Riemann zeros and a fractal potential

Hua Wu and D. W. L. Sprung
Department of Physics and Astronomy, McMaster University, Hamilton, Ontario, Canada L8S 4M1
(Received 1 July 1993)

The nontrivial Riemann zeros are reproduced using a one-dimensional local-potential model. A close look at the potential suggests that it has a fractal structure of dimension $d = 1.5$.

PACS number(s): 05.45.+b

The implications of classical chaos for quantum systems are of considerable interest in physics. One of the signatures most carefully studied is the energy-level statistics of a quantum system. Gutzwiller's formula [1–3] links this level density to the periodic orbits of the corresponding classical dynamical system in a semiclassical approach. It has been noticed that the Riemann ζ function [4, 5], well known in number theory, has an intriguing relationship with chaotic systems [6, 7]. The so-called nontrivial zeros of the Riemann zeta function exhibit an intrinsically random distribution of the type seen in the spectrum of a random Hermitian matrix (Gaussian unitary ensemble or GUE), and there is a sum rule formula for the density of these zeros which is analogous to Gutzwiller's formula. All this is taken to suggest that there may be a chaotic dynamical system whose quantum eigenvalues are the Riemann zeros. The discovery of such a system would have great significance to the study of "quantum chaos;" and in any case it is interesting to see that a classical mathematical function can have a direct connection with modern physics. One of the related systems is the hyperbolic billiard [8], whose average level density agrees with that of the Riemann zeros in the leading order, and the fluctuations are of Gaussian orthogonal ensemble (GOE) type.

The Riemann ζ function is defined for $\mathrm{Re}(z) > 1$ by

$$\zeta(z) = \sum_{r=1}^{\infty} \frac{1}{r^z} = \prod_{p \in \text{primes}} \frac{1}{1 - p^{-z}} \tag{1}$$

and for $\mathrm{Re}(z) \leq 1$ by analytic continuation. It has so-called trivial zeros at the real points $z = -2n, n = 1, 2, ..., \infty$. All its other zeros are complex and lie in a narrow strip about the vertical line $\mathrm{Re}(z) = 1/2$. According to Riemann's celebrated hypothesis (1856) the nontrivial zeros all lie symmetrically on that line, i.e., $\zeta(1/2 \pm iE_n) = 0$, where the E_n are real. Riemann's hypothesis is supported by numerical tests up to very large values of E_n, but mathematicians are still unable to prove or disprove it. For brevity we will refer to the E_n as being the Riemann zeros, instead of their imaginary parts. Assuming Riemann's hypothesis to be correct, the density of the zeros obeys:

$$d(E) - \bar{d}(E) \approx -\frac{1}{2} \sum_{p} \sum_{\substack{k=-\infty \\ (k \neq 0)}}^{\infty} \ln(p) \exp(-\tfrac{1}{2}|k| \ln p) \exp(iEk \ln p), \tag{2}$$

where $\bar{d}(E) = \frac{1}{2\pi} \ln(\frac{E}{2\pi})$ is the average density of zeros. It is this sum rule which is an analog to the Gutzwiller formula for the level density of dynamical systems. The sum over prime numbers p corresponds to the primitive periodic orbits and k is the repetition index.

As in spectroscopy of energy levels, the spectral density of the E_n can be divided into a smooth part and a fluctuating part. The smooth part gives the number of zeros below E as

$$\mathcal{N}(E) = \frac{E}{2\pi} \ln \frac{E}{2\pi e} + \frac{7}{8} \quad , \tag{3}$$

while the fluctuating part has been shown to obey GUE statistics [9]. This suggests that the E_n are eigenvalues of a quantum Hamiltonian which is quantized from a classical chaotic system without time-reversal invariance [6, 10]. However, such a system has yet to be found.

Some time ago, we developed an algorithm to construct a one-dimensional local potential model obeying GOE level statistics [11]. This algorithm will be applied here to the Riemann zeros, so as to reproduce the low lying E_n. By so doing, we have effectively found a system which has the imaginary parts of (a finite number of) the Riemann zeros as its eigenvalues. However, it is not what was expected, as the system is time reversible, and being one-dimensional, it is apparently integrable. This conundrum is partially resolved when we examine the fractal structure of the resulting potential. To begin, we describe its construction.

Consider a particle of mass m moving in a reflection-symmetric one-dimensional local potential $V(x)$,

$$H = \frac{p^2}{2m} + V(x). \tag{4}$$

In the semiclassical approximation, the accumulated number of states below E is

$$\mathcal{N}(E) = \frac{1}{h} \int\int_{H \leq E} dx\, dp = \frac{2}{\pi} \int_0^{x_{\max}} \sqrt{E - V(x)}\, dx, \tag{5}$$

where we have chosen $2m/\hbar^2 = 1$, and $x_{\max}$ is the x value of the right-hand turning point.

Our first task is to find the potential V which gives rise to $\mathcal{N}(E)$ as in Eq. (3). Let $V(0) = V_0$, $f(V) = dx/dV$, and differentiating Eq. (5) with respect to E, we have

$$\int_{V_0}^{E} \frac{f(V)}{\sqrt{E - V}} dV = \frac{1}{2} \ln\left(\frac{E}{2\pi}\right). \tag{6}$$

This is called Abel's integral equation and its solution is

$$x = \frac{1}{\pi}\left\{\sqrt{V - V_0}\ln\frac{V_0}{2\pi e^2} + \sqrt{V}\ln\frac{\sqrt{V} + \sqrt{V - V_0}}{\sqrt{V} - \sqrt{V - V_0}}\right\}. \tag{7}$$

Because V_0 appears inside a logarithm, it should be nonnegative. In the limit $V_0 \to 0$,

$$x = \frac{\sqrt{V}}{\pi}\ln\frac{2V}{\pi e^2} \quad . \tag{8}$$

However, considering V as a function of x, this solution is double valued for small x. We found there is a critical value, $V_0 = 2\pi$, beyond which $V(x)$ is single valued. The choice of V_0 affects the potential at small x, but when x is large, they all approach Eq. (8). A practical way to fix V_0 is to demand that our potential model fit the first zero (14.134 725); this leads to $V_0 = 3.100\,73\pi$, somewhat above the critical value. In Fig. 1 we plot $V(x)$ for several V_0 values.

Our second task is to construct a potential which exactly reproduces the E_n. Obviously, to reproduce all (an infinite number of) the E_n is beyond our ability. A more manageable goal is to reproduce the first N zeros. For this we require numerical values of the $E_n, n = 1, ..., N$, which can be found following Odlyzko's work [9]. For us, the simplest way is to compute them using MATHEMATICA, in which the ζ function is defined in the complex plane.

Starting from the potential given by Eq. (7), one can solve the Schrödinger equation to have a set of eigenvalues $e_n, n = 1, ..., N$. The least-squares function is taken to be

$$F = \sum_n (e_n - E_n)^2 \quad . \tag{9}$$

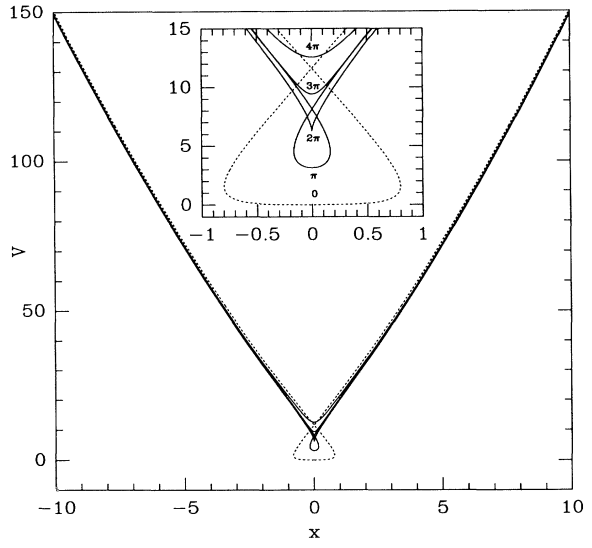

FIG. 1. Potentials $V(x)$ which generate the smooth part of the Riemann zero spectrum. The label in the inset is the V_0 value for the corresponding curve.

We must now vary $V(x)$ to minimize F. To do this, we take the functional derivative of F with respect to $V(x)$. Since

$$\frac{\delta e_n}{\delta V(x)} = \frac{\delta}{\delta V(x)}\langle n|H|n\rangle = \phi_n^2(x) \quad , \tag{10}$$

where $\phi_n(x)$ is the normalized wave function, one has

$$\frac{\delta F}{\delta V(x)} = 2\sum_n (e_n - E_n)\phi_n^2(x). \tag{11}$$

In practice, the functional derivative is replaced by the partial derivative of F with respect to $V(x)$ at specified grid points. Thus the problem is transformed into minimization of a function in a high order parameter space. We have used the conjugate gradient method to perform this minimization.

We have fitted up to $N = 500$ zeros. The Schrödinger equation is solved by the Numerov method on a grid of 8392 points with an intergrid distance 0.005 on the positive real axis. Because the potential is assumed to have reflection symmetry, only the $x \geq 0$ part is independent. We achieved a fit of all N zeros with average deviation less than 4×10^{-6}. However, the accuracy of solution of the differential equation by the Numerov method, for the last levels, is only 3×10^{-4}, so this is the dominant error. This is 0.02% of the average energy level spacing. With smaller intergrid distance, this could be further reduced.

In Fig. 2 we show the potential fitting 500 levels, over a region wide enough to include the smooth extension of $V(x)$ beyond the turning point of the last level. In the inset we show, on an expanded scale, the fluctuations of $V(x)$ for the cases of 100, 300, and 500 levels.

We make the following observations: (1) there is no

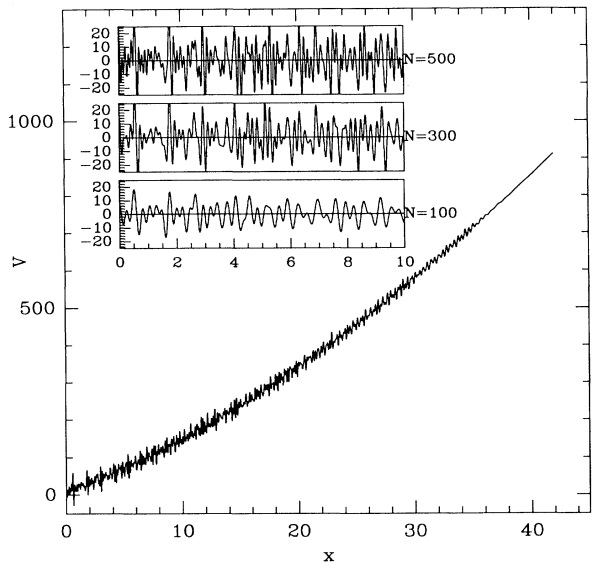

FIG. 2. Potential $V(x)$ which fits the first 500 Riemann zeros. Shown in the inset are the fluctuations for the potentials fitting the first 100, 300, and 500 zeros. By fluctuation, we mean the fitted potential minus the smooth part given by Eq. (7) with $V_0 = 3.100\,73\pi$.

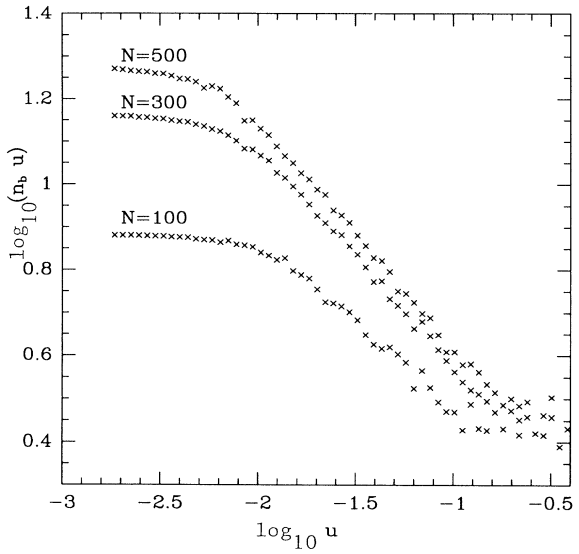

FIG. 3. Box-counting-method analysis of the fractal dimension of the potential curve for $N = 100, 300$, and 500. Logarithms are to the base 10. The negative slope of the curve is the fractal dimension minus 1. The box size for $\log_{10}(u) = -1$ and -2 in the original x space is 1 and 0.1, roughly the range in which the fractal dimension is valid.

fluctuation beyond the last level's turning point; (2) for a fixed N, the fluctuations of the fitted potential decrease as energy increases; (3) in a fixed x range, finer structure appears in the potential built on the existing structure as N increases. All these can be qualitatively understood with the help of Eq. (11). The functional derivative of F with respect to $V(x)$ determines the direction one moves in the parameter space. All the wave functions decay exponentially beyond the turning point, and thus quickly die away. This implies that the lower energy levels have very small influence on the potential beyond their turning points. Therefore the part of the potential beyond the last level's turning point remains unchanged. Clearly, the potential in the low energy range has greater "responsibility" than that in a higher energy range, thus it has more structure. The wave function inside the potential well is basically an oscillatory wave of wave number $k(x) = \sqrt{E - V(x)}$. The amplitude of the wave decreases as the energy rises. In a fixed x range, a higher level has a more finely oscillating wave function. Thus as the number of fitted levels increases, we see more and more fine detail in the potential. Based on this argument, the finest structure will be determined by the wave number of the last wave function.

This leads to the following question [12]: what will happen if an infinite number of levels are to be fitted? For any fixed range of x, there will be an infinite number of higher levels which affect the determination of the potential; thus there will be infinitely fine structure in the resulting potential. These observations suggest the fitted potential might have a fractal structure, to be described by its fractal dimension. Since we cannot prove strict self-resemblance in the refining process, the fractal geometry is only defined weakly, as is the case for example for a coastline map.

Figure 3 shows our fractal dimension analysis using the box counting method. We picked that part of the potential curve within $0 \leq x \leq 10$ (2000 points). We then rescaled the curve so that it fits in a 1×1 box. This area is then divided into boxes of side u. The number of boxes containing any part of the curve is counted to be n_b. For a given N, there is a lower limit on the width of the potential fluctuations. If the box size is smaller than this smallest structure, one will have $n_b \sim 1/u$. Then on a log-log plot, $\log_{10}(n_b)$ vs $\log_{10}(u)$ will be a straight line of slope -1. For a clearer view, we plot $\log_{10}(n_b u)$ vs $\log_{10}(u)$ which for dimension 1 should be a flat line with no slope. Conversely, there will be a maximum size of the fluctuations. If the box size is coarser than this, the box counting method sees the curve as a thick line, and one will again see zero slope. In between these two limits, finer structure will be increasingly picked up as the box size decreases. If the fractal dimension is a good prescription, $n_b \sim 1/u^d$, then $\log_{10}(n_b u) = (1-d)\log_{10}(u) + \text{const}$. We have shown the results for $N = 100, 300$, and 500, and it can be seen that there is a good linear region in between the two limits in all three cases. As expected, the small box limit moves to the left as N increases. The slope for the $N = 500$ curve yields $d = 1.5$. If we go even higher in N, we expect that d will not change much, but the linear region where the fractal dimension is valid will expand.

In summary, we have found analytically a one-dimensional local potential which generates the smooth average level density obeyed by the Riemann zeros. We have then shown how any finite number of low lying Riemann zeros can be reproduced by introducing fluctuations on top of the potential. The mystery of how a one-dimensional integrable system can produce a "chaotic" spectrum is resolved by adopting the concept of a fractal potential which, in the infinite N limit, would lead to the system having a dimension larger than one.

We are grateful to NSERC Canada for continuing support under research Grant No. OGP00-3198. We also thank Professor R. K. Bhaduri, Professor D. A. Goodings, Professor O. Bohigas, and Professor E. B. Bogomolny for fruitful discussions and for providing some references.

[1] M. C. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer-Verlag, Berlin, 1990).
[2] M. C. Gutzwiller, J. Math. Phys. **12**, 343 (1971).
[3] R. Balian and C. Bloch, Ann. Phys. (N.Y.) **69**, 76 (1972).
[4] H. M. Edwards, *Riemann's Zeta Function* (Academic, New York, 1974).
[5] P. Ribenboim, *The Book of Prime Number Records*, 2nd ed. (Springer-Verlag, New York, 1989).
[6] M. V. Berry, in *Quantum Chaos and Statistical Nuclear Physics*, edited by T. H. Seligman and H. Nishioka, Springer Lecture Notes in Physics Vol. 263 (Springer-Verlag, Berlin, 1986), pp. 1–17.

[7] M. V. Berry, in *Chaos and Quantum Physics*, edited by M-J. Giannoni, A. Voros, and J. Zinn-Justin, Les Houches Lecture Series LII (North-Holland, Amsterdam, 1991), pp. 251–304.
[8] M. Sieber and F. Steiner, Physica **D 44** 248 (1990); Phys. Lett. A **148**, 415 (1990); F. Steiner and J. P. Trillenberg, J. Math. Phys. **31** 1670 (1990).
[9] A. M. Odlyzko, Math. Comput. **48**, 273 (1987).
[10] M. V. Berry, Proc. R. Soc. London A **400**, 229 (1985).
[11] Hua Wu, M. Vallieres, D. H. Feng, and D. W. L. Sprung, Phys. Rev. A **42**, 1027 (1990).
[12] O. Bohigas, in *Chaos and Quantum Physics*, edited by M-J. Giannoni, A. Voros, and J. Zinn-Justin, Les Houches Lecture Series LII (North-Holland, Amsterdam, 1991), pp. 87–199.