PHYSICAL REVIEW E VOLUME 48, NUMBER 3 SEPTEMBER 1993

# Transport coefficients of quantum plasmas

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(Received 28 September 1992)

Transport coefficients of fully ionized plasmas with a weakly coupled, completely degenerate electron gas and classical ions with a wide range of coupling strength are expressed within the Bloch transport equation. Using the Kohler variational principle the collision integral of the quantum Boltzmann equation is derived, which accounts for quantum effects through collective plasma oscillations. The physical implications of the results are investigated through comparisons with other theories. For practical applications, electrical and thermal conductivities are derived in simple analytical formulas. The relation between these two transport coefficients is expressed in an explicit form, giving a generalized Wiedemann-Franz law, where the Lorentz ratio is a dependent function of the coupling parameter and the degree of degeneracy of the plasma.

PACS number(s): 52.25.Fi, 72.10.Bg, 71.45.−d

## I. INTRODUCTION

High-density-plasma properties have been the subject of several studies for some time now. They usually raise interest for such practical applications as laser-compressed plasmas and the interiors of heavy planets and of degenerate stars. The impact of electron degeneracy on transport properties has been shown to be of great importance and can play a significant role in target performance in laser fusion plasmas [1].

Several authors have taken into account the electron degeneracy for the calculation of transport coefficients in strongly coupled plasmas [2–4]. In most existing models, the effects of interparticular correlations are often taken into account in a qualitative manner. Electronic transport of completely ionized dense hydrogen plasma has been considered with the help of the Ziman formula [5,6] where ionic structure factors were obtained by numerical simulation [5] of a one-component plasma (OCP) model.

The quantum Lenard-Balescu transport equation has been used [7] for thermal and electrical conductivities of a plasma of highly degenerate, weakly coupled electrons and nondegenerate, weakly coupled ions. This latter model ended up with an unexpected significance of electron-electron collisions. This type of collision can augment heat resistance either by deflection of colliding electrons or by transfer of energy from the faster to the slower particle. These mechanisms both are ineffective in the electron-ion collisions and have no significant effect on electric conductivity since the electron-electron collisions conserve the electric current.

Moreover, the energy exchanged in an electron-electron collision, in a sufficiently degenerate plasma, is small compared to the electron kinetic energy (Fermi energy), but is still typically of order $k_B T$. Since the distribution function varies on the scale of $k_B T$, energy redistribution by electron-electron collisions becomes an important mechanism for thermal conductivity over a significant range of high temperatures and densities.

For greater degeneracy, however, the Pauli exclusion principle drastically restricts the phase space for the electron-electron collisions and entirely prevents them at absolute zero temperature (completely degenerate plasma).

For a high density, strongly coupled, partially or completely degenerate plasma, where the concept of the Debye shielding breaks down, statistical theories containing the Debye length as a characteristic parameter would be physically meaningless for such a nonideal plasma. Moreover, transport phenomena, the understanding of which has been attempted via the model of discrete interacting particles, i.e., where electrons are elastically deflected at the surface of the Fermi sphere by the ionic density fluctuations, do not give a rigorous account of an essential characteristic property of such a medium: excitation of collective oscillations. For that, collective behavior effects are treated through Coulomb collisions and separated as long-and short-range interactions and are arbitrarily splitted in the integration process.

A number of investigations [8,9] developed transport theories of stellar interiors, using an ordinary two-body Boltzmann equation for electron-ion scattering with the Born approximation for the unshielded Coulomb potential. In order to eliminate the long-range Coulomb potential divergence, the Coulomb potential has been cut off at the mean interionic distance. These models assumed static shielding.

Unification of both treatments of short- and long-range Coulomb collision is a more comprehensive approach [10] to the very-high-density plasmas, where the shielding distance (Debye length) loses its physical meaning as an upper impact parameter. Some authors [11,12] proposed such a unification scheme of Coulomb collision theory in the limit of the weak coupling but introduced artificial splitting, in the integrations, into two separate parts.

For quantum plasmas, the Lenard-Balescu equation has been used. In its derivation, an assumption is made that all the many-body correlations (collective effects) incorporated in the dielectric function to account for the dynamic Coulomb shielding and both multiquasiparticle collisions and quasiparticle interaction energy are neglected, but it is valid only for weak coupling [7]. Density fluctuations for dense plasma can be split into two approximately independent components associated, respectively, with the collective and individual particle aspects of the system.

The collective component, which is present only for wavelengths greater than the Debye length, represents organized oscillations brought about by the long-range part of the Coulomb interactions [13]. When such an oscillation is excited, each individual particle suffers a small perturbation of its velocity and position, arising from the combined potential of all the other particles. The contribution to the density fluctuations resulting from these perturbations is in phase with the potential producing it, so that in an oscillation we find a small organized wave-like perturbation superposed on the random thermal motion of the particle.

For the high density, partially or completely degenerate plasma, i.e., at low temperature, the thermal motion no longer plays the dominant role. Instead, the cumulative potential of all the particles will be considerable because the long range of the force permits a very large number of particles to contribute to the potential at a given point. Hence the collective aspect would be dominant and particularly governs the transport phenomena.

Herein, we extend and apply the Bloch [14] transport theory to the very-high-density plasma based on the concepts similar to those used for solids and liquid metals [15–17]. The development of such a model to be applied to degenerate plasmas is justified since the medium may be described as a distorted lattice. The strong interparticle correlations keep the ions in positions that resemble a lattice structure. Since the system is fluid, the average position of the particles changes slowly with time, unlike the real lattice that appears with still stronger correlations [18].

As the plasma goes from low ($\Gamma < 1$) ($\Gamma$ is the ratio of the Coulomb interaction energy to the thermal energy of the ions) to high ($\Gamma > 1$) ion coupling with increasing density, it undergoes a transition from a nonideal classical plasma to a quasicrystalline plasma, with an incomplete ordering comparable to that of a liquid [19–21]. A range of the coupling parameter has been proposed for such a crystalline transition by several authors [22,23] such as $178 \leq \Gamma \leq 196$. In the case of complete degeneracy ($T=0$ K), the crystalline transition is observed in the range [24] $1.3 \leq r_s \leq 1.8$, where $r_s = a_e m_e e^2/\hbar^2$ is the Wigner-Seitz radius for the electron ($a_e$) in units of Bohr radius and depends only on the electron density.

For the above reasons, it appears more adequate to calculate the transport coefficients of very dense plasmas from the picture of electron and ion oscillations.

In analogy to the Bloch theory, the role of the longitudinal phonons, in the theory of metals, is played here by the quanta of the plasma oscillations, plasmons and ion sound waves, with some idealization made by an adequate choice of the dispersion relations and an extrapolation of the spectrum to large wave numbers. The theory

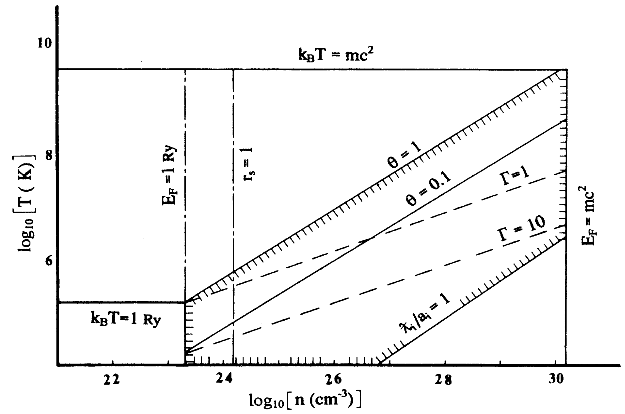


FIG. 1. Region of validity (area with dashed boundaries) of the present quantum model for $Z=1$. $\Gamma \equiv (Ze)^2/a_i k_B T$, $a_i = (3Z/4\pi n)^{1/3}$, $\theta \equiv k_B T/E_F$, $E_F = (\hbar^2/2m_e)(3\pi^2 n)^{2/3}$, $r_s \equiv a_e m_e e^2/\hbar^2$, $a_e = (3/4\pi n)^{1/3}$, $\lambda_i \equiv \hbar/(m_i k_B T)^{1/2}$.

to be presented is able to take into account, in its formulation, both electron-electron and electron-ion contributions. The results are compared with previous theories and simulations.

The domain of validity of the present model is defined in Fig. 1 (area with dashed boundaries) where, in the temperature-density plane, the characteristic quantities are represented for a completely degenerate electron gas (for the Fermi degeneracy parameter $\Theta \leq 1$) with classical ions ($\lambda_i/a_i < 1$), $\lambda$ is the de Broglie wavelength for the ions, over a wide range of coupling strength $\Gamma$.

## II. BOLTZMANN COLLISION TERM FOR QUANTUM PLASMAS

### A. Electron-plasmon and electron–sound wave interaction

In consideration are transport coefficients of high-density completely degenerate plasmas due to the many-body interactions of the electron with longitudinal plasma waves. The motion of electrons in a continuum of volume $\Omega$ is affected by the continuum oscillations (many-body interactions). In ideal plasmas, the change of motion is caused by binary collisions of the electrons with the plasma particles. In nonideal plasmas, however, the electrons interact with the Coulomb field of all charged particles. Therefore, this interaction can be treated as a scattering of the electrons by the random longitudinal waves of the plasma continuum, which are thermally excited. As in the theory of metals [15,16], we are considering a free-electron model, applied to dense plasmas with $Z$ electrons per ion where the electron wave functions are approximated by plane waves.

The electron energy $E$ is given in terms of the wave vector $\mathbf{k}$ by $E = \hbar^2 k^2/2m_e$, so that the Fermi surface is spherical. Let $\omega_e(\mathbf{q})$ be the $e$th eigenoscillation with wave vector $\mathbf{q}$ of an electron wave ($e$ plasmon) and $\omega_i(\mathbf{q})$ and $i$th eigenoscillation with wave vector $\mathbf{q}$ of an ion sound wave ($i$ plasmon).

Taking into account conservation of energy and momentum, i.e., $\hbar\omega(\mathbf{q})=E'-E$ and $\hbar\mathbf{q}=\mathbf{p}'-\mathbf{p}$, where $E,\mathbf{p}$ and $E',\mathbf{p}$, are, respectively, the energy and momentum of an electron before and after a collision with a plasmon of energy $\hbar\omega(\mathbf{q})$ and momentum $\hbar\mathbf{q}$, we see that an electron interacting with the plasma as a whole can emit and absorb plasmons and ion sound waves which are quasiparticles obeying Bose-Einstein statistics and their distribution function is

$$\bar{N}_q=\frac{1}{\exp\left[\dfrac{\hbar\omega(\mathbf{q})}{k_BT}\right]-1} . \tag{1}$$

Let $P(\mathbf{k},\mathbf{k}')$ be the transition probability per unit time that, upon a collision of the electron with a plasmon, the electron in a state $\mathbf{k}$ moves to another state $\mathbf{k}'$ which is not occupied by any other electron. If $f(\mathbf{k})$ is the distribution function of the electron occupying state $\mathbf{k}$ and $f(\mathbf{k}')$ the distribution function of the electron in the state $\mathbf{k}'$, the number of electrons which move from state $\mathbf{k}$ to state $\mathbf{k}'$ is (Pauli principle) $P(\mathbf{k},\mathbf{k}')f(\mathbf{k})[1-f(\mathbf{k}')]$.

Since there always exists an inverse transition of the above forward interaction, the total rate of change in time of $f(\mathbf{k})$ due to electron-wave interactions is obtained by summing over all $\mathbf{k}$:

$$\frac{\partial f(\mathbf{k})}{\partial t}=\sum_{k'}\{P(\mathbf{k}',\mathbf{k})f(\mathbf{k}')[1-f(\mathbf{k})]-P(\mathbf{k},\mathbf{k}')f(\mathbf{k})[1-f(\mathbf{k}')]\} . \tag{2}$$

The interaction processes are calculated by perturbation theory.

The probability of a transition [25,26] from an initial state $\mathbf{k}$ to a final state $\mathbf{k}'$ is

$$P(\mathbf{k},\mathbf{k}')=\frac{2\pi}{\hbar}|M_{kk'}|^2\delta(E_{k'}-E_k) , \tag{3}$$

where $E_{k'}$ and $E_k$ are the energies of the electron in the state $\mathbf{k}'$ and $\mathbf{k}$, respectively. $|M_{kk'}|$ is the matrix element of the transition $\mathbf{k}\to\mathbf{k}'$. For the absorption of a plasmon, $|M_{kk'}|$ is proportional to $\bar{N}_q$, and for the emission of a plasmon it is proportional to $(\bar{N}_q+1)$.

With the Fourier transform of the potential $U_s(r)$, by means of which the particles in the plasma interact, for electron ($s=e$) and ion ($s=i$) oscillations with different frequencies $\omega_s$, which is described here by a Yukawa potential with a shielding radius $\delta_s$ [$\delta_s\sim n_s^{-1/3}$, see (A6) and (A7)],

$$U_s(r)=e_se_e\exp(-r/\delta_s)/r,\quad e_e=-e,\ e_i=Ze , \tag{4}$$

the matrix element becomes

$$|M_{kk'}|^2=|\alpha_q^s|^2q^2|U_s(q)|^2,\quad s=e,i , \tag{5}$$

$$|\alpha_q^s|^2=\begin{cases}\dfrac{\hbar\bar{N}_q}{2m_sn_s\omega_s(\mathbf{q})} & \text{for the absorption of an oscillation ,}\\[2ex] \dfrac{\hbar(\bar{N}_q+1)}{2m_sn_s\omega_s(\mathbf{q})} & \text{for the emission of an oscillation .}\end{cases} \tag{6}$$

For plane waves normalized in a unit volume, the Fourier transform of $U_s(r)$ is given by

$$|U_s(q)|=\frac{4\pi n_se|e_s|}{\delta_s^{-2}+q^2},\quad q=|\mathbf{k}'-\mathbf{k}| , \tag{7}$$

$m_s$ is the mass of the particles, $n_s$ their density, $|\alpha_q^s|^2$ is the mean-square amplitude of the $q$th mode of an oscillation of frequency $\omega_s(\mathbf{q})$, and $\mathbf{e}_\mathbf{q}$ is the unit vector in the direction of propagation of the vector $\mathbf{q}$.

The distribution function $f(\mathbf{k})$ in (2) is not symmetric with respect to the origin in $\mathbf{k}$ space, since it is "polarized" by the electric field $\mathscr{E}$.

Taking $\mathscr{E}$ in the $\mathbf{x}$ direction, $\mathscr{E}=(\mathfrak{E},0,0)$, a first-order perturbation gives to $f(\mathbf{k})$ the form

$$f(E)=f_0(E)+\varphi,\quad \varphi=-\frac{\partial f_0}{\partial E}\Phi,\quad -\frac{\partial f_0}{\partial E}=\frac{f_0(1-f_0)}{k_BT} , \tag{8}$$

where $f_0(E)$ is the Fermi distribution describing the thermal equilibrium of the electron, $f_0(E)=(1+\exp[(E-\xi)/k_BT])^{-1}$, $\xi=\xi(T)$ is the Fermi energy [$\xi(T=0\ \mathrm{K})=(\hbar^2/2m_e)(3\pi^2n)^{2/3}$ is the Fermi edge], and $\Phi$ is a function of the energy $E$ to be determined as

$$\Phi=e|\mathscr{E}|v_xg(E) , \tag{9}$$

where $v_x$ is the velocity of the electrons in the $\mathbf{x}$ direction.

If Eq. (2) is changed from a discrete summation over $\mathbf{k}'$ to an integral and the volume $\Omega$ is normalized to unity, then the collision term becomes

$$\left.\frac{\partial f}{\partial t}\right|_c=\frac{1}{(2\pi)^3k_BT}\int W(\mathbf{k}',\mathbf{k})[\Phi(\mathbf{k}')-\Phi(\mathbf{k})]d^3\mathbf{k}' , \tag{10}$$

where the detailed microscopic balance for direct and inverse interactions has been used, i.e.,

$$W(\mathbf{k},\mathbf{k}')=P(\mathbf{k}',\mathbf{k}')f_0(\mathbf{k}')[1-f_0(\mathbf{k})]=P(\mathbf{k},\mathbf{k}')f_0(\mathbf{k})[1-f_0(\mathbf{k}')]=W(\mathbf{k}',\mathbf{k}) . \tag{11}$$

Analytical results can be obtained if a set of the oscillation frequencies is properly defined.

### B. Dispersion relations

The plasma under consideration is a continuum of volume $\Omega$ containing $N$ electrons, $N/Z$ ions, and an electron density $n=N/\Omega$, which exhibits $3N$ (high-frequency branch) and $3N/Z$ (low-frequency branch) characteristic frequencies $\omega_s(\mathbf{q})$ of longitudinal oscillations $(s=e,i)$. The high-frequency branch corresponds to electron plasma oscillations and the low-frequency branch is the ion sound waves.

The dispersion relation $\omega_{e,i}=\omega_{e,i}(\mathbf{q})$ for the electron oscillations and ion sound waves are (a) extended to strongly coupled plasmas by redefining the specific-heat ratios $\kappa_{e,i}$ and (b) extrapolated to large wave numbers $q\sim n^{1/3}$ for which Landau damping is very small.

The oscillation frequencies are the high-frequency branch $(s=e)$ of the space charge waves and for $n\ll\tilde{n}$ [$\tilde{n}=2(2\pi m_e k_B T)^{2/3}/h^3$ is the critical density] are taken to be

$$\omega_e(\mathbf{q})=\omega_p(1+\alpha^2q^2)^{1/2} , \tag{12}$$

where $\alpha^2=C_m^2/\omega_p^2$, with $C_m=(\kappa_e k_B T/m_e)^{1/2}$ the speed of sound of the electron gas, $\kappa_e=c_p/c_v=\frac{5}{3}$, and $\omega_p$ the plasma frequency [$\omega_p=(4\pi ne^2/m_e)^{1/2}$]. For a $n>\tilde{n}$, degenerate plasma, the high-frequency branch dispersion relation is given by [27]

$$\omega_e(\mathbf{q})=[\omega_p^2(1+\alpha^2q^2)+(\hbar q^2/2m_e)^2]^{1/2} , \tag{13}$$

where $\alpha^2=(\frac{3}{5})v_F^2/\omega_p^2$ and $v_F=(\hbar/m_e)(3\pi^2n)^{1/3}$ is the Fermi speed.

Here we consider that $q\ll\omega_p/v_F$ and $\omega_p\gg\hbar q^2/2m_e$. Thus, in this region, the spectrum of the longitudinal oscillations is similar to that of a weakly degenerate (high temperature) plasma, with the distinction, however, that in the present case, the chaotic motions of the electrons are due to Fermi energy rather than to the temperature. In addition, Eq. (13) takes quantum effects into account.

The low-frequency branch $(s=i)$ of the space charge waves is due to ion sound waves which are coupled with the electrons. Their dispersion relation for $n\ll\tilde{n}$ is given by

$$\omega_i(\mathbf{q})=\nu(\mathbf{q})C_s\mathbf{q} , \tag{14}$$

where $C_s$ is the speed of sound in the ion gas, $C_s=(\kappa_i k_B T/m_i)^{1/2}$, $\kappa_i=c_p/c_v=\frac{5}{3}$, and $\nu(\mathbf{q})$ is a correlation factor which shows the influence of the electrons on the ion oscillations [20]. Since the dispersion factor $\nu(\mathbf{q})$ is a bounded function varying very little with $q$ such that $1\le\nu(\mathbf{q})\le(1+Z)^{1/2}$ for $q\in(0,\hat{q}_i)$ [$\hat{q}_i\sim(3Z/4\pi n)^{1/3}$], it can be approximated by an average value of order unity [20].

For a completely degenerate plasma, the speed of sound in the ion gas is no longer a function of temperature, it does depend on the Fermi speed of electrons. Accordingly for $n\gg\tilde{n}$, the ion sound waves dispersion relation becomes

$$\omega_i(q)=(Zm_e/3m_i)^{1/2}v_Fq . \tag{15}$$

The damping factor of these sound waves is small in comparison with $\omega_i$; this permits the propagation of sound to great distances. It should be noted in this system that the kinetic energy will depend both on the temperature or the Fermi energy which determines the random thermal motion and on the amplitude of organizing oscillations (plasmons). Normally, the thermal energy is much greater than the organized oscillation energy, so that for all practical purposes, frequency does not depend appreciably on the amplitude of organized oscillations. The entire effect is in the domain of the nonlinear aspects of the problem, and therefore, can be neglected in a linear approximation. So, the average energy appearing in the dispersion relation should be the value existing in the absence of organizing oscillations.

## III. TRANSPORT COEFFICIENTS

### A. Boltzmann equation resolution by the variational method

This method has been proposed by Kohler [28], where the collision equation is reduced to a variational principle which can be interpreted as a principle of maximum entropy production. The variational principle is based on the linearized Boltzmann equation for an electric field and a temperature field. The Boltzmann equation contains implicitly the equilibrium between the reduction in entropy by external fields and the increase of entropy by collisions. The solution of the transport equation is so designed that the entropy production caused by the interaction of electrons with plasmons and ion sound waves will be a maximum.

The collision term [Eq. (10)], established through the interaction model proposed, is related to the field term to express the Boltzmann equation as follows:

$$-(\mathbf{v}\cdot\mathbf{A})\frac{\partial f_0}{\partial E}=\frac{1}{8\pi^3k_BT}\int W(\mathbf{k}',\mathbf{k})[\Phi(\mathbf{k})-\Phi(\mathbf{k}')]d^3\mathbf{k}' , \tag{16}$$

where the left-hand side has an explicit form:

$$v_x\frac{\partial f_0}{\partial E}\left[-e\mathfrak{E}-\frac{d\xi}{dx}-\frac{E-\xi}{T}\frac{dT}{dx}\right] . \tag{17}$$

$\mathfrak{E}$ is the electrical field component in the $x$ direction and $T$ the plasma temperature.

Equation (16) can be written in short as

$$\left.\frac{\partial f}{\partial t}\right|_{\text{field}}=-\left.\frac{\partial f}{\partial t}\right|_c\leftrightarrow F=\mathcal{L}\Phi , \tag{18}$$

where $\mathcal{L}$ is a definite position integral operator which connects the unknown function $\Phi$ with the known field function $F$. In addition, $\mathcal{L}$ is linear and we can write

$$\mathcal{L}\Phi=\mathcal{L}(a\Phi_1+b\Phi_2)=a\mathcal{L}\Phi_1+b\mathcal{L}\Phi_2 . \tag{19}$$

Let

$$F_1=\mathcal{L}\Phi_1=-v_x\frac{\partial f_0}{\partial E} , \tag{20}$$

$$F_x=\mathcal{L}\Phi_2=-v_x\frac{\partial f_0}{\partial E}(E-\xi) , \tag{21}$$

where $\Phi_1$ and $\Phi_2$ are due to the fact that the electrical and temperature fields are independent of each other and related through

$$\Phi = \left[ -e\mathcal{E} - \frac{d\xi}{dx} \right] \Phi_1 + \left[ -\frac{1}{T}\frac{dT}{dx} \right] \Phi_2 . \quad (22)$$

Moreover, $\Phi$ is one of the $\psi$ trial functions which satisfy the relation

$$\langle \psi, F \rangle = \langle \psi, \mathcal{L}\psi \rangle , \quad (23)$$

The variational principle provides a solution of the integral equation (18) which gives to the product $\langle \Phi, \mathcal{L}\Phi \rangle$ its maximum value:

$$\langle \Phi, \mathcal{L}\Phi \rangle \geq \langle \psi, \mathcal{L}\psi \rangle . \quad (24)$$

Hence, the transport equation, in consideration here, is reduced to an extremum problem. For this purpose, $\Phi_1$ and $\Phi_2$ of Eqs. (20) and (21) can be chosen as power series of $E-\xi$ where the coefficients will be determined by the optimization principle.

### B. Transport coefficients

In the Boltzmann theory, the electrical current and energy fluxes are given as [29,30]

$$\mathbf{J} = \frac{e}{4\pi^3} \int \mathbf{v}\varphi \, d^3\mathbf{k} , \quad (25)$$

$$\mathbf{Q} = \frac{1}{4\pi^3} \int \mathbf{v}(E-\xi)\varphi d^3\mathbf{k} . \quad (26)$$

In the presence of a weak electrical field and a small temperature gradient, the current density $\mathbf{J}$ and the rate of flow of heat $\mathbf{Q}$ are given in the form

$$\mathbf{J} = L_{11}\frac{\mathbf{E}^*}{T} + L_{21}\left[ -\frac{\nabla T}{T^2} \right] , \quad (27)$$

$$\mathbf{Q} = L_{12}\frac{\mathbf{E}^*}{T} + L_{22}\left[ -\frac{\nabla T}{T^2} \right] , \quad (28)$$

with $\mathbf{E}^* = \mathbf{E} - (\nabla\xi/e)$. From Eqs. (8) and (22) with (25) and (26), the kinetic coefficients of (27) and (28) are defined as

$$L_{11} = \frac{-eT}{4\pi^3} \int \Phi_1 \frac{\partial f_0}{\partial E} v_x d^3k , \quad (29)$$

$$L_{12} = \frac{eT}{4\pi^3} \int \Phi_2 \frac{\partial f_0}{\partial E} v_x d^3k , \quad (30)$$

$$L_{21} = \frac{-T}{4\pi^3} \int \Phi_1 (E-\xi) \frac{\partial f_0}{\partial E} v_x d^3k , \quad (31)$$

$$L_{22} = \frac{-T}{4\pi^3} \int \Phi_2 (E-\xi) \frac{\partial f_0}{\partial E} v_x d^3k . \quad (32)$$

In Eqs. (27)–(32), the transport coefficients are defined such as for $\nabla T = 0$, the electrical conductivity $\sigma$ is the ratio of $\mathbf{J}$ to $\mathbf{E}^*$, and the thermal coefficient $\mathcal{K}$ is the ratio of $\mathbf{Q}$ to $\nabla T$ for $\mathbf{E}^* = 0$. The Peltier effect $\beta$ and the thermoelectric effect $\alpha$ are defined from Eqs. (28) and (27) for $\nabla T = 0$ and $\mathbf{E}^* = 0$, respectively. It should be noted that if a temperature gradient is present in a steady state, but no steady current is flowing, the electrostatic field $\mathcal{E}$ will build up to such a value that $\mathbf{J}$ vanishes. Hence, $\lambda$, the effective coefficient of heat conductivity (thermal conductivity), is defined as the ratio of $\mathbf{Q}$ to $\nabla T$ for $\mathbf{J} = 0$ hence

$$\sigma = \frac{L_{11}}{T} , \quad (33)$$

$$\mathcal{K} = \frac{L_{22}}{T^2} , \quad (34)$$

$$\alpha = -\frac{L_{21}}{T^2} , \quad (35)$$

$$\beta = -\frac{L_{12}}{T} , \quad (36)$$

$$\lambda = \frac{L_{11}L_{22} - L_{12}L_{21}}{L_{11}T^2} . \quad (37)$$

The kinetic coefficients are defined by Eqs. (29)–(32) as functions of $\Phi_1$ and $\Phi_2$ which will be studied shortly.

The typical electron energy of the plasma in consideration here is $E \simeq \xi$ (about the Fermi energy) so the most convenient trial function $\psi$ for $\Phi_1$ and $\Phi_2$ would be

$$\Phi_1 = v_x C(E) = v_x \sum_{\mu} c_\mu (E-\xi)^\mu , \quad (38)$$

$$\Phi_2 = v_x B(E) = v_x \sum_{\mu} b_\mu (E-\xi)^\mu , \quad (39)$$

where $c_\mu$ and $b_\mu$ are parameters to be determined by the variational method. Now that the trial functions are defined, the two equations (20) and (21) can be solved by an optimization procedure of the relation (23).

First, Eq. (38) in (10) would give, with the help of (23),

$$(\psi, \mathcal{L}\psi) = \sum_{\mu,\nu} c_\mu c_\nu D_{\mu\nu} , \quad (40)$$

$$(\psi, F_1) = \sum_{\mu} c_\mu N_\mu , \quad (41)$$

with

$$D_{\mu\nu} = \int v_x (E-\xi)^\mu \mathcal{L}[v_x (E-\xi)^\nu] d^3k = D_{\nu\mu} , \quad (42)$$

$$N_\mu = -\int v_x^2 \frac{\partial f_0}{\partial E} (E-\xi)^\mu d^3k . \quad (43)$$

The symmetry of $D_{\mu\nu}$ follows from the fact that $\langle \psi, \mathcal{L}\Phi \rangle = \langle \psi, \mathcal{L}\psi \rangle$. The variational principle is now used to find the maximum of $\langle \psi, \mathcal{L}\psi \rangle$ with the supplementary condition $\langle \psi, \mathcal{L}\psi \rangle = \langle \psi, F_1 \rangle$, i.e.,

$$\sum_{\mu,\nu} c_\mu c_\nu D_{\mu\nu} = \sum_{\mu} c_\mu N_\mu . \quad (44)$$

These calculations are carried out in the Appendix.

For the solution of the second equation [Eq. (21)] with (39), we obtain in the same way

$$\sum_{\nu} b_\nu D_{\mu\nu} = B_\mu = N_{\mu+1} , \quad (45)$$

with

$$B_\mu = \int -v_x^2 \frac{\partial f_0}{\partial E}(E-\xi)^{\mu+1} d^3k \ . \tag{46}$$

The more terms of the series (38) and (39) that are taken into consideration the more laborious the method will be. But for the plasma in consideration where $E \simeq \xi$ the series can let us manage with $\mu = 0,1$.

With these expressions, the transport coefficients can be written in the form

$$\sigma = \frac{e^2}{4\pi^3} \sum_\mu c_\mu N_\mu \ , \tag{47}$$

$$\alpha = \frac{e}{4\pi^3} \sum_\mu c_\mu B_\mu \ , \tag{48}$$

$$\beta = \frac{e}{4\pi^3} \sum_\mu b_\mu N_\mu = \alpha T \ , \tag{49}$$

$$\mathcal{K} = \frac{e^2}{4\pi^3 T} \sum_\mu b_\mu B_\mu \ . \tag{50}$$

$\lambda$ is obtained such that the Onsager [29] relation is verified by

$$\lambda = \frac{1}{4\pi^3 T} \left[ \sum_\mu b_\mu B_\mu - \frac{\left[ \sum_\nu c_\nu B_\nu \right]^2}{\sum_\rho c_\rho N_\rho} \right] , \tag{51}$$

which coincides with the relation $\lambda - \mathcal{K} = -\alpha\beta/\sigma$ (as in Ref. [31]).

The coefficients $c_\mu$ and $b_\mu$ are determined from the system of equations (see the Appendix) as follows:

$$N_\mu = \sum_\nu c_\nu D_{\mu\nu} \ , \tag{52}$$

$$B_\mu = \sum_\nu b_\nu D_{\mu\nu} = N_{\mu+1} \ , \tag{53}$$

$$D_{\mu\nu} = \frac{1}{(2\pi)^3 k_B T} \int\int v_x (E-\xi)^\mu W(\mathbf{k}',\mathbf{k})[v_x(E-\xi)^\nu - v_x'(E'-\xi)^\nu] d^3\mathbf{k}\, d^3\mathbf{k}' \ . \tag{54}$$

$D_{\mu\nu}$ is a function of $W(\mathbf{k},\mathbf{k}')$ which contains the transition probability $P(\mathbf{k},\mathbf{k})$. The latter function has different forms for the absorption and for the emission of a plasmon [Eq. (6)]. In the case of the absorption of a plasmon, $D_{\mu\nu} = D_{\mu\nu}^+$ with $E' = E + \hbar\omega_s$ and $\mathbf{k}' = \mathbf{k} + \mathbf{q}$. For the emission transition, $\mathbf{k}' = \mathbf{k} - \mathbf{q}$, $E = E' - \hbar\omega_s$, and $D_{\mu\nu} = D_{\mu\nu}^-$ where $D_{\mu\nu}^-$ is obtained from $D_{\mu\nu}^+$ by replacing $\mathbf{q}$ by $-\mathbf{q}$ and $\omega_s$ by $-\omega_s$. Details of the explicit calculations of these expressions are in the Appendix.

## IV. RESULTS AND COMPARISONS

### A. Electrical and thermal conductivities of quantum plasma

For the electrical conductivity, the Bloch approximation ($\mu = 0$) is used to deduce $\sigma$ from Eq. (47). The coefficients $c_\mu$ and $N_\mu$ ($\mu = 0$) are given in the Appendix by (A18) and (A13) as functions of $D_{00}$ (A8). Explicitly,

$$\sigma = \frac{ne^2}{m_e} \frac{3\pi\hbar^7\kappa_i^3}{4Ze^4 m_e m_i^2 \delta_i^4 I_5(\hat{\epsilon}_i)(k_B T)^2} \ , \tag{55}$$

where

$$\delta_i = (4\pi n/3z)^{-1/3}, \quad \hat{\epsilon}_i = 2\pi\hbar C_s/\delta_i k_B T \ , \tag{56}$$

and $I_5$ is given by (for $n = 5$)

$$I_n(\hat{\epsilon}_i) = \int_0^{\hat{\epsilon}_i} \frac{\epsilon^n e^\epsilon}{(e^\epsilon - 1)^2[1 + 4\pi^2(\epsilon/\hat{\epsilon}_i)^2]^2} d\epsilon \ . \tag{57}$$

The Bloch approximation ($\mu = 0$), which was adequate for the electrical conductivity, is no longer sufficient for the thermal conductivity and other thermoelectric effects. For that, higher order in the variational method is necessary for these coefficients.

An approximation of order one ($\mu = 0,1$) gives the thermal conductivity as a function of the coefficients $b_0$, $b_1$, $c_1$, $B_0$, $B_1$, $N_1$, which are given in the Appendix by Eqs. (A20), (A21), (A19), (A16), (A17), and (A14), respectively.

As a function of the degree of the electron degeneracy $\Theta$ which is defined as the ratio of thermal energy $k_B T$ to the Fermi edge $E_F$ [$E_F = \xi(T = 0\ \text{K}) = (\hbar^2/2m_e)(3\pi^2 n)^{2/3}$], the thermal conductivity of degenerate plasmas is

$$\begin{aligned}
\lambda = \pi^4 (k_B/e)^2 T\sigma\theta \Bigg[ & \frac{2\kappa_i m_e}{9m_i} + \Theta\left\{ \frac{2}{9}\left[\frac{\kappa_i m_e}{m_i}\right]^2 + \frac{\pi^2}{27} - \frac{1}{54}\frac{I_7(\hat{\epsilon}_i)}{I_5(\hat{\epsilon}_i)} \right\} + \Theta^2 \left\{ -\frac{2\pi^2}{9}\frac{\kappa_i m_e}{m_i} - \frac{1}{36}\frac{\kappa_i m_e}{m_i}\frac{I_7(\hat{\epsilon}_i)}{I_5(\hat{\epsilon}_i)} \right\} \\
& + \Theta^3 \left\{ -\frac{\pi^4}{18} - \frac{\pi^2}{6}\left[\frac{\kappa_i m_e}{m_i}\right]^2 + \frac{\pi^2}{36}\frac{I_7(\hat{\epsilon}_i)}{I_5(\hat{\epsilon}_i)} \right\} + \Theta^4 \left\{ \frac{\pi^4}{24}\frac{\kappa_i m_e}{m_i} + \frac{\pi^2}{48}\frac{\kappa_i m_e}{m_i}\frac{I_7(\hat{\epsilon}_i)}{I_5(\hat{\epsilon}_i)} \right\} \\
& + \Theta^5 \left\{ \frac{\pi^6}{48} - \frac{\pi^4}{96}\frac{I_7(\hat{\epsilon}_i)}{I_5(\hat{\epsilon}_i)} \right\} \Bigg] \left[ 2\frac{\kappa_i m_e}{m_i} + \Theta\left\{ \frac{\pi^2}{3} - \frac{1}{6}\frac{I_7(\hat{\epsilon}_i)}{I_5(\hat{\epsilon}_i)} \right\} \right]^{-1} \\
\times \Bigg[ & 2\frac{\kappa_i m_e}{m_i} + \theta\left\{ \frac{\pi^2}{3} - \frac{1}{6}\frac{I_7(\hat{\epsilon}_i)}{I_5(\hat{\epsilon}_i)} \right\} + \frac{\pi^2}{2}\Theta^2 \left\{ \frac{\pi^2}{2}\Theta - \frac{\kappa_i m_e}{m_i} \right\} \Bigg]^{-1} ,
\end{aligned} \tag{58}$$

where $\sigma$ and $I_7(\hat{\epsilon}_i)$ are given by Eqs. (55) and (57), respectively, with $n=7$.

Numerical computation showed for strong degeneracy ($\Theta$ sufficiently small) that all the terms in $\Theta^3$ and higher can be neglected. Similarly it has been shown that terms with $(m_e/m_i)^2$ are insignificant compared to $\pi^2/27-\frac{1}{6}I_7(\hat{\epsilon}_i)/I_5(\hat{\epsilon}_i)$, hence Eq. (58) becomes

$$\lambda=(\pi^4/9)(k_B/e)^2T\sigma\frac{\Theta}{2\frac{\kappa_i m_e}{m_i}+\Theta\left\{\frac{\pi^2}{3}-\frac{1}{6}\frac{I_7(\hat{\epsilon}_i)}{I_5(\hat{\epsilon}_i)}\right\}} . \tag{59}$$

Before we will discuss these results further, it should be noted that Eqs. (55)–(59) are valid for $\Theta<1$. Moreover, the electron-electron interaction contribution is being carried out, however, by Matthiessen rule $(1/\sigma=1/\sigma_{ee}+1/\sigma_{ei})$. We showed that $\sigma\simeq\sigma_{ei}$, in agreement with the Pauli exclusion principle due to electron degeneracy.

These transport coefficients are better discussed in terms of strongly coupled plasma parameters which are the following: The coupling parameter $\Gamma$ is defined as

$$\Gamma=\frac{Z^2e^2}{a_ik_BT} , \tag{60}$$

where $Z^2e^2/a_i$ is the Coulomb interaction energy and $a_i$ the Wigner-Seitz radius of the ion $[a_i=(3Z/4\pi n)^{1/3}]$. Another parameter $r_s$ is similarly defined for the electrons:

$$r_s=a_em_ee^2/\hbar^2 , \tag{61}$$

where $a_e=(3/4\pi n)^{1/3}$ is the Wigner-Seitz radius of the electron and $\hbar^2/m_ee^2$ is the Bohr radius. Hence the Fermi degeneracy parameter $\Theta$ takes the form

$$\Theta=2(4/9\pi)^{2/3}Z^{5/3}(r_s/\Gamma) . \tag{62}$$

In order to discuss these results as functions of the pertinent plasma parameters, the integrals in Eq. (58) can be replaced by the series $J_n(\hat{\epsilon}_i)$ $(n=5,7)$. This can be done if use is made of the mean value theorem [32] for integrals to take $U_s(q)$ at its mean value since this function is bounded in the interval $(0,\hat{q}_s)$; $\hat{q}_s$ is the maximum wave number of the oscillation of type $s$.

In terms of $r_s$ and $\Gamma$, the electrical conductivity becomes

$$\sigma=\frac{9\pi^4\kappa_i^3e^4m_e^3}{16Z^{17/3}\hbar^3J_5(\hat{\epsilon}_i)m_i^2}\frac{\Gamma^2}{r_s^5} , \tag{63}$$

with $\hat{\epsilon}_i=2\pi(\kappa_im_e/m_i)^{1/2}(\Gamma^{1/2}/r_s^{1/2})$. For $\hat{\epsilon}_i\ll1$, i.e., for small $\Gamma$, $J_5(\hat{\epsilon}_i)\simeq\hat{\epsilon}_i^4/4$ and hence $\sigma\sim r_s^{-3}$. In this range of nonideality, the electrical conductivity of dense plasma is a linear function of the electron density, similar to that of metals at low temperature. For $\hat{\epsilon}_i\gg1$, i.e., for high $\Gamma$, $J_5(\hat{\epsilon}_i)\simeq5!\sum_{s=1}^{\infty}(1/s^5)=124.43$, so that

$$\sigma\sim\Gamma^2/r_s^2\sim n^{7/3}/T^2 .$$

A similar analysis of the thermal conductivity shows

$$\lambda=\frac{(4/9\pi)^{2/3}(\pi^8\kappa_im_e^4e^6k_B)/16\hbar^5Z^{7/3}m_i^2J_5(\hat{\epsilon}_i)r_s^5}{\frac{\kappa_im_e}{m_i}+(4/9\pi)^{2/3}Z^{5/3}(r_s/\Gamma)\left[\frac{\pi^2}{3}-\frac{1}{6}\frac{J_7(\hat{\epsilon}_i)}{J_5(\hat{\epsilon}_i)}\right]} . \tag{64}$$

For $\hat{\epsilon}_i\ll1$ or for weak coupling, $J_5(\hat{\epsilon}_i)\sim\hat{\epsilon}_i^4/4$ and $J_7(\hat{\epsilon}_i)\sim\hat{\epsilon}_i^6/6$. Hence

$$\lambda\simeq\frac{\pi^2}{3}\left[\frac{k_B}{e}\right]^2\sigma T . \tag{65}$$

Here again, the thermal conductivity is that of metals since $\lambda/\sigma T=(\pi^2/3)(k_B/e)^2$, the Wiedemann-Franz law, gives the ideal Sommerfeld number. For $\hat{\epsilon}_i\gg1$, $\Gamma>1$, strong coupling, $J_5(\hat{\epsilon}_i)\simeq5!\sum_{s=1}^{\infty}(1/s^5)=124.43$ and $J_7(\hat{\epsilon}_i)\simeq7!\sum_{s=1}^{\infty}1/s^7=5082.1$.

The thermal conductivity, in these extreme conditions, behaves as

$$\lambda\simeq\frac{\pi^4}{18\frac{m_e}{m_i}\kappa_i}\Theta\sigma T . \tag{66}$$

$\lambda$ is independent of $T$ and varies as $n^{-5/3}$ for high $\Gamma$.

Most of the existing calculation schemes, for coupled plasmas, cannot in principle predict a correct value for the transport coefficients for $\Theta\leq1$ because classical statistics is used for the electrons. When Fermi degeneracy is weak $(\theta>1)$, however, several models exist and are in good agreement with experiment [33–35].

On the other hand, the electrical and thermal conductivities of dense plasmas with strong Coulomb coupling and with a high degree of degeneracy (for the electrons) by Minoo, Deutsh, and Hansen (MDH) [5] and by Itoh *et al.* [36], on the basis of Ziman formulas, gave fairly good formulas for these coefficients, but often they included heavy numerical simulation for the calculations of structure factors.

Moreover, Lampe [7] developed a quantum model for highly degenerate, high-temperature plasmas establishing formulas for electrical and thermal conductivities where dynamic shielding, in the random phase approximation, is treated correctly. In Figs. 2 and 3, we compare the

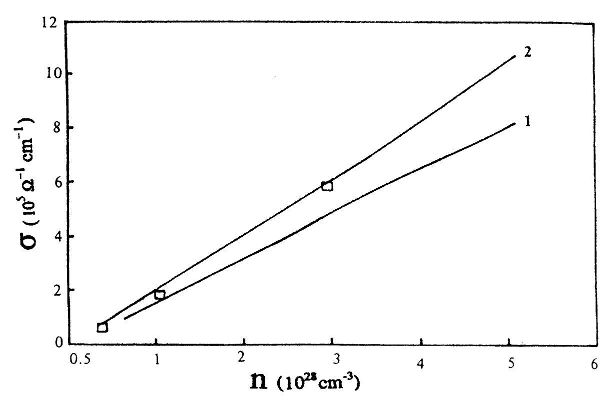


FIG. 2. Electrical conductivity behavior as a function of the density $n$ of hydrogen plasma at $T=10^7$ K, curve 1, Ref. [5]; curve 2, present theory, Eq. (55); and □, Ref. [7].

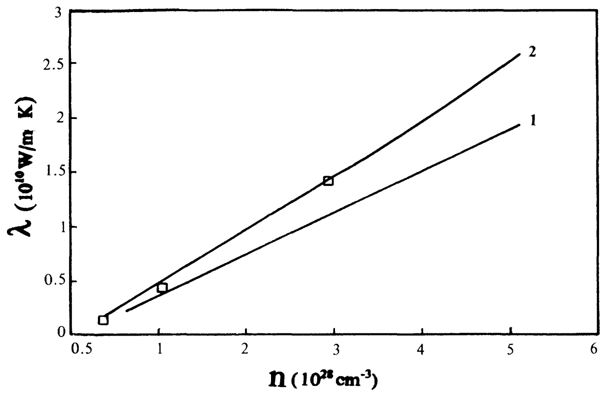


FIG. 3. Thermal conductivity for the same conditions as Fig. 2 where for curve 2 the formula is given by Eq. (58); curve 1, Ref. [5]; and □, Ref. [7]. The similar behavior between $\sigma$ and $\lambda$ shows that the Lorentz ratio varies little with the plasma parameters in this range of density and temperature.

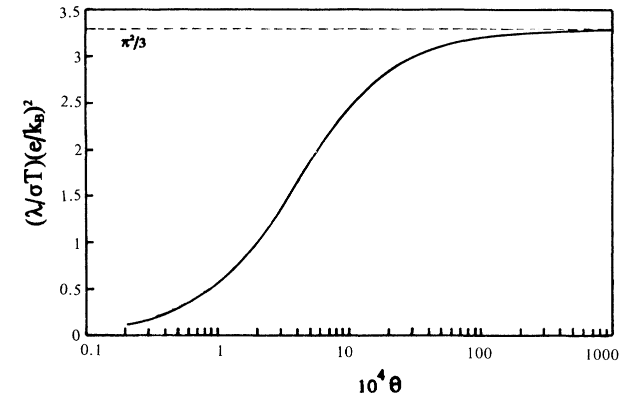


FIG. 4. Lorentz ratio equation (68) vs the degree of degeneracy ($\theta$) for a hydrogen plasma. The normalized ratio goes to $\pi^2/3$ for $\theta \geq 0.1$.

present results [Eqs. (55) and (59)]; with those of Lampe [7] (weak coupling) and MHD [5] (numerical simulation) where a very good agreement is observed for the particular range of degeneracy $(0.017 \leq \theta \leq 0.1)$ and coupling strength $(0.4 \leq \Gamma \leq 1)$ for which the three models are simultaneously valid.

Thereby, we have thus shown that the quantum collective approach is capable of describing accurately, in analytical forms, the transport coefficients of a degenerate plasma over a wide range of plasma parameters.

### B. Wiedemann-Franz-type law for degenerate plasmas

The relation between the electrical and thermal conductivities can be written in the form

$$\frac{\lambda}{\sigma T} = L(\Theta, \Gamma) \ . \tag{67}$$

This relation is a fundamental parameter in the transport phenomena studies. In Eq. (67), if $L$ is constant, the relation is the Wiedemann-Franz law valid for an interacting electron system where the interactions are elastic, i.e., the electron suffers no loss of energy. For classical, high-temperature, low-density plasmas $(\Gamma \ll 1, \ \theta \gg 1)$, for which a common relaxation time to the electrical and thermal conductivities exists, the ratio $L$ is a constant and equals the Lorentz number [35] $[4(k_B/e)^2]$. For a degenerate coupled plasma $(\Gamma \geq 1$ and $\theta \leq 1)$, most existing models, based on the Ziman formulas, give a constant ratio $L$, the same as that of metals and equal to the ideal Sommerfeld number $[(\pi^2/3)(k_B/e)^2]$. However, for this latter type of plasma (strongly coupled plasma), Bernu and Hansen [2] suggested that correct transport coefficients should be related by a Lorentz ratio depending on the coupling parameter, MDH [5] found a departure of $L$ from the ideal Sommerfeld value which has been attributed to inelastic diffusion effects of the electrons. In the present model, the Wiedemann-Franz-type law is obtained from Eqs. (55) and (59), which gives

$$\frac{\lambda}{\sigma T} = \frac{\pi^2}{3} \left[ \frac{k_B}{e} \right]^2 \frac{1}{1 + r(\theta)} \ , \tag{68}$$

where

$$r(\theta) = \frac{6}{\pi^2} \kappa_i \frac{m_e}{m_i \theta} - \frac{1}{2\pi^2} \frac{I_7(\theta)}{I_5(\theta)} \ . \tag{69}$$

$r(\theta)$ represents the effects of strong coupling and degeneracy. Figures 4 and 5 show the behavior of Eq. (68) as a function of the pertinent parameters $\theta$, $\Gamma$, and $r_s$. It is clear, from these curves, that the Lorentz ratio is a heavily dependent function of the coupling parameter and the degree of degeneracy for $\Gamma \gg 1$ and $\theta \ll 1$. In such a range of strong degeneracy $(\theta \ll 1)$ and coupling $(\Gamma \gg 1)$, Lampe, [7] theory, which is valid for only very-high-temperature plasma, provides a constant Lorentz ratio which equals, precisely, the ideal Sommerfeld number $(\pi^2/3)(k_B/e)^2$. Only in the weak-coupling regime, of both species, and with an intermediate degeneracy strength of the electron gas does his theory show a depar-

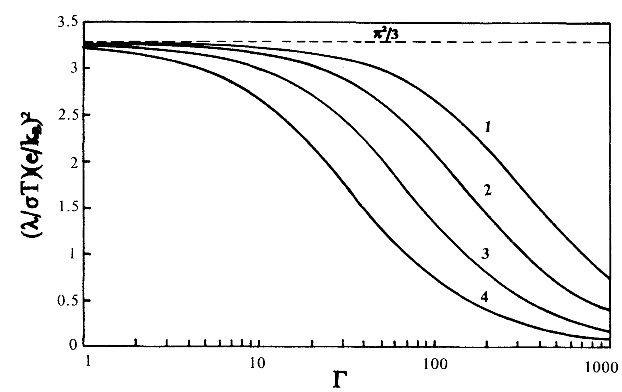


FIG. 5. Lorentz ratio equation (68) vs the coupling parameter ($\Gamma$) for a hydrogen plasma. The normalized ratio goes to $\pi^2/3$ for $\Gamma \leq 1$. From curve 1 to 4, $r_s = 0.252$, 0.117, 0.054, 0.025, respectively.

ture of the Lorentz ratio from the ideal Sommerfeld value.

The ratio $L$ of the present theory, which is reduced to a function of a single variable $\Theta$, tends asymptotically to $(\pi^2/3)(k_B/e)^2$ (Figs. 4 and 5), as $\Gamma$ becomes less than unity at different $r_s$, and when $\Theta$ $(\Theta \sim r_s/\Gamma)$ increases to values greater than $\sim 0.1$ at any coupling strength.

## V. SUMMARY

Presented in this paper is an electron conductivity model for dense plasmas which gives a complete set of transport coefficients. The model is useful for describing Coulomb systems where a common relaxation time between electrical and thermal conductivities is no longer justified. To be consistent, these coefficients satisfy the Onsager symmetry relation. The coefficients are reasonably accurate over a wide range of plasma temperature and density and are expressed in computationally simple forms.

For the quantitative calculations, the theory presented does not contain the Debye length $d_D$, which no longer exists in the range of the extreme conditions of densities and temperature. In this respect, our theory differs from most of the previously existing nonideal plasma models.

Moreover, we should see that only quantum formulation of the transport phenomena can show the degeneracy effects on the Lorentz ratio through the Wiedemann-Franz law. This latter result is explicitly quantified here; in this respect, the present theory has an advantage over the existing theories.

## ACKNOWLEDGMENTS

The authors wish to thank the Centre de Developpement des Technologies Avancées for the formal support during the present work.

## APPENDIX

Determination of the coefficients $c_\mu$ and $b_\mu$ is as follows. Define the trial function $\psi$ for $\Phi_1$ as

$$\psi = v_x \sum_\mu c_\mu (E-\xi)^\mu \; . \tag{A1}$$

We seek the maximum of $\langle \psi, \mathcal{L}\psi \rangle$ with the supplementary condition $\langle \psi, \mathcal{L}\psi \rangle = \langle \psi, F_1 \rangle$, i.e.,

$$\sum_{\mu,\nu} c_\mu c_\nu D_{\mu\nu} = \sum_\mu c_\mu N_\mu \; , \tag{A2}$$

where $D_{\mu\nu}$ and $N_\mu$ are defined in Eqs. (52) and (53) with (45). For this purpose we add $\langle \psi, \mathcal{L}\psi \rangle$ to $\langle \psi, F_1 \rangle$, multiplied by a Lagrangian parameter $\lambda$, and obtain the maximum from the condition

$$\frac{d}{dc_\mu}\left[\sum_{\mu,\nu} c_\mu c_\nu D_{\mu\nu} + \lambda \sum_\mu c_\mu N_\mu\right] = 2\sum_\nu c_\nu D_{\mu\nu} + \lambda N_\mu = 0 \; , \tag{A3}$$

multiplying by $c_\mu$ and summing over $\mu$ and comparing with (A2), we obtain $\lambda = -2$, so

$$N_\mu = \sum_\nu c_\nu D_{\mu\nu} \; . \tag{A4}$$

This is the system of equations which determines the coefficients $c_\nu$.

The same procedure is used for the calculation of the function $\Phi_2$.

The system of equations that determines the coefficients $b_\nu$ is

$$B_\mu = \sum_\nu b_\nu D_{\mu\nu} = N_{\mu+1} \; . \tag{A5}$$

Expressions of $D_{\mu\nu}$ are as follows. $D^+_{\mu\nu}$ for the absorption of a plasmon is evaluated by expressing all the variables as functions of $E$ and $\mathbf{q}$, as well as the angle between $\mathbf{k}$ and $\mathbf{k}'$. The angular integrals are readily carried out. $D^-_{\mu\nu}$ for the emission of a plasmon is obtained from $D^+_{\mu\nu}$ by replacing $\mathbf{q}$ by $-\mathbf{q}$ and $\omega_s$ by $-\omega_s$. With the change of variables $\eta = (E-\xi)/k_B T$ and $\epsilon = \hbar\omega_s/k_B T$, we obtain for the $e$-$e$ interaction

$$D^\pm_{\mu\nu}|_{e-e} = \pm \frac{2^5 \pi^2 n e^4 \delta_e^4}{3\hbar^3 \alpha^2 \epsilon_p^2}(k_B T)^{\mu+\nu-1}$$

$$\times \int_{\epsilon=\epsilon_p}^{\epsilon_e} \int_{\eta=-\xi/k_B T}^{+\infty} (k_B T\eta + \xi)\frac{\epsilon^2-\epsilon_p^2}{\alpha^2 \epsilon_p^2}\eta^\mu \left\{ \eta^\nu - (\eta \pm \epsilon)^\nu \left[ 1 \pm \frac{k_B T \epsilon \mp \dfrac{\hbar^2}{2m_e \alpha^2}\left[\dfrac{\epsilon^2-\epsilon_p^2}{\epsilon_p^2}\right]}{2(\eta k_B T + \xi)} \right] \right\}$$

$$\times \frac{1}{e^\eta + 1}\,\frac{1}{e^{-\eta\mp\epsilon}+1}\,\frac{1}{e^{\pm\epsilon}-1}\,\frac{1}{\left[1+4\pi^2 \dfrac{\epsilon^2-\epsilon_p^2}{\epsilon_e^2-\epsilon_p^2}\right]^2} d\epsilon\, d\eta \; , \tag{A6}$$

where $\alpha^2=\frac{3}{5}v_F^2/\omega_p^2$, $\hat{\epsilon}_e=\epsilon_p(1+\alpha^2\hat{q}_e^2)^{1/2}$, $\epsilon_p=\hbar\omega_p/k_BT$, and $\delta_e=2\pi/\hat{q}_e=2\pi(18\pi^2n)^{-1/3}$.

$D^{\pm}_{\mu\nu}$ is defined in Eq. (38), where $\psi$ is a trial function of the unknown $\phi_1$ and $\phi_2$, and is used in a variational principle to determine the integral collision term of Eq. (15). The different parts of $D^{\pm}_{\mu\nu}$ of (A6) are essentially energy ratios, which are integrated over convenient dimensionless variables ($\eta$ and $\epsilon$). The + and − signs come from the matrix element of the transitions, where the electron distribution may absorb (+) or emit (−) a plasmon or a sound wave during the interaction processes.

For the $e$-ion interaction,

$$D^{\pm}_{\mu\nu}|_{e\text{-ion}}=\pm\frac{2^5\pi^2ne^4\delta_i^4}{3\hbar^7C_s^4}(k_BT)^{\mu+\nu+3}\int_{\epsilon=0}^{\hat{\epsilon}_i}\int_{\eta=-\xi/k_BT}^{+\infty}(k_BT\eta+\xi)\epsilon^2\eta^{\mu}\left[\eta^{\nu}-(\eta\pm\epsilon)^{\nu}\left[1\pm\frac{k_BT\epsilon\mp\dfrac{k_B^2T^2\epsilon^2}{2m_eC_s^2}}{2(\eta k_BT+\xi)}\right]\right]\frac{1}{e^{\eta}+1}$$
$$\times\frac{1}{e^{-\eta\mp\epsilon}+1}\,\frac{1}{e^{\pm\epsilon}-1}\,\frac{1}{[1+4\pi^2\epsilon^2/\hat{\epsilon}_i^2]^2}d\epsilon\,d\eta\ , \qquad \text{(A7)}$$

where $C_s^2=\kappa_ik_BT/m_i$, $\delta_i=(4\pi n/3Z)^{-1/3}$, $\hat{\epsilon}_i=2\pi\hbar C_s/\delta_ik_BT$. Knowing that the electrons are completely degenerate, the limit $\eta=\xi/k_BT$ tends to $-\infty$.

The contributions due to the $e$-$e$ interactions being negligible and noting that we can generally manage with the case $\mu=(0,1)$, we obtain

$$D_{00}\simeq D_{00}|_{e\text{-ion}}=\frac{2^4ne^4Z\delta_i^4(k_BT)^5}{3\hbar^7m_iC_s^6}I_5(\hat{\epsilon}_i)\ , \qquad \text{(A8)}$$

$$D_{10}\simeq D_{10}|_{e\text{-ion}}=m_eC_s^2D_{00}\ , \qquad \text{(A9)}$$

$$D_{11}\simeq D_{11}|_{e\text{-ion}}=\left[2m_e\xi C_s^2+\frac{\pi^2}{3}(k_BT)^2-\frac{1}{6}(k_BT)^2\frac{I_7(\hat{\epsilon}_i)}{I_5(\hat{\epsilon}_i)}\right]D_{00}\ , \qquad \text{(A10)}$$

where $I_n(\hat{\epsilon}_i)$ is given by Eq. (57).

Expressions of $N_\mu$ are as follows:

$$N_\mu=-\int_0^{\infty}\frac{\partial f_0}{\partial E}(E-\xi)^{\mu}F(E)dE\ ,$$

for integrals of Fermi type, where

$$F(E)=\int_{E=\text{const}}\frac{v_x^2dS}{|\nabla_kE|}\ . \qquad \text{(A11)}$$

$N_\mu$ is given by the following extension near the Fermi energy [30]:

$$N_\mu=(E-\xi)^{\mu}F(E)|_{\xi}+\frac{\pi^2}{6}(k_BT)^2\left[(E-\xi)^{\mu}\frac{d^2F(E)}{dE^2}+2\mu(E-\xi)^{\mu-1}\frac{dF(E)}{dE}+\mu(\mu-1)(E-\xi)^{\mu-2}F(E)\right]_{\xi}+\cdots\ . \qquad \text{(A12)}$$

Knowing that in degenerate plasmas

$$\xi\simeq E_F\left[1-\frac{\pi^2}{12}\left[\frac{k_BT}{E_F}\right]^2\right]\simeq E_F\ ,$$

$E_F=\xi(T=0\text{ K})$, hence

$$N_0=F(E_F)+\frac{\pi^2}{6}(k_BT)^2\frac{d^2F}{dE^2}\bigg|_{E_F}\simeq F(E_F)\simeq\frac{4\pi^2n}{m_e}\ , \qquad \text{(A13)}$$

$$N_1=\frac{\pi^2}{3}(k_BT)^2\frac{dF}{dE}\bigg|_{E_F}\simeq\frac{\pi^2}{2}\frac{(k_BT)^2}{E_F}N_0\ , \qquad \text{(A14)}$$

$$N_2=\frac{\pi^2}{3}(k_BT)^2F(E_F)\simeq\frac{\pi^2}{3}(k_BT)^2N_0\ , \qquad \text{(A15)}$$

$$B_0=N_1\ , \qquad \text{(A16)}$$

$$B_1=N_2\ . \qquad \text{(A17)}$$

Finally, we obtain with the resolution of the systems (A5) and (A6)

$$c_0=\frac{N_0}{D_{00}}\ , \qquad \text{(A18)}$$

$$c_1=\frac{1}{E_F}\,\frac{\dfrac{\pi^2}{3}\Theta-\kappa_i\dfrac{m_e}{m_i}}{2\kappa_i\dfrac{m_e}{m_i}+\left[\dfrac{\pi^2}{3}-\dfrac{1}{6}-\dfrac{I_7}{I_5}\right]\Theta}\,\frac{N_0}{D_{00}}\ , \qquad \text{(A19)}$$

$$b_0=\pi^2E_F\frac{\left[\dfrac{2}{3}\kappa_i\dfrac{m_e}{m_i}\Theta^2+\dfrac{\pi^2}{3}\Theta^3-\dfrac{1}{12}\dfrac{I_7}{I_5}\Theta^3\right]}{2\kappa_i\dfrac{m_e}{m_i}+\left[\dfrac{\pi^2}{3}-\dfrac{1}{6}\dfrac{I_7}{I_5}\right]\Theta}\,\frac{N_0}{D_{00}}\ , \qquad \text{(A20)}$$

$$b_1=\frac{\pi^2}{3}\,\frac{\theta}{2\kappa_i\dfrac{m_e}{m_i}+\left[\dfrac{\pi^2}{3}-\dfrac{1}{6}\dfrac{I_7}{I_5}\right]\theta}\,\frac{N_0}{D_{00}}\ . \qquad \text{(A21)}$$

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