

where k and ω are related by the dispersion relation

$$\frac{B_0^2 \omega^2}{R_i^2} = \left(B_0^2 - \frac{\omega^2}{k^2}\right)^2. \tag{1.6}$$

The question arises whether nonlinear modulational instabilities of such waves can lead to strong hydrodynamic effects like shock waves. The latter could dramatically affect the energetic budget of media like interstellar clouds. In contrast, finite amplitude Alfvén plane or solitary waves are only weakly dissipative. A considerable amount of literature has been devoted to the effect of longitudinal perturbations in the long wavelength limit. When the direction of propagation makes a finite angle with the ambient magnetic field, a standard reductive perturbation expansion leads to the modified Korteweg–De Vries equation, although in this equation the nonlinear term is only relevant when the transverse magnetic perturbation is as strong as the transverse component of the ambient field [2,3]. In contrast, when the propagation is quasiparallel, the phase velocities of the Alfvén wave and, depending on β, the fast or slow magnetosonic wave tend to coincide in the linear nondispersive limit. The so-called derivative nonlinear Schrödinger equation (DNLS) is then obtained. This equation which is integrable by inverse scattering [4] was derived using the kinetic theory [5] and subsequently from the two-fluid model [6]–[10].

Perturbations with a dependency in the transverse variables were considered by Mjølhus and Wyller [11,12] and Malara and Elaoufir [13]. Mjølhus and Hada [14] performed a modulation analysis that displays a degeneracy for purely transverse perturbations. In the present paper, we show that this degeneracy is suppressed when the coupling between Alfvén and magnetosonic waves is taken into account. This effect, which drastically modifies the response of the Alfvén waves to a nonlinear modulation, was only partly included in Hoshino's analysis of purely transverse perturbations [15]. In Sec. II, starting from the plasma equations in the two-fluid approximation, we derive a multidimensional version of the DNLS equation which reduces to the equation given by Mjølhus and Wyller [11,9] in the case of localized solitary waves but includes coupling to magnetosonic waves when wave trains are considered. The nonlinear modulational analysis is performed in Sec. III. The stability conditions are discussed in Sec. IV.

II. NONLINEAR DYNAMICS OF LONG WAVELENGTH ALFVÉN WAVES

We concentrate here on the nonlinear dynamics of Alfvén waves with a typical wavelength of order ϵ^{-1}, large compared to the gyromagnetic radius R_i. Our aim is to show that in the case of periodic Alfvén wave trains, the coupling to magnetosonic waves is relevant to leading order. We assume that the Alfvén waves propagate in a direction (taken as the x axis) making a small angle $\epsilon^{1/2}\theta$ with the ambient magnetic field assumed in the (x,z) plane. Proceeding as in [12], we rescale the time and space variables by defining

$$\xi = \epsilon(x-t); \quad \eta = \epsilon^{3/2}y; \quad \zeta = \epsilon^{3/2}(z-\epsilon^{1/2}\theta t); \quad \tau = \epsilon^2 t. \tag{2.1}$$

By this change of variables and the condition $\epsilon \ll 1$, we simultaneously use a reference frame moving with the phase velocity of the Alfvén wave, restrict ourselves to long wavelengths, and permit an even weaker variation in the transverse directions. Furthermore, we expand

$$\begin{aligned}
\rho &= 1+\epsilon\rho_1+\epsilon^2\rho_2+\cdots,\\
u_x &= \epsilon(u_{x1}+\epsilon u_{x2}+\cdots),\\
u_y &= \epsilon^{1/2}(u_{y1}+\epsilon u_{y2}+\cdots),\\
u_z &= \epsilon^{1/2}(u_{z1}+\epsilon u_{z2}+\cdots),\\
b_x &= 1+\epsilon(b_{x1}+\epsilon b_{x2}+\cdots),\\
b_y &= \epsilon^{1/2}(b_{y1}+\epsilon b_{y2}+\cdots),\\
b_z &= \epsilon^{1/2}(b_{z1}+\epsilon b_{z2}+\cdots),
\end{aligned}$$

where $b_{z1} = \theta + \tilde{b}_{z1}$ includes both the rescaled z component θ of the ambient magnetic field and the perturbation $\tilde{b}_{z1}$. The different scalings for the transverse and longitudinal components of the velocity and magnetic field insures the selection of the Alfvén waves by making magnetosonic waves subdominant. Indeed, to leading order, the transverse components of the velocity and magnetic field obey

$$\partial_\xi u_{y1} + \partial_\xi b_{y1} = 0, \tag{2.2}$$

$$\partial_\xi u_{z1} + \partial_\xi b_{z1} = 0\ , \tag{2.3}$$

which corresponds to the Alfvén wave

$$u_{y1} = -b_{y1}, \tag{2.4}$$

$$u_{z1} = -b_{z1}\ . \tag{2.5}$$

The equations for the density and the longitudinal velocity and magnetic field components give

$$-\partial_\xi\rho_1 + \partial_\xi u_{x1} + \partial_\eta u_{y1} + \partial_\zeta u_{z1} = 0, \tag{2.6}$$

$$\partial_\xi u_{x1} - \beta\partial_\xi\rho_1 - \tfrac{1}{2}\partial_\xi(b_{y1}^2+b_{z1}^2) = 0, \tag{2.7}$$

$$\partial_\xi b_{x1} - \partial_\eta u_{y1} - \partial_\zeta u_{z1} = 0\ . \tag{2.8}$$

Because of Eqs. (2.4) and (2.5), Eq. (2.8) reads

$$\partial_\xi b_{x1} + \partial_\eta b_{y1} + \partial_\zeta b_{z1} = 0, \tag{2.9}$$

which also results from the solenoidal character of the magnetic field. Eliminating $\partial_\eta v_{y1}$ from (2.6) and (2.8), we get

$$b_{x1} - \rho_1 + u_{x1} = V_1\ . \tag{2.10}$$

Furthermore Eq. (2.7) rewrites

$$u_{x1} = \beta\rho_1 + \frac{|B_1|^2}{2} + \Pi_1, \tag{2.11}$$

where $|B_1|^2 = b_{y1}^2 + b_{z1}^2$. Here $V_1 = V_1(\eta,\zeta,\tau)$ and $\Pi_1 = \Pi_1(\eta,\zeta,\tau)$ arise as integration constants. When dealing with localized solutions whose decay as $\xi \to \pm\infty$ is fast enough to make b_{x1}, ρ_1, and $|B_1|^2$ absolutely integrable, the functions V_1 and Π_1 vanish identically and are indeed neglected in [12] and [13]. In contrast, these quantities cannot be ignored when dealing with periodic wave trains, for example when the modulational stability of a plane Alfvén wave is considered.

At the next order in ϵ, the equations for the velocity and magnetic field components along the y axis lead, respectively, to equations of the form

$$\partial_\xi u_{y2} + \partial_\xi b_{y2} = f, \tag{2.12}$$

$$\partial_\xi u_{y2} + \partial_\xi b_{y2} = g, \tag{2.13}$$

where the right hand sides f and g are expressed using the leading order terms in the expansion of the density, velocity, and magnetic field. The solvability condition of Eqs. (2.12) and (2.13) reduces to the identification of f and g, and reads

$$\partial_\tau b_{y1} + \partial_\xi\left[\left(V_1+\frac{\rho_1}{2}\right)b_{y1}\right] - \partial_\eta P_1 - \frac{1}{2R_i}\partial_{\xi\xi}b_{z1} = 0, \tag{2.14}$$

where we used Eq. (2.10) and denote by

$$P_1 = \frac{|B_1|^2}{2} + \beta\rho_1 + b_{x1} \tag{2.15}$$

the leading order of the total pressure fluctuation. The density fluctuation ρ_1 is obtained from (2.10) and (2.11) in the form

$$\rho_1 = \frac{1}{1-\beta}\left[\frac{|B_1|^2}{2} + b_{x1} + \Pi_1 - V_1\right]\ . \tag{2.16}$$

Similarly, the equations for the z-component of the velocity and magnetic field give

$$\partial_\tau b_{z1} + \partial_\xi\left[(V_1+\frac{\rho_1}{2})b_{z1}\right] - \partial_\zeta P_1 + \frac{1}{2R_i}\partial_{\xi\xi}b_{y1} = 0\ . \tag{2.17}$$

In order to get the evolution of V_1, we first write the equations at the next order in ϵ, for the density

$$\partial_\tau\rho_1 - \partial_\xi\rho_2 + \partial_\xi(\rho_1 u_{x1} + u_{x2}) + \partial_\eta(-\rho_1 b_{y1} + u_{y2}) + \partial_\zeta(-\rho_1 b_{z1} + u_{z2}) = 0 \tag{2.18}$$

and that for the longitudinal magnetic field component

$$\partial_\tau b_{x1} - \partial_\xi b_{x2} = \partial_\eta(u_{x1}b_{y1} + b_{y1}b_{x1} - u_{y2}) + \partial_\zeta u_{x1}b_{z1} + b_{z1}b_{x1} - u_{z2} - \frac{1}{R_i}(\partial_\eta\partial_\xi b_{z1} + \partial_\zeta\partial_\xi b_{y1})\ . \tag{2.19}$$

The unknown quantities are eliminated by considering

$$\partial_\tau\langle b_{x1}-\rho_{x1}\rangle-\partial_\eta\langle u_{x1}b_{y1}+b_{x1}b_{y1}\rangle$$
$$-\partial_\zeta\langle u_{x1}b_{z1}+b_{x1}b_{z1}\rangle+\partial_\eta\langle\rho_1 b_{y1}\rangle+\partial_\zeta\langle\rho_1 b_{z1}\rangle=0, \tag{2.20}$$

where $\langle\cdots\rangle$ denotes the average $\frac{1}{2L}\int_{-L}^{L}\cdots\, d\xi$ taken on the domain of the ξ variable. At this step, we either assume that the functions are periodic in ξ, as it is the case when dealing with an Alfvén wave train (see Sec. III), or that they decay fast enough at infinity to make the mean values $\langle\cdots\rangle$ finite. Since V_1 is independent of ξ, it identifies with its average

$$\langle V_1\rangle=\langle b_{x1}-\rho_1\rangle+\langle u_{x1}\rangle.$$

We thus now write an equation for $\langle u_{x1}\rangle$

$$\partial_\tau\langle u_{x1}\rangle+\langle u_{y1}\partial_\eta u_{x1}\rangle+\langle u_{z1}\partial_\zeta u_{x1}\rangle$$
$$-\langle b_{y1}\partial_\eta b_{x1}\rangle-\langle b_{z1}\partial_\zeta b_{x1}\rangle-\left\langle\rho_1\partial_\xi\frac{|B_1|^2}{2}\right\rangle=0\,. \tag{2.21}$$

In Eq. (2.21), we rewrite

$$\begin{aligned}\langle u_{y1}\partial_\eta u_{x1}\rangle&=\partial_\eta\langle u_{y1}u_{x1}\rangle+\langle u_{x1}\partial_\eta b_{y1}\rangle\\&=\partial_\eta\langle u_{y1}u_{x1}\rangle-\langle u_{x1}(\partial_\xi b_{x1}+\partial_\zeta b_{z1})\rangle\\&=\partial_\eta\langle u_{y1}u_{x1}\rangle-\langle u_{x1}\partial_\xi(\rho_1-u_{x1})\rangle-\langle u_{x1}\partial_\zeta b_{z1}\rangle\\&=\partial_\eta\langle u_{x1}u_{y1}\rangle-\langle u_{x1}\partial_\xi\rho_1\rangle-\langle u_{x1}\partial_\zeta b_{z1}\rangle,\end{aligned} \tag{2.22}$$

where we have used (2.9) and (2.10). From (2.11), we get

$$\left\langle\rho_1\partial_\xi\frac{|B_1|^2}{2}\right\rangle=\langle\rho_1\partial_\xi u_{x1}\rangle\,. \tag{2.23}$$

Using again (2.9), we have

$$\begin{aligned}\langle b_{y1}\partial_\eta b_{x1}\rangle&=\partial_\eta\langle b_{y1}b_{x1}\rangle-\langle b_{x1}\partial_\eta b_{y1}\rangle\\&=\partial_\eta\langle b_{y1}b_{x1}\rangle+\langle b_{x1}\partial_\zeta b_{z1}\rangle\end{aligned} \tag{2.24}$$

and we finally obtain

$$\partial_\tau\langle u_{x1}\rangle-\partial_\eta\langle u_{x1}b_{y1}\rangle-\partial_\zeta\langle u_{x1}b_{z1}\rangle$$
$$-\partial_\eta\langle b_{x1}b_{y1}\rangle-\partial_\zeta\langle b_{x1}b_{z1}\rangle=0\,. \tag{2.25}$$

The equation for V_1 then reads

$$\partial_\tau V_1-\partial_\eta(2V_1\langle b_{y1}\rangle+\langle\rho_1 b_{y1}\rangle)$$
$$-\partial_\zeta(2V_1\langle b_{z1}\rangle+\langle\rho_1 b_{z1}\rangle)=0\,. \tag{2.26}$$

In order to compute b_{x1} from Eq. (2.9), we need $\langle b_{x1}\rangle$. Furthermore, to close the system, an equation for Π_1 is also required. From Eq. (2.16), we have

$$\Pi_1-V_1=(1-\beta)\hat\rho_1-\beta\langle b_{x1}\rangle-\left\langle\frac{|B_1|^2}{2}\right\rangle, \tag{2.27}$$

where $\hat\rho_1=\langle\rho_1-b_{x1}\rangle$. The time evolution of $\hat\rho_1$ is obtained from Eqs. (2.20) and (2.10)

$$\partial_\tau\hat\rho_1+\partial_\eta(V_1\langle b_{y1}\rangle)+\partial_\zeta(V_1\langle b_{z1}\rangle)=0\,. \tag{2.28}$$

In order to estimate $\langle b_{y1}\rangle$ and $\langle b_{z1}\rangle$, we go back to Eqs. (2.14)–(2.17) and average on the ξ variable. Differentiating the resulting equations with respect to η and ζ, and using the divergenceless condition (2.9), we get

$$(\partial_{\eta\eta}+\partial_{\zeta\zeta})\langle P_1\rangle=0\,. \tag{2.29}$$

Under the assumption that $\langle P_1\rangle$ remains bounded at infinity, we conclude that $\langle P_1\rangle$ is constant in space. It thus identifies with its average on the full spatial domain. It is easily seen that this average is constant in time. This results from the conservative form of the equations for ρ_1 and b_{x1}, together with the conservation of the full space average of $\frac{|B_1|^2}{2}$, as seen from eqs. (2.14)–(2.17). As a consequence $\langle P_1\rangle$ stays equal to its initial value, and so do $\langle b_{y1}\rangle$ and $\langle b_{z1}\rangle$ that we shall denote by $\langle b_{y1}(0)\rangle$ and $\langle b_{z1}(0)\rangle$.

The average on the full domain of ρ_1 and b_{x1} can be taken to be zero initially (otherwise, these averages are included in the unperturbed zeroth-order quantities). In this case $\langle P_1\rangle$ identifies with the mean magnetic energy perturbation E^M given by the full domain average of the initial value $\frac{|B_1(0)|^2}{2}$. Consequently

$$\left\langle\frac{|B_1|^2}{2}\right\rangle+\beta\langle\rho_1\rangle+\langle b_{x1}\rangle=E^M\,, \tag{2.30}$$

or

$$\langle b_{x1}\rangle=-\frac{1}{1+\beta}\left(\left\langle\frac{|B_1|^2}{2}\right\rangle+\beta\hat\rho_1-E^M\right)\,. \tag{2.31}$$

From Eqs. (2.27) and (2.31), Eq. (2.16) becomes

$$\rho_1=\frac{1}{1-\beta}\left[\frac{|B_1|^2}{2}+b_{x1}\right.$$
$$\left.-\frac{1}{1+\beta}\left(\left\langle\frac{|B_1|^2}{2}\right\rangle-\hat\rho_1+\beta E^M\right)\right]\,. \tag{2.32}$$

We finally rewrite the equations governing long-wavelength Alfvén waves in the form

$$\partial_\tau B_1 + \partial_\xi \left[(V_1 + \tfrac{\rho_1}{2})B_1\right] - \tfrac{1}{2}(\partial_\eta + i\partial_\zeta)P_1 + \tfrac{i}{2R_i}\partial_{\xi\xi}B_1 = 0, \tag{2.33}$$

$$P_1 = \tfrac{|B_1|^2}{2} + \beta\rho_1 + b_{x1}, \tag{2.34}$$

$$\rho_1 = \tfrac{1}{1-\beta}\left[b_{x1} + \tfrac{|B_1|^2}{2} - \tfrac{1}{1+\beta}\left(\left\langle \tfrac{|B_1|^2}{2}\right\rangle - \hat{\rho}_1 + \beta E^M\right)\right], \tag{2.35}$$

$$\partial_\xi b_{x1} + \partial_\eta b_{y1} + \partial_\zeta b_{z1} = 0, \tag{2.36}$$

$$\langle b_{x1}\rangle = -\tfrac{1}{1+\beta}\left(\left\langle \tfrac{|B_1|^2}{2}\right\rangle + \beta\hat{\rho}_1 - E^M\right), \tag{2.37}$$

$$\partial_\tau\hat{\rho}_1 + \partial_\eta(V_1\langle b_{y1}(0)\rangle) + \partial_\zeta(V_1\langle b_{z1}(0)\rangle) = 0, \tag{2.38}$$

$$\partial_\tau V_1 - 2\partial_\eta(V_1\langle b_{y1}(0)\rangle) - 2\partial_\zeta(V_1\langle b_{z1}(0)\rangle) = \partial_\eta\langle\rho_1 b_{y1}\rangle + \partial_\zeta\langle\rho_1 b_{z1}\rangle \tag{2.39}$$

with $B_1 = b_{y1} + ib_{z1}$.

In the absence of transverse variation, $\langle\frac{|B_1|^2}{2}\rangle = E^M$, while $\langle b_{x1}\rangle = \langle\rho_1\rangle = V_1 = 0$. We thus recover the usual (one-dimensional) DNLS equation since the constant E^M can be removed by a phase shift. Note that in the dissipative analogous of DNLS, known as the Cohen-Kulsrud equation [16]–[18], the energy E^M decays in time and thus cannot be eliminated. In the multidimensional case, Eqs. (2.33)–(2.39) simplify for localized solutions with a fast enough decay at infinity since in this case $\langle\frac{|B_1|^2}{2}\rangle = E^M = V_1 = \langle b_{x1}\rangle = \langle\rho_1\rangle = 0$, and $\rho_1 = \frac{1}{1-\beta}\left(b_{x1} + \frac{|B_1|^2}{2}\right)$. We then recover the equations given in [12]. In contrast, Eqs. (2.33)–(2.39) should be the starting point when dealing with periodic wave trains.

III. MODULATION OF WEAKLY NONLINEAR ALFVÉN WAVE TRAINS

In this section we consider the modulation of plane Alfvén wave trains. As in Sec. II, we are interested in the long-wavelength limit so that Eqs. (2.33)–(2.39) are a possible starting point. These equations admit exact solutions in the form of circularly polarized monochromatic plane waves that could be modulated [14]. However, transverse perturbations generate on a short time scale a longitudinal correction to the magnetic field, which produces a coupling of all harmonics. Therefore, the modulation analysis cannot be restricted to the case of monochromatic waves but has to be performed for arbitrary solutions that will be approximated by a truncated Galerkin series. Whitham's method can be implemented to this problem. The algebra is nevertheless cumbersome because the usual equations describing the conservation of the phase and of the action are coupled to an equation for the mean transverse magnetic field. For simplicity, we thus choose to restrict ourselves to low amplitude solutions and to perform the usual weakly nonlinear analysis which leads to nonlinear Schrödinger-like equations.

The multidimensional DNLS equations derived in Sec. II contain quantities such as V which are averages over the ξ variable and thus only retain the mean effect of the magnetosonic waves. These equations can thus be viewed as a quasistatic approximation. In a context of modulation the approximation consisting in keeping V independent of the longitudinal coordinate is certainly valid as long as the modulation occurs at comparable scales in all the directions. It is not the case when the longitudinal scale is significantly larger than the transverse one. In order to develop a formalism which embodies such a situation, we explicitly introduce a dependency in the large-scale longitudinal variable $X = \epsilon\xi$ which will allow us to resolve the local dynamics of magnetosonic waves. The resulting equations, although unclosed, will be the proper starting point for a modulational analysis.

It is easily seen that the only modification due to the introduction of this additional variable in the DNLS equation concerns equations for the longitudinal field components. On the left-hand side (lhs) of Eq. (2.18), the extra terms are $-\partial_X\rho_1 + \partial_X u_1$ and on the lhs of Eq. (2.21), we add $-\partial_X b_{x1}$. Denoting by an overbar the average on the fast variable ξ, Eq. (2.28) is replaced by

$$\partial_\tau\hat{\rho}_1 + \partial_X V_1 + \partial_\eta(V_1\overline{b_{y1}}) + \partial_\zeta(V_1\overline{b_{z1}}) = 0\ . \tag{3.1}$$

In the same way, we get instead of Eq. (2.26)

$$\begin{aligned}\partial_\tau V_1 - 2\partial_X V_1 - 2\partial_\eta(V_1\overline{b_{y1}}) - 2\partial_\zeta(V_1\overline{b_{z1}})\\ = \partial_X\left[(1-\beta)\hat{\rho}_1 - \beta\overline{b_{x1}} - \frac{\overline{|B_1|^2}}{2}\right]\\ +\partial_\eta\overline{\rho_1 b_{y1}} + \partial_\zeta\overline{\rho_1 b_{z1}}\ .\end{aligned} \tag{3.2}$$

Equations (2.33) and (2.36) for the magnetic field are not modified. The density fluctuation ρ_1 is again given by Eq. (2.16) where $\Pi_1 - V_1$ is given by Eq. (2.27). Note that Eq. (2.31) is no longer valid because quantities averaged on the ξ variable are still dependent on X.

For convenience, we rename the variables by replacing τ by t, ξ by x, X by ϵx, η by y and ζ by z, drop the subscript 1, replace b_x by a and define $\partial = \partial_y + i\partial_z$. We get

$$\partial_t b + \partial_x\left[(V + \frac{\rho}{2})b\right] - \frac{1}{2}\partial P + \frac{i}{2R_i}\partial_{xx}b = 0, \tag{3.3}$$

$$P = \frac{|b|^2}{2} + \beta\rho + a, \tag{3.4}$$

$$\rho = \frac{1}{1-\beta}\left[\frac{|b|^2}{2} + a + (1-\beta)\hat{\rho} - \beta\overline{a} - \frac{\overline{|b|^2}}{2}\right], \tag{3.5}$$

$$\partial_x a + \tfrac{1}{2}(\partial^* b + \partial b^*) = 0, \tag{3.6}$$

$$\partial_t\hat{\rho} + \frac{1}{\epsilon}\partial_x V + \frac{1}{2}\left[\partial^*(V\overline{b}) + \partial(V\overline{b^*})\right] = 0, \tag{3.7}$$

$$\partial_t V - \frac{2}{\epsilon}\partial_x V - \partial^*(V\bar{b}) - \partial(V\overline{b^*})$$
$$= \frac{1}{\epsilon}\partial_x\left[(1-\beta)\hat{\rho} - \beta\bar{a} - \frac{\overline{|b|^2}}{2}\right] + \frac{1}{2}\left[\partial^*(\overline{\rho b}) + \partial(\overline{\rho b^*})\right], \tag{3.8}$$

where the star stands for the complex conjugate. In the limit $\epsilon \to 0$, we recover (2.33)–(2.39).

In order to perform a modulation analysis on Eqs. (3.3)–(3.8), we define μ as the small parameter measuring the magnitude of b, and introduce the slow variables $X = \mu x$, $Y = \mu y$, $Z = \mu z$, $\tau = \mu t$. We then look for solutions of Eqs. (3.3)–(3.8) such that b, a, ρ, and P depend on x, t, X, Y, Z, and τ, while $\hat{\rho}$ and V only depend on X, Y, Z, and τ. In this multiple-scale procedure, ϵ is considered as given. We substitute $\partial_x \to \partial_x + \mu\partial_X$, $\partial \to \mu\nabla_\perp$, $\partial_t \to \partial_t + \mu\partial_\tau$, and expand b, a, ρ, $\hat{\rho}$, V, and P in powers of μ, in the form $b = \mu b_1 + \mu^2 b_2 + \cdots$.

At order μ (linear theory), we get

$$\partial_t b_1 + \frac{1}{2R_i}\partial_{xx} b_1 = 0. \tag{3.9}$$

Furthermore, we have $a_1 = \rho_1 = P_1 = \hat{\rho}_1 = V_1 = 0$. As a solution of (3.9), we choose a circularly polarized wave of the form $b_1 = B_1(X,Y,Z,\tau)e^{i(kx-\omega t)}$ with $\omega = \frac{-k^2}{2R_i}$. By this choice, we restrict ourselves to waves propagating parallel to the ambient magnetic field.

At order μ^2, the equation for the transverse magnetic field gives

$$\partial_t b_2 + \frac{1}{2R_i}\partial_{xx} b_2 = -\left(\partial_\tau B_1 - \frac{k}{R_i}\partial_X B_1\right)e^{i(kx-\omega t)} \tag{3.10}$$

and the associated solvability condition reads

$$\partial_\tau B_1 + v_g \partial_X B_1 = 0 . \tag{3.11}$$

This equation means that for times τ of order unity, the (complex) amplitude B_1 is simply advected with the group velocity $v_g = -\frac{k}{R_i}$ of the linear Alfvén wave. It follows that $b_2 = B_2 e^{i(kx-\omega t)}$ where B_2 is not determined at this order. To describe the dynamics of the magnetic field on a longer time scale $T = \mu^2 t$, we must consider the equation for the magnetic perturbation at order $O(\mu^3)$ where the elimination of resonant oscillating modes yields

$$\partial_\tau B_2 - \frac{k}{R_i}\partial_X B_2 + ik\left(V_2 + \frac{\hat{\rho}_2}{2} + \frac{\bar{a}_2}{2}\right)B_1$$
$$-\tfrac{1}{2}\nabla_\perp \tilde{P}_2 + \frac{i}{2R_i}\partial_{XX} B_1 = 0 \tag{3.12}$$

with

$$P_2 = \frac{|B|^2}{2} + \beta\rho_2 + a_2 \tag{3.13}$$

and

$$\rho_2 = \frac{1}{1-\beta}\left[a_2 + (1-\beta)\hat{\rho}_2 - \beta\bar{a}_2\right] . \tag{3.14}$$

Here, the tilde refers to the coefficient of $e^{i(kx-\omega t)}$ in a Fourier-mode expansion, while the overbar denotes the nonoscillating mode.

At order $O(\mu^2)$, the divergenceless condition for the magnetic field gives

$$\partial_x a_2 + \tfrac{1}{2}(\partial^* b_1 + \partial b_1^*) = 0 \tag{3.15}$$

which yields

$$a_2 = \tilde{a}_2 e^{i(kx-\omega t)} + \tilde{a}_2^* e^{-i(kx-\omega t)} + \bar{a}_2 \tag{3.16}$$

with $\tilde{a}_2 = -\frac{1}{2ik}\nabla_\perp B_1$. We thus have

$$\tilde{P}_2 = \beta\tilde{\rho}_2 + \tilde{a}_2 = -\frac{1}{2(1-\beta)ik}\nabla_\perp B_1 . \tag{3.17}$$

We now need equations for V_2, $\hat{\rho}_2$, and $\bar{a}_2$. The solvability condition at order $O(\mu^3)$ corresponding to the elimination of nonoscillating secular terms for the transverse magnetic field gives

$$-\partial_\tau \bar{b}_2 + \tfrac{1}{2}\nabla_\perp \bar{P}_2 = 0 \tag{3.18}$$

with $\bar{P}_2 = \frac{|B_1|^2}{2} + \beta\rho_2 + (1+\beta)\bar{a}_2$, whereas the solenoidal character of the magnetic field implies

$$\partial_X \bar{a}_2 + \tfrac{1}{2}(\nabla_\perp^* \bar{b}_2 + \nabla_\perp \bar{b}_2^*) = 0 . \tag{3.19}$$

The equations for $\hat{\rho}_2$ and V_2 are obtained at order $O(\mu^3)$ in the form

$$\partial_\tau \hat{\rho}_2 + \frac{1}{\epsilon}\partial_X V_2 = 0, \tag{3.20}$$
$$\partial_\tau V_2 - \frac{2}{\epsilon}\partial_X V_2 = \frac{1}{\epsilon}\partial_X\left[(1-\beta)\hat{\rho}_2 - \beta\bar{a}_2 - \frac{|B_1|^2}{2}\right] . \tag{3.21}$$

Note that the system (3.11), (3.12), and (3.18)–(3.21) is closed on the time scale τ but not on the time scale T of the transverse magnetic field. In order to describe the dynamics of the magnetosonic waves on this longer time scale, we write equations similar to (3.18)–(3.21) at order $O(\mu^4)$. Summing these equations with (3.18)–(3.21) and denoting $\hat{\rho} = \hat{\rho}_2 + \mu\hat{\rho}_3$, $V = V_2 + \mu V_3$, $B = B_1 + \mu B_2$, $a = a_2 + \mu a_3$, and $\bar{b} = \bar{b}_2 + \mu\bar{b}_3$, we get

$$\partial_\tau \bar{b} - \tfrac{1}{2}\nabla_\perp\left(\frac{|B|^2}{2} + \beta\hat{\rho} + (1+\beta)\bar{a}\right)$$
$$+\mu\left(\tfrac{1}{2}\partial_X(\overline{\rho_2 b_1}) + \frac{i}{2R_i}\partial_{XX}\bar{b}\right) = 0, \tag{3.22}$$

$$\partial_X \bar{a} + \tfrac{1}{2}(\nabla_\perp^* \bar{b} + \nabla_\perp \bar{b}^*) = 0, \tag{3.23}$$
$$\partial_\tau \hat{\rho} + \frac{1}{\epsilon}\partial_X V = 0, \tag{3.24}$$
$$\partial_\tau V - \frac{2}{\epsilon}\partial_X V = \frac{1}{\epsilon}\partial_X\left[(1-\beta)\hat{\rho} - \beta\bar{a} - \frac{|B|^2}{2}\right]$$
$$+\frac{\mu}{2}\left[\nabla_\perp^*(\overline{\rho_2 b_1}) + \nabla_\perp(\overline{\rho_2 b_1^*})\right] . \tag{3.25}$$

Similarly, summing Eqs. (3.11) and (3.12) we obtain

$$(\partial_\tau + v_g\partial_X)B + \mu i\left[k\left(V + \frac{\hat\rho}{2} + \frac{\bar a}{2}\right)B - \frac{1}{4k(1-\beta)}\Delta_\perp B + \frac{1}{2R_i}\partial_{XX}B\right] = 0. \quad (3.26)$$

In Eqs. (3.22) and (3.25),

$$\overline{\rho_2 b_1} = \frac{1}{1-\beta}\tilde a_2^* B_1 = \frac{1}{2ik(1-\beta)}(\nabla_\perp B_1^*)B_1\ , \quad (3.27)$$

so that the order μ coefficient in Eq. (3.25) reads

$$\frac{\mu}{2}\left[\nabla_\perp\left(-\frac{1}{2ik(1-\beta)}(\nabla_\perp^* B_1)B_1^*\right) + \nabla_\perp^*\left(-\frac{1}{2ik(1-\beta)}(\nabla_\perp B_1^*)B_1\right)\right]$$

or

$$\frac{i\mu}{4k(1-\beta)}\left[(\Delta_\perp B_1)B_1^* - (\Delta_\perp B_1^*)B_1\right]\ .$$

This closes the system (3.22)–(3.25) for the dynamics on the time scale T. Nevertheless, it is not always possible to choose a reference frame in which the τ variable can be eliminated. Indeed, even if B_1 only depends on $X - v_g\tau$, this is not the case for B_2 which is coupled to the magnetosonic waves $\hat\rho$ and V.

The system (3.22)–(3.25) simplifies when $\epsilon \to 0$, this limit being nonuniform with respect to the ratio of the transverse and longitudinal scales. When the modulation scales are comparable in all directions, it is possible to average over the magnetosonic waves and consider only the long time-scale dynamics of B. Indeed Eq. (3.24) leads to $V = V(Y,Z,T)$, while Eq. (3.25) gives

$$\hat\rho = \frac{1}{1-\beta}\left[\beta\bar a + \frac{|B|^2}{2} + F(Y,Z,T)\right]\ . \quad (3.28)$$

In this equation the function F is determined by the condition $\langle\hat\rho\rangle = \hat R_0(Y,Z)$, where $\langle\ \rangle$ denotes average over X and $\hat R_0$ is the initial value of $\hat\rho$. The latter condition derives from Eq. (3.24), which shows that at this order of approximation $\langle\hat\rho\rangle$ remains constant in time, the terms involving $\langle b_y\rangle$ and $\langle b_z\rangle$ entering only at the next order in ϵ. Averaging Eq. (3.25) over the variable X, we get

$$\partial_T V = \frac{i}{4k(1-\beta)}\langle B^*\Delta_\perp B - B\Delta_\perp B^*\rangle\ . \quad (3.29)$$

Noting that all the quantities now depend on $\xi = X - v_g\tau$ and T, we can rewrite Eq. (3.22) in the form

$$v_g\partial_\xi\bar b + \tfrac{1}{2}\nabla_\perp\left(\frac{|B|^2}{2} + \beta\hat\rho + (1+\beta)\bar a\right) = O(\mu)\ . \quad (3.30)$$

Deriving Eq. (3.23) with respect to X and using Eq. (3.30), we get

$$\frac{2k}{R_i}\partial_{\xi\xi}\bar a + \Delta_\perp\left(\beta\hat\rho + \frac{|B|^2}{2} + (1+\beta)\bar a\right) = 0\ . \quad (3.31)$$

Averaging Eq. (3.31) with respect to ξ and assuming that the total volume averages of $\hat\rho$ and $\bar a$ vanish, we recover Eq. (2.31). It follows that

$$\hat\rho = \frac{1}{1-\beta}\left[\beta\bar a + \frac{|B|^2}{2} - \frac{1}{(1+\beta)}\left(\left\langle\frac{|B|^2}{2}\right\rangle + \beta E^M - \hat R_0\right)\right]\ . \quad (3.32)$$

Furthermore, in the frame of reference moving with the group velocity v_g, Eq. (3.26) reduces to

$$\partial_T B + i\left[k\left(V + \frac{\hat\rho}{2} + \frac{\bar a}{2}\right)B - \frac{1}{4k(1-\beta)}\Delta_\perp B + \frac{1}{2R_i}\partial_{XX}B\right] = 0\ . \quad (3.33)$$

Combining Eqs. (3.29) and (3.33), we find

$$\partial_T V = \partial_T\langle|B|^2\rangle \quad (3.34)$$

or

$$V = \langle|B|^2\rangle + V_0(Y,Z)\ . \quad (3.35)$$

In the case where V vanishes initially, $V_0(Y,Z) = -\langle|B(0)|^2\rangle$. Let us denote

$$U(Y,Z) = V_0(Y,Z) + \frac{1}{2(1-\beta^2)}\hat R_0(Y,Z)\ . \quad (3.36)$$

This leads to the following system for the "outer" solution valid outside the cone in Fourier space defined by $|K_X|/|K_\perp| < \epsilon\mu$, where K_X and $K_\perp = K_Y + iK_Z$ denote the longitudinal and (complex) transverse wavenumbers

$$\frac{1}{i}\partial_T B + \frac{1}{2R_i}\partial_{\xi\xi}B - \frac{1}{4k(1-\beta)}\Delta_\perp B + k\left[\frac{1}{2(1-\beta)}\left(\frac{|B|^2}{2} + \bar a\right) + \frac{3-4\beta^2}{2(1-\beta^2)}\left\langle\frac{|B|^2}{2}\right\rangle + U(Y,Z) - \frac{\beta}{2(1-\beta^2)}E^M\right]B = 0, \quad (3.37)$$

$$\left(\frac{2k}{R_i}\partial_{\xi\xi} + \frac{1}{1-\beta}\Delta_\perp\right)\bar a = -\frac{1}{1-\beta}\Delta_\perp\left(\frac{|B|^2}{2} - \frac{\beta}{1+\beta}\left\langle\frac{|B|^2}{2}\right\rangle\right), \quad (3.38)$$

$$\langle\bar a\rangle = -\frac{1}{1+\beta}\left(\left\langle\frac{|B|^2}{2}\right\rangle - E^M\right)\ . \quad (3.39)$$

When the typical longitudinal scale of the solution is much larger than the transverse scale, Eqs. (3.37) and (3.39) are not valid. In this case Eqs. (3.22) and (3.23) simplify to

$$\Delta_\perp \left(\frac{|B|^2}{2} + \beta\hat{\rho} + (1+\beta)\bar{a} \right) = 0 \; , \qquad (3.40)$$

while Eqs. (3.24) and (3.25) become

$$\partial_T \hat{\rho} + \partial_{\tilde{X}} V = 0, \qquad (3.41)$$

$$\partial_T V - 2\partial_{\tilde{X}} V = \partial_{\tilde{X}} \left[(1-\beta)\hat{\rho} - \beta\bar{a} - \frac{|B|^2}{2} \right] + \frac{i}{4k(1-\beta)} \left(B^* \Delta_\perp B - B \Delta_\perp B^* \right), \quad (3.42)$$

with $\tilde{X} = \epsilon\mu X$. Equation (3.26) rewrites

$$\frac{1}{i}\partial_T B + k\left(V + \frac{\hat{\rho}}{2} + \frac{\bar{a}}{2} \right) B - \frac{1}{4k(1-\beta)} \Delta_\perp B = 0 \; . \qquad (3.43)$$

Note here that Eq. (3.43) is written in the same reference frame as Eqs. (3.3)–(3.8). Equations (3.40)–(3.43) are our "inner" expansion.

A system of equations simpler than (3.22)–(3.26) but nevertheless uniformly valid in the full range of scales is obtained by matching Eqs. (3.37)–(3.39) and (3.40)–(3.43). There indeed exists a matching region where both the outer and inner expansions reduce to Eqs. (3.37)–(3.39) without the ξ-derivative terms. This leads to the "uniform" expansion

$$\frac{1}{i}\left(\partial_T B + \frac{v_g}{\mu}\partial_X B \right) + k\left(V + \frac{\hat{\rho}}{2} + \frac{\bar{a}}{2} \right) B - \frac{1}{4k(1-\beta)} \Delta_\perp B + \frac{1}{2R_i}\partial_{XX} B = 0, \quad (3.44)$$

$$\partial_T \hat{\rho} + \frac{1}{\epsilon\mu}\partial_X V = 0, \qquad (3.45)$$

$$\partial_T V - \frac{2}{\epsilon\mu}\partial_X V = \frac{\partial_X}{\epsilon\mu} \left[(1-\beta)\hat{\rho} - \beta\bar{a} - \frac{|B|^2}{2} \right] + \frac{i}{4k(1-\beta)} \left(B^* \Delta_\perp B - B \Delta_\perp B^* \right), \qquad (3.46)$$

$$\frac{2k}{R_i}\partial_{XX}\bar{a} + \Delta_\perp \left(\beta\hat{\rho} + (1+\beta)\bar{a} + \frac{|B|^2}{2} \right) = 0 \; . \qquad (3.47)$$

Several remarks are in order. First the coefficients ϵ and μ disappear when the above system is rewritten using the primitive variables for coordinates and fields. Furthermore in the absence of transverse modulation, we recover the Zakharov-type equations [19] given by Ovenden, Shah, and Schwartz [20] when coming back to the laboratory frame of reference. In this special case, the longitudinal magnetic field vanishes and $\hat{\rho}$ reduces to the density perturbation. In addition, in the limit where $\epsilon\mu \to 0$, these one-dimensional equations reduce to the usual nonlinear Schrödinger equation. According to the polarization of the circularly polarized carrying wave, the wave packet will be focusing or defocusing [8].

When the modulation is not purely longitudinal, two mean drift terms come into play in the form of a Doppler shift in the equation for the envelope of the transverse magnetic field (3.44). One due to the longitudinal magnetic field $\bar{a}$ has the same character than the mean flow in the Davey-Stewartson equations for surface gravity waves [21,22]. The other one due to magnetosonic velocity perturbation V is analogous to the mean flow obtained at the onset of Rayleigh-Bénard convection [23].

The limit equations (3.37)–(3.39) could have been derived directly from Eqs. (2.33)–(2.39) by standard modulation techniques. They are analogous to the weakly nonlinear modulation analysis performed by Mjølhus and Hada [14], except that our equations include the average effect of the magnetosonic waves. The influence of this effect on the modulational instability is discussed in Sec. IV. Equations (3.37)–(3.39) do not include the small parameters ϵ nor μ because magnetosonic waves have been averaged out. This simplification is due to the smallness of the parameter ϵ. In situations where the group velocities of the two waves are different but comparable ($\epsilon \sim 1$) the small parameter μ can also be eliminated. A system of coupled equations, still including averages, is then obtained for the two waves in their own characteristic coordinates. An example is provided by counterpropagating waves discussed in [24,25].

Finally, the specific case of purely transverse modulation addressed by Hoshino [15] is recovered by removing the X dependence in Eqs. (3.44)–(3.47). Hoshino's analysis is performed in the low frequency MHD limit where dispersion is ignored. As a consequence, circularly polarized waves cannot be selected in contrast with the case we consider here. Furthermore the cubic NLS equation he obtains for one of the components of the magnetic field (that in the direction of the perturbation) omits the effect of the mean flow V.

IV. MODULATIONAL INSTABILITY AND SELFFOCUSING

The modulational stability of Alfvén waves is studied by performing a linear analysis of Eqs. (3.37)–(3.39). These equations are valid when the characteristic longitudinal and transverse scales are of the same order of magnitude, i.e., outside the cone in Fourier space, defined in Sec. III. It is nevertheless easily seen that these equations are still valid when the fields are independent of ξ; in this case the mean value $\langle\,\rangle$ identifies with the function itself and Eqs. (3.40)–(3.43) with $\frac{\partial}{\partial \tilde{X}} = 0$ are recovered.

Equations (3.37)–(3.39) have a solution of the form $\bar{a}_0 = 0$ and $B = b_0$ for any complex value b_0. In order to study its stability we write the perturbed field $B =$

$b_0(1+b)e^{i\phi}$. For non-purely-transverse perturbations, the linearized equations for the real quantities a, b, and ϕ read

$$\partial_T\phi + \frac{1}{2R_i}\partial_{\xi\xi}b - \frac{1}{4k(1-\beta)}\Delta_\perp b + \frac{k}{2(1-\beta)}(|b_0|^2 b + a) = 0, \quad (4.1)$$

$$-\partial_T b + \frac{1}{2R_i}\partial_{\xi\xi}\phi - \frac{1}{4k(1-\beta)}\Delta_\perp\phi = 0, \quad (4.2)$$

$$\frac{2k}{R_i}\partial_{\xi\xi}a + \frac{1}{1-\beta}\Delta_\perp a = -\frac{1}{1-\beta}|b_0|^2\Delta_\perp b, \quad (4.3)$$

where we used the property $\langle b\rangle = \langle a\rangle = 0$. For harmonic perturbations b, ϕ, and $a \propto e^{i(K_X\xi+K_YY+K_ZZ)-i\Omega T}$, we obtain the dispersion relation

$$\Omega^2 = \left(\frac{|K_\perp|^2}{4k(1-\beta)} - \frac{1}{2R_i}K_X^2\right) \times \left[-\frac{1}{2R_i}K_X^2 + \frac{k}{2(1-\beta)}|b_0|^2 + \frac{|K_\perp|^2}{2(1-\beta)}\left(\frac{1}{2k} - \frac{k|b_0|^2}{(1-\beta)(\frac{2k}{R_i}K_X^2 + \frac{1}{1-\beta}|K_\perp|^2)}\right)\right]. \quad (4.4)$$

Writing $|K_\perp| = K\sin\phi$ and $K_X = K\cos\phi$, we see that in the limit where K tends to zero, Ω^2 takes the sign of $k(1-\beta)(\tan^4\phi - \frac{4k^2(1-\beta)^2}{R_i^2})$. This agrees with the results of Mjølhus and Hada [14] in the case of non-purely-transverse perturbations ($\phi \neq \frac{\pi}{2}$). However the analysis of these authors becomes singular in the limit $\phi \to \frac{\pi}{2}$, and they suggested going back to the full fluid model. In fact, as we show here, it is enough to stay in the long-wavelength limit described by the DNLS equations, provided the effect of magnetosonic waves is included. For purely transverse perturbation, the latter reduces to a mean flow whose effect survives in the linear approximation. In this case, the dispersion relation reads

$$\Omega^2 = \frac{3+4\beta}{1-\beta^2}K^2\frac{|b_0|^2}{8} + \frac{K^4}{16k^2(1-\beta)^2}, \quad (4.5)$$

whereas in the limit $\phi = \frac{\pi}{2}$, Eq. (4.4) only gives the second term in the rhs of Eq. (4.5). It misses the mean drift effect, which is in fact predominant in the long-wavelength limit. Due to this term, the carrying wave is stable with respect to purely transverse perturbations for $\beta < 1$ and unstable for $\beta > 1$, regardless of its polarization.

In the limit $\epsilon\mu \to 0$ where the "cone" reduces to the plane $K_X = 0$, the frequency Ω undergoes a discontinuity at $\phi = \frac{\pi}{2}$. The introduction of magnetosonic waves in Eqs. (3.44)–(3.47) allows a smooth although sharp transition.

When considering the fully nonlinear regime, the situation is more complex. Inspection of (3.37)–(3.39) shows a competition between focusing and defocusing effects. If focusing in the transverse directions takes place, the magnetosonic waves should play a dynamic role not only through an average effect, but pointwise. It is then necessary to turn to Eqs. (3.44)–(3.47). This problem deserves a comprehensive study which will be addressed in a forthcoming paper. The situation, however, simplifies in the case of a purely transverse modulation. In contrast with the approach of Mjølhus and Hada [14], where, due to the omission of the mean flows all the nonlinear terms cancel out, our equations reduce to

$$\frac{1}{i}\partial_T B - \frac{1}{4k(1-\beta)}\Delta_\perp B + k\left(\frac{4\beta+3}{2(1+\beta)}\frac{|B|^2}{2} + U(Y,Z)\right)B = 0, \quad (4.6)$$

where the term proportional to the magnetic energy E^M has been eliminated by a phase shift, while U defined in (3.36) can be viewed as a time independent external potential. It is well known that in the case $\beta > 1$, this two-dimensional nonlinear Schrödinger equation can develop a singularity in a finite time [19,26,27]. The nature of this singularity is discussed in [28] and references therein.

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