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Quasi-isochronous storage rings

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A study is made of the single-particle dynamics of an electron-positron storage ring where the phase-slip factor is made small in order to make the ring nearly isochronous and reduce the bunch length. What is found is that a quasi-isochronous ring makes it possible to obtain a bunch length in the millimeter range, about one order of magnitude shorter than present values. In this study we have extended the work of others on isochronous storage rings by quantitatively including higher-order terms in the longitudinal equations of motion. Scaling laws are then derived relating the linear term with the next-highest-order term. These scaling laws, which are derived from a two-dimensional Hamiltonian (one dimension of position and one of momentum), establish criteria for stability. These scaling laws are then checked with full six-dimensional tracking on one particular lattice.

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I. INTRODUCTION

The exploration of the structure of matter at smaller and smaller distances follows two paths. The first is the construction of higher-energy accelerators, like the Superconducting Super Collider (SSC), large electron-positron (LEP) storage rings, and future linear colliders, to allow for a direct investigation of very short distances and more massive particles; the second is the study, usually at lower energies, of the violation of symmetry principles, or detailed tests of the validity of the standard model. Examples of this second approach are the study of *CP* violation in *K*-meson or *B*-meson systems, and the related proposals of *K* factories, *B* factories, and ϕ factories. In these factories, the most important parameter is the collider luminosity. To reach their goal these systems must have an ever larger luminosity; in the case of *B* factories, the required value is greater than 10^{33} $\text{cm}^{-2}\,\text{s}^{-1}$, and values in excess of $10^{34}\,\text{cm}^{-2}\,\text{s}^{-1}$ would be desirable.

One strategy for increasing the collider luminosity is to increase the average electron and positron beam current. An alternative to this approach is to make the luminosity larger by increasing the beam densities at the interaction point [1]. This requires a reduction in the bunch length and a strong beam transverse focusing to a β function of the order of the bunch length. In this case the luminosity scales like the inverse of the bunch length.

In this paper, we focus on the possibility of reducing the bunch length in an electron-positron storage ring collider by making the storage ring nearly isochronous, i.e., with a revolution time independent of particle energy. This is done by reducing the linear term in the ring phase-slip factor to nearly zero. We study the beam dynamics in this essentially nonlinear situation and establish the condition for stable single-particle motion.

We find that, by considering only the longitudinal degree of freedom (corresponding to the direction in which the beam is traveling), it is possible to arrive at an analytical formula which describes the size of the stable longitudinal phase-space area, including the effects of nonlinear terms. The size of the stable phase-space area in an accelerator is important because it has a direct bearing on the lifetime of a beam of particles in a storage ring. The larger this area is, the smaller the chance that a particle can "visit" an unstable region of phase space and get lost from the beam.

From this analytical formula we derive scaling laws which determine how large the nonlinear terms can be and still provide a sufficiently large enough stable phase-space area for a good beam lifetime. We also show how it is possible to control the nonlinear terms in the equations of motion with sextupoles and higher-order magnets.

Finally, our scaling laws, which are derived from a two-dimensional Hamiltonian, are checked on a specific accelerator lattice with six-dimensional (6D) tracking. The accelerator lattice which we chose as an example of a quasi-isochronous ring is the synchrotron at the Ultraviolet Synchrotron Orbital Radiation (UVSOR) facility at the Institute for Molecular Science in Okazaki, Japan [2]. The results of 6D tracking code give us confidence that the scaling laws do give a good measure of the size of the stable phase-space area for that lattice.

Reference frame

Prior to beginning a discussion of the equations of motion of a particle in a storage ring collider, we will first define the reference frame to be used throughout this discussion. There exists in all storage rings a closed orbit called the ideal or reference or design orbit of the ring. This design orbit is the orbit of the "ideal" particle for which the machine is designed. The ideal particle has the reference energy E_0 and the proper phase with respect to the radio-frequency cavity, and follows this design orbit.

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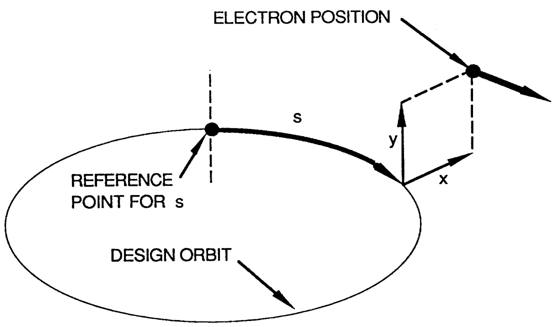

FIG. 1. Coordinate system.

It is convenient to use a coordinate system where a particle's position is measured with respect to this design orbit. The instantaneous position of a particle in the ring can be specified in terms of (s,x,y), where s is the azimuthal coordinate of the particle measured along the design orbit from some reference point, and x and y are the respective radial and vertical distances of the particle from the design orbit. This coordinate system is illustrated in Fig. 1.

II. LONGITUDINAL EQUATIONS OF MOTION (TWO-DIMENSIONAL THEORY)

Our discussion of quasi-isochronous storage rings will be preceded by a short summary of the general equations of motion for the longitudinal degree of freedom of a storage ring, after which the differences between conventional rings and quasi-isochronous rings may be more clearly illustrated. The main difference between a conventional and a quasi-isochronous storage ring lies in the longitudinal beam dynamics; the transverse beam dynamics are not strongly influenced, except for the synchrobetatron-coupling effects. In particular, the synchrotron oscillation frequency is assumed to be very small. Defining what we mean by very small is one of the key questions to be addressed here.

Let us first define what we mean when we say that a storage ring is isochronous. A storage ring is isochronous when the time it takes for a particle to make one revolution around the ring is independent of its energy. The degree to which a storage ring approaches the isochronous condition is described by the parameter η, the phase-slip factor. The phase-slip factor is defined as the relative difference of the revolution time which an arbitrary particle and the reference particle take to go around the ring, divided by the arbitrary particle's relative energy deviation from the reference particle:

$$\eta=\frac{(T_a-T_0)/T_0}{(E_a-E_0)/E_0} . \tag{1}$$

In the limit that η goes to zero, the machine is operating in an isochronous mode. When η is small, we say the machine is operating in a "quasi-isochronous" mode. What we mean by small is several orders of magnitude smaller than what exists in machines presently. This means values of the first-order phase-slip factor η_{c_1} [see Eqs. (11) and (12)] of 10^{-4}–10^{-6}. The bunch length in a storage ring is proportional to the square root of η_{c_1} [1]. Reducing η_{c_1} by two orders of magnitude results in a bunch-length reduction of one order of magnitude. Reducing the phase-slip factor is the method by which the quasi-isochronous ring accomplishes the decrease in the bunch length. Of interest is the fact that, when we make η_{c_1} small, nonlinear terms which are usually neglected in the equations of motion can become important. To allow for this possibility, we assume in the equation of motion that η is a function of the particle energy, $\eta=\eta(\delta)$, where $\delta=(E_a-E_0)/E_0$ is an arbitrary particle's relative energy deviation from the reference particle. We define the phase distance Ψ as the difference between the arbitrary particle's and the reference particle's time of arrival at the rf cavity multiplied by $2\pi/T_0$. Therefore, in one turn the change in the phase distance is $\Delta\Psi=2\pi(T_a-T_0)/T_0$. Using δ and Ψ as variables, we can write the equations of motion for electrons in the presence of synchrotron-radiation energy losses and a radio-frequency system that can compensate these losses as

$$\Psi'=\eta(\delta)\delta , \tag{2}$$

$$\delta'=\frac{eV_0}{2\pi E_0}\sin(h\Psi+\phi_0)-\frac{U_0}{2\pi E_0}(1+J_E\delta) + (\text{fluctuations}) , \tag{3}$$

where V_0 is the rf peak voltage, U_0 is the energy radiated per turn from the reference particle, J_E is the radiation damping partition number [3] and with (fluctuations) we indicate the term arising from quantum fluctuations in the emission of synchrotron radiation. The prime superscript implies a derivative with respect to $\omega_0 t$, where t is time and $\omega_0=2\pi/T_0$ is the revolution frequency of the reference particle around the ring. The harmonic number h is the ratio of the rf frequency to the revolution frequency $(h=\omega_{\text{rf}}/\omega_0)$.

Phase slip factor

The phase-slip factor $\eta(\delta)$, as discussed earlier, is dependent upon two quantities: the difference in velocity between the test particle and the ideal particle, and the difference in path length between the test particle and the ideal particle as they travel around the ring. The faster the test particle moves, the farther it moves, tending to decrease Ψ; however, the longer the path length, the longer it will take to move around the ring, tending to increase Ψ. The information concerning these effects is embodied in the phase-slip factor and can be rewritten as

$$\eta=\frac{(T_a-T_0)/T_0}{(E_a-E_0)/E_0}=\frac{\Delta T/T_0}{\Delta E/E_0}=\frac{\Delta\Psi/2\pi}{\Delta E/E_0} , \tag{4}$$

where $\Delta\Psi$ is the change in Ψ per turn for given ΔE.

The full expression for η is [4]

$$\eta=\frac{1}{L_0\delta}\int_0^{L_0}ds\left\{\left[\left[1+\frac{x}{\rho_s}\right]^2+(x')^2+(y')^2\right]^{1/2}-1\right\}$$
$$-\frac{\Delta\beta_p}{\beta_p\delta}\ , \tag{5}$$

where ρ_s is the local radius of curvature of the design trajectory, and β_p is the velocity of the particles in the laboratory frame divided by the velocity of light. To simplify this initial discussion of a quasi-isochronous ring, we will expand η in successive orders of δ and assume that δ is constant for each particle during one revolution. This is a reasonable assumption in the absence of synchrotron radiation and in the limit that the synchrotron oscillation frequency ν_{s_0} [see Eq. (23)], is small. We do this in two steps. First, x, y, x', and y' are written in terms of a series expansion in powers of δ:

$$x=x_\beta+D_{x_0}\delta+D_{x_1}\delta^2+\cdots\ , \tag{6}$$

$$y=y_\beta\ , \tag{7}$$

$$x'=x'_\beta+D'_{x_0}\delta+D'_{x_1}\delta^2+\cdots\ , \tag{8}$$

$$y'=y'_\beta\ , \tag{9}$$

where x_β is the betatron amplitude of the oscillation and D_{x_0} and D_{x_1} are the first- and second-order components of the dispersion function. We assume for simplicity that the dispersion is zero in the vertical (y) direction. Second, the square root in Eq. (5) is expanded in powers of δ. The phase-slip factor can then be written

$$\eta=\frac{\eta_0}{\delta}+\eta_1+\eta_2\delta+\cdots\ . \tag{10}$$

At this point we define the closed-orbit phase-slip factor η_c, which is the phase-slip factor without any betatron-oscillation terms (i.e., $x_\beta=y_\beta=0$). In other words, η_c is the difference in revolution time that a particle with an energy offset δ traveling on its closed orbit takes to circulate around the ring relative to the reference particle. For the remainder of this discussion we will discuss only η_c, neglecting betatron oscillations. However, betatron oscillations will be introduced in the numerical tracking.

The closed-orbit phase-slip factor η_c can be expressed as a power-series expansion in δ:

$$\eta=\eta_{c_1}+\eta_{c_2}\delta+\cdots\ . \tag{11}$$

The term η_{c_1} is given by

$$\eta_{c_1}=\frac{1}{L_0}\int_0^{L_0}ds\left[\frac{D_{x_0}}{\rho_s}\right]-\frac{1}{\beta_0^2\gamma_0^2}\ . \tag{12}$$

The η_{c_2} term is given by

$$\eta_{c_2}=\frac{1}{L_0}\int_0^{L_0}ds\left[\frac{D'^2_{x_0}}{2}+\frac{D_{x_1}}{\rho_s}-\frac{1}{\beta_0^2\gamma_0^2}\frac{D_{x_0}}{\rho_s}\right]$$
$$+\frac{3}{2\beta_0^2\gamma_0^2}\left[1+\frac{1}{\beta_0^2\gamma_0^2}\right]\ . \tag{13}$$

Normally, the η_{c_1} term is the dominant term in determining the particle's motion. For highly relativistic particles it is usually positive, but can be made nearly zero or negative by having regions of inverted bending in the ring, $\rho_s<0$, or of negative dispersion $D_{x_0}<0$ [5].

As examples of typical values of the emittance, first-order phase-slip factor, dispersion, and β_x, we give their values in the smooth approximation [3]. In the smooth approximation the emittance ε, the phase-slip factor η_{c_1}, the dispersion D_{x_0}, and the horizontal beta function β_x are

$$\varepsilon=\frac{J_E}{J_x}\frac{R}{\nu_x^3}\delta^2_{\rm rms}\ , \tag{14}$$

$$\eta_{c_1}=\frac{1}{\nu_x^2}-\frac{1}{\beta_0^2\gamma_0^2}\ , \tag{15}$$

$$D_{x_0}=\frac{R}{\nu_x^2}\ , \tag{16}$$

$$\beta_x=\frac{R}{\nu_x}\ , \tag{17}$$

where ν_x is the horizontal tune of the ring, R is the average radius of the ring, J_x is the horizontal betatron radiation damping partition number, and $\delta_{\rm rms}$ is the relative rms energy spread of particles in the ring.

For a ring which has a 8-m radius, an rms energy spread of 3.5×10^{-4} and a horizontal tune of 3:$\varepsilon=7.3\times10^{-8}$ m rad, $D_{x_0}=0.89$ m, $\eta_{c_1}=0.11$, and $\beta_x=2.67$ m having assumed $J_E/J_x=2$.

III. QUASI-ISOCHRONOUS STORAGE RINGS

The value of η_{c_1} can be adjusted to be zero or negative by having regions of negative dispersion or inverted bending in the ring [see Eq. (12)]. The effects of the higher-order terms of the phase-slip factor become important when the linear phase-slip factor η_{c_1} is made small.

As a first step toward understanding the behavior of a quasi-isochronous storage ring, we study the equations of motion where the phase-slip factor is given as

$$\eta=\eta_{c_1}+\eta_{c_2}\delta \tag{18}$$

and ignore higher-order terms in δ.

We begin with a discussion of what is important for good beam stability and lifetime in the ring from a single-particle-dynamics point of view. A serious consideration when designing a storage ring which has good beam lifetime is that there should be a "large" three-dimensional volume, the dynamic aperture, in which particles can stably circulate around the ring, oscillating around the reference trajectory. For a ring circulating electrons or positrons, this volume should be at least ten times the rms value in all three dimensions [3]. The reason for this is that sudden changes in the momentum of a particle can result from the emission of a photon, and this change can shift the particle to a much different region of phase space than it occupied previously. The particle will then tend to damp down to the reference particle's position because of radiation damping. The whole region in which the particle "lives" must be stable or the particle will be lost. Because the longitudinal equations of motion can be rather nonlinear in a quasi-isochronous storage ring, we had concerns about the size of the stable longitudinal phase-space area. We have derived general scaling laws which give the size of the longitudinal phase space in terms of η_{c_1} and η_{c_2}.

In the development of these scaling laws, four approximations are made. The first approximation is that the longitudinal and transverse motion are uncoupled. So, when looking at longitudinal phase space, only the longitudinal equations of motions (2) and (3) need to be considered. The second approximation made is that the transverse displacement of a particle is only a function of its energy, and can be written

$$x = D_{x_0} + D_{x_1}\delta \ . \tag{19}$$

In other words, the particle's betatron oscillations are ignored (i.e., $\eta = \eta_c$). The third approximation is based upon the assumption that there is no longitudinal damping in the system. The fourth approximation is that there are no energy fluctuations due to photon emissions.

These approximations are made for several reasons. The first reason is that treating the motion as completely decoupled makes arriving at an analytical expression for the size of the dynamic aperture possible. This assumption of decoupled motion is reasonable, especially if the transverse motion is relatively linear and the betatron oscillations are small.

The second reason is that the particles in the ring will perform synchrotron oscillations about a stable fixed point. This stable fixed point varies for particles with betatron oscillations of different amplitudes. Particles with a large betatron oscillation will oscillate about a point with a larger value of δ than particles with small betatron oscillations. As long as the stable fixed point is not shifted too much, the assumption of zero betatron amplitude should not affect the stable phase-space area.

The third reason for making these approximations is that longitudinal damping provides a stabilizing presence. In our calculations, we are concerned with beam loss due to leaving the dynamic aperture. In such a case this would lead to rapid particle loss, usually in a time much less then a damping time. Hence we neglect damping and stochastic fluctuation processes because they operate on a slower time scale. This is justifiable if the dynamic aperture is much larger then the beam emittances. The results which have been derived from this analysis should serve as guidelines.

The longitudinal equations of motion [Eq. (2) and (3)] in the absence of damping and fluctuations can be rewritten as

$$\phi' = h(\eta_{c_1}\delta + \eta_{c_2}\delta^2) \ , \tag{20}$$

$$\delta' = \frac{eV_0}{2\pi E_0}[\sin(\phi+\phi_0) - \sin(\phi_0)] \ , \tag{21}$$

where $\phi = h\Psi$. In this case the system can be described by the Hamiltonian

$$H = \tfrac{1}{2}h\eta_{c_1}\delta^2 + \tfrac{1}{3}h\eta_{c_2}\delta^3 + \frac{eV_0}{2\pi E_0}[\cos(\phi+\phi_0) + \phi\sin(\phi_0)] \ . \tag{22}$$

From the Hamiltonian, it is now possible to distinguish stable from the unstable regions of phase space. Now we would like to single out two different longitudinal phase-space regimes: the rf bucket and the α-bucket regime.

A. rf bucket

For accelerators with a large η_{c_1}, η_{c_2} can be ignored in the equations of motion. The stable phase space is bounded by a separatrix which can be seen in Fig. 2. The trajectories are characterized by one stable and one unstable fixed point.

For the sake of an example, let us assume that the ring is operating above transition, i.e., $\eta_{c_1} > 0$. That means that the higher-energy particles have a smaller revolution frequency than the lower-energy particles. In this case, $\cos\phi_0 < 0$. The synchrotron frequency or small oscillation frequency around the stable fixed point is

$$\nu_{s_0} = \left[\frac{h\eta_{c_1} eV_0|\cos\varphi_0|}{2\pi E_0}\right]^{1/2} . \tag{23}$$

The separatrix which passes through the unstable fixed point encloses the stable phase-space area (see Fig. 2). The fixed point is located at $\phi = \pi - 2\phi_0$ and $\delta = 0$. The maximum stable energy displacement is

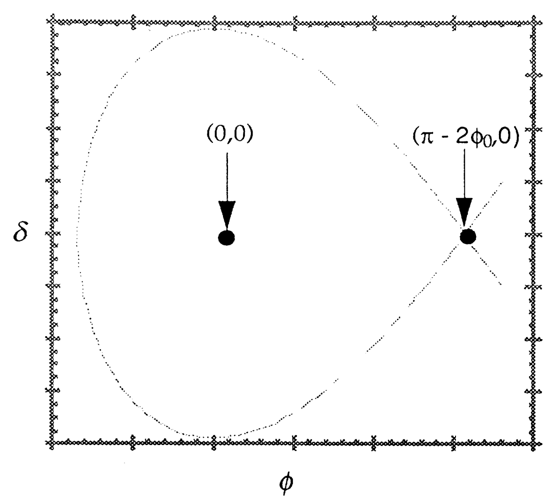

FIG. 2. rf-bucket regime.

$$\delta_m = \left[\left\{ \frac{2eV_0}{\pi h \eta_{c_1} E_0} \left[\left[\frac{\pi}{2} - \phi_0 \right] \sin_0 - \cos\phi_0 \right] \right\} \right]^{1/2} . \quad (24)$$

Now that the stable phase-space area is defined, it can be compared with the rms bunch length and rms energy spread. In this rf-bucket regime, where η_{c_1} is large and η_{c_2} is small, the conditions for a good lifetime are

$$\sigma_{L_{\rm rms}} < \frac{1}{20} \frac{R(\pi - 2\phi_0)}{h} , \quad (25)$$

$$\delta_{\rm rms} < \frac{1}{10} \delta_m . \quad (26)$$

B. α bucket

What happens as we decrease η_{c_1}? When we reduce the value of η_{c_1}, the energy acceptance given by Eq. (24) becomes larger. However, the longitudinal chromaticity term η_{c_2}, if nonzero, becomes important, and the phase-space trajectories are modified. With a nonzero value of η_{c_2}, there are now two stable fixed points and two unstable ones. The stable fixed points are ($\phi=0$, $\delta=0$) and ($\phi=\pi-2\phi_0$, $\delta=-\eta_{c_1}/\eta_{c_2}$). The unstable fixed points are ($\phi=\pi-2\phi_0$, $\delta=0$) and ($\phi=0$, $\delta=-\eta_{c_1}/\eta_{c_2}$).

There are two phase-space regimes corresponding to whether the distance between the stable and unstable fixed points is larger or smaller than the linear maximum-energy displacement defined in Eq. (24), and they are separated by the condition

$$\frac{\eta_{c_1}}{\eta_{c_2}} = \left[\left\{ \frac{2eV_0}{\pi h \eta_{c_1} E_0} \left[\left[\frac{\pi}{2} - \phi_0 \right] \sin\phi_0 - \cos\phi_0 \right] \right\} \right]^{1/2} . \quad (27)$$

These two regimes can be seen in Figs. 3(a) and 3(c), respectively, where 3(b) is the case which lies on the boundary between the two regimes and occurs when Eq. (27) is satisfied.

The first regime is the rf-bucket regime, where there are two stable phase-space areas which lie over each other [see Fig. 3(a)]. One bucket is just that which is illustrated in Fig. 2, and the other is one which is directly below it. These stable phase-space buckets are sometimes described as "fish," where the stable fixed points represents the eye and the unstable fixed points represents the tail. If the machine is operated above transition, the upper fish is "swimming" in the negative ϕ direction, while the lower fish is "swimming" in the positive ϕ direction.

The effect of decreasing the ratio of η_{c_1}/η_{c_2} is that the lower fish will rise toward the upper fish. At the point where $\eta_{c_1}/\eta_{c_2}=\delta_m$, the two fish are both "sharing" the two unstable fixed points. Each separatrix goes through the two unstable fixed points. By decreasing the ratio still further, the result is that the fish "exchange" tails. Now the fish are "swimming" up and down. The fish whose "eye" is at (0,0) is "swimming" in the positive δ direction, and the fish whose "eye" is at ($\pi-2\phi_0$, $-\eta_{c_1}/\eta_2$) is "swimming" in the negative-δ direction.

We refer to the regime in which the fish are "swimming" up and down as the α-bucket regime [see Fig. 3(c)]. In the rf-bucket regime, the rf cavity determines

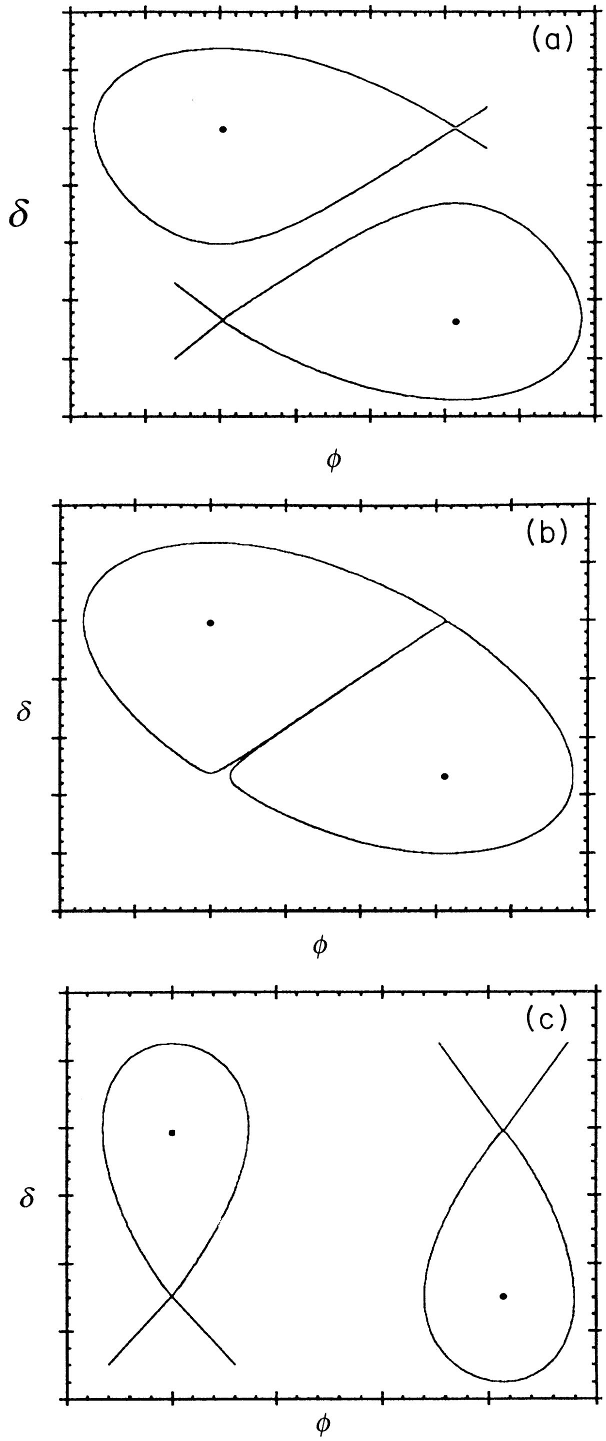

FIG. 3. (a) rf-bucket regime. (b) Boundary between rf bucket and α-bucket regime. (c) α-bucket regime.

the size of the buckets, whereas in the α-bucket regime the phase-slip factor determines the size of the buckets.

In the α-bucket regime the condition for a good lifetime is

$$\delta_{\rm rms} < \left| \frac{1}{20} \frac{\eta_{c_1}}{\eta_{c_2}} \right| . \quad (28)$$

For a given value η_{c_2}, this gives a limit for the smallest value of η_{c_1}, which the ring can support with a good lifetime. In order to have a smaller value of η_{c_1}, we would need to first reduce the value of η_{c_2}. In other words, the longitudinal chromaticity needs to be reduced in order to decrease η_{c_1}. For a given η_{c_1}, the phase space will be largest when $\eta_{c_2}=0$. Therefore it is desirable to set η_{c_2} to zero.

The term η_{c_2} acts as a change in the longitudinal or synchrotron tune of a particle in the ring as a function of its energy. The term η_{c_2} is related to what we call the "longitudinal chromaticity" of the ring, and it corresponds to a change in the synchrotron tune with energy

$$\nu_{s_1} = \frac{\partial \nu_s(\delta=0)}{\partial \delta} = \nu_{s_0} \frac{\eta_{c_2}}{\eta_{c_1}} . \quad (29)$$

The first and last two terms in the expression [Eq. (13)] for η_{c_2} are always positive. In the absence of sextupoles, the first term in Eq. (13) which is a function of $D_{x_0}'^2$ is always positive, and for highly relativistic particles is the dominant term for determining η_{c_2}. The second term in Eq. (13) which is a function of D_{x_1} can be made positive or negative with sextupoles and can balance out the other terms. Therefore the longitudinal chromaticity can be set to zero with sextupoles (setting $\eta_{c_2}=0$) in the same manner as transverse chromaticity. The use of sextupoles to control η_{c_2} has been suggested by others in connection with minimizing beam loss during transition crossing in hadron synchrotrons [6].

We have derived an expression to determine how effective a sextupole is at changing the value of η_{c_2}. By transforming the equations of motion around the δ-dependent fixed point, the only terms contributing to the δ-dependent closed orbit come from a strictly δ-dependent function. This is a result of the Hamiltonian nature of the flow. Hence in the case of a sextupole, the second-order phase-slip factor η_{c_2} must be a cubic function of the dispersion and nothing else. The change in η_{c_2} resulting from a thin sextupole with an integrated strength of S located at a longitudinal position $s(S)$, where D_{x_0} and D_{y_0} are the respective horizontal and vertical dispersions at $s(S)$, is

$$\Delta\eta_{c_2} = -\frac{SL_0}{3}(D_{x_0}^3 - 3D_{x_0}D_{y_0}^2) , \quad (30)$$

where L_0 is the length of the trajectory around the ring of the reference particle.

In order to independently control the chromaticity both transversely and longitudinally without effecting the linear lattice functions, three families of sextupoles are needed. This third family of sextupoles is important if the machine is going to operate at very small values of η_{c_1}. In Eq. (30), we give an expression for the effect of a sextupole on η_{c_2}. One can also determine the proper field strengths for all three sextupole families from the one-turn transfer map of the ring [7].

The two-dimensional theory can be generalized to include higher-order δ terms in the phase-slip factor. The unstable fixed point in the α-bucket regime is found by setting $\eta_c=0$. The maximum value of δ_m which is stable is the smallest solution of Eq. (31) which is real:

$$\eta_{c_1} + \eta_{c_2}\delta_m + \eta_{c_3}\delta_m^2 + \cdots = 0 . \quad (31)$$

In the development of these laws, we have ignored any synchrobetatron coupling. Therefore these results need to be verified for any given quasi-isochronous lattice with full 6D tracking.

IV. COMPARISON OF THE TWO-DIMENSIONAL THEORY WITH SIX-DIMENSIONAL TRACKING

We have made a comparison of the two-dimensional theory with six-dimensional tracking for one particular lattice. As mentioned earlier, the storage ring which we chose at an example of a quasi-isochronous ring is the lattice of the UVSOR ring [8]. A list of parameters of the ring in a normal and low η_c operation are given in Table I.

The lattice of the ring is a double-bend achromat of periodicity four. There are four families of quadrupoles and two families of sextupoles. When going from the normal to the low-η_{c_1} configuration, the four families of quadrupoles were adjusted to keep the transverse tunes constant, to keep the β functions at the end of each period nearly constant, and to vary η_{c_1}. The sextupoles

TABLE I. UVSOR ring parameters in normal operation and small-η configurations. Parameters were supplied by H. Hama.

Parameters	Normal η	Small η
Length of the ring (m)	52.3	52.3
Energy of the beam (MeV)	600	600
Horizontal tune	3.16	3.16
Vertical tune	2.62	2.62
Phase-slip factor, η_{c_1}	3.5×10^{-2}	1.297×10^{-3}
Peak voltage of rf cavity (V)	47.5×10^3	47.5×10^3
Central frequency (MHz)	90.115	90.115
Harmonic number	16	16
Synchronous angle (rad)	-0.111	-0.111
(Energy loss)/(turn) (eV)	5.2×10^3	5.2×10^3
Synchrotron tune (kHz)	14.8	2.849
Synchrotron period (no. of turns)	381	1979
rms energy spread (rel)	3.46×10^{-4}	3.46×10^{-4}
Bunch length (mm)	39	8

were adjusted to keep the transverse chromaticities constant. In the low-η_{c_1} configuration, η_{c_1} is 1.3×10^{-3} and η_{c_2} is 0.16. The first-order phase-slip factor in the low η_{c_1} is $\frac{1}{30}$ of the normal configuration. We used this low η_{c_1} configuration for our tracking comparison.

The tracking was done with an explicit symplectic integrator [9]. The integrator was derived from the full six-dimensional Hamiltonian [10] for a particle in an isomagnetic guide field with a thin-lens cavity. Also, this code utilizes automatic differentiation [11] to calculate Taylor series relative to the synchronous particle making one revolution around the ring. From these Taylor series or one-turn map, we can extract both linear and nonlinear properties of the map such as chromaticity, η_{c_1}, η_{c_2}, η_{c_3}, etc. [7].

We chose the following criteria for determining whether or not a particle is outside the dynamic aperture: A particle which ventures more than 1 m transversely from the reference orbit is considered lost and is thus outside of the dynamic aperture.

We used the following procedure when tracking. Particles were launched with initial transverse coordinates x and y and an initial relative energy offset δ, but no initial transverse momenta ($p_x=p_y=0$) or initial longitudinal offset ($s=0$). These particles were "pushed" around the ring until they were either lost from the dynamic aperture or survived five synchrotron oscillations (10 000 turns). The particles with the largest initial values of x and y which survived were recorded. This gives us a fairly good idea of the size of the dynamic aperture.

We initially tracked particles with small betatron oscillations. The results can be seen in Fig. 4. The cusp of the "fish" is at $\delta=0.008$, which is the value predicted by η_{c_1}/η_{c_2}. This means that the higher-order terms in η_c, i.e., η_{c_3}, η_{c_4}, etc., do not contribute significantly to the bucket shape.

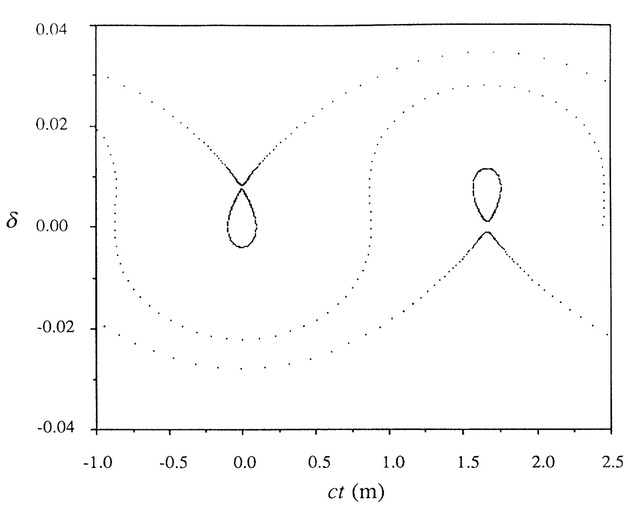

FIG. 4. Longitudinal phase space for particles with a small emittance in the UVSOR ring. (The speed of light is designated by c.)

The results of tracking particles with large betatron oscillations can be seen in Fig. 5. In Fig. 5, one can see a three-dimensional closed surface viewed from three different angles. Particles that had initial coordinates inside the aperture survived and those outside were lost. This surface gives a rough idea of how large the dynamic aperture is. What is found is that the scaling laws give an accurate prediction of the length of the stable phase-space area in δ. The longitudinal aperture only begins to shrink appreciably at very large betatron amplitudes.

We can conclude from these results that the emittance

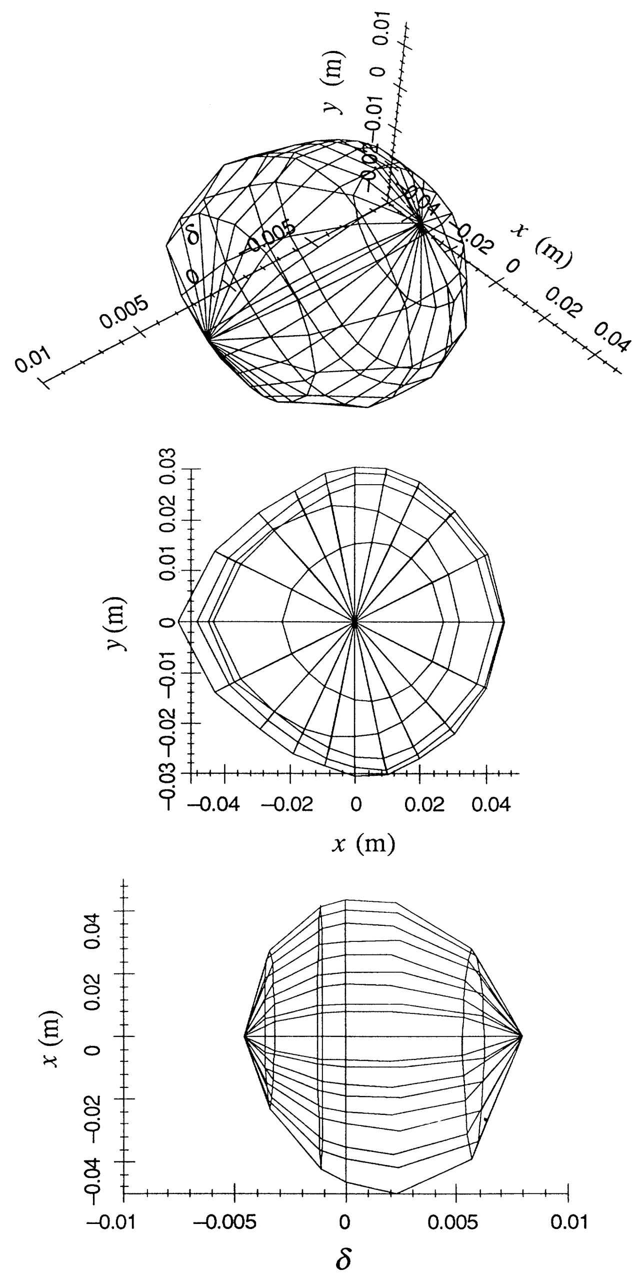

FIG. 5. Three-dimensional dynamic aperture in the UVSOR ring viewed from three different angles.

of the ring contributed very little to determining the size of the longitudinal phase-space area. The longitudinal phase space in this ring is determined primarily by the η_{c_1} and η_{c_2} terms. This gives us confidence that, for the ring, the simple scaling law Eq. (28) agrees well with the 6D tracking and is thus a good guide to determining the size of the stable longitudinal phase-space area for that particular lattice.

In order to proceed to lower values of η_{c_1}, it is necessary to adjust the sextupoles in the ring to lower η_{c_2}. At UVSOR they were experimentally able to adjust η_{c_2} by varying sextupole strengths [12]. Being able to control η_{c_2} allowed them to operate the ring at a lower value of η_{c_1}.

V. CONCLUSION

We have demonstrated that the size of the longitudinal phase space is governed by the strengths of the higher-order δ terms of the phase-slip factor. We derived simple scaling laws to give a quantitative estimate of how large this phase-space area is. We also showed that it is possible to correct the higher-order terms in the equation of motion with higher-order magnets, and to give an expression for the effect of a sextupole on η_{c_2}. It is thus possible from the point of view of single-particle dynamics to operate a ring with a small value of η_{c_1} and still have a sufficiently large stable longitudinal phase-space area. Therefore, storage rings should be able to produce short bunch lengths just by decreasing η_{c_1}.

We have previously studied the effect of the longitudinal microwave instability on the collective stability of the beam [1]. We found that the threshold peak current should not decrease as we lower η_{c_1}. In fact, with the inclusion of radiation damping, the bunch should be able to tolerate a larger peak current than when operating in a larger-η_{c_1} regime. This work was done assuming a broad-band impedance and Stanford positron-electron accelerator ring (SPEAR) scaling. We are in the process of studying the effect of the vacuum impedance [13] or the effect of coherent radiation which increases as the bunch length decreases. We hope to report on this soon.

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