Resonant shape oscillations and decay of a soliton in a periodically inhomogeneous nonlinear optical fiber

Boris A. Malomed*
Department of Applied Mathematics, School of Mathematical Sciences, Tel Aviv University, Ramat Aviv 69978, Israel

David F. Parker and Noel F. Smyth
Department of Mathematics and Statistics, University of Edinburgh, The King's Buildings, Mayfield Road, Edinburgh EH9 3JZ, Scotland
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The propagation of a soliton in a nonlinear optical fiber with a periodically modulated but sign-preserving dispersion coefficient is analyzed by means of the variational approximation. The dynamics are reduced to a second-order evolution equation for the width of the soliton that oscillates in an effective potential well in the presence of a periodic forcing induced by the imhomogeneity. This equation of motion is considered analytically and numerically. Resonances between the oscillations in the potential well and the external forcing are analyzed in detail. It is demonstrated that regular forced oscillations take place only at very small values of the amplitude of the inhomogeneity; the oscillations become chaotic as the inhomogeneity becomes stronger and, when the dimensionless amplitude attains a threshold value which is typically less than $\frac{1}{4}$, the soliton is completely destroyed by the periodic inhomogeneity.

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I. INTRODUCTION

Propagation of solitons in nonlinear optical fibers is a challenging physical problem with very promising practical applications [1,2]. As is well known, the evolution of an envelope of electromagnetic waves in a monomode fiber is described, with high accuracy, by the nonlinear Schrödinger (NLS) equation

$$iu_z+\tfrac{1}{2}u_{tt}+|u|^2u=0\ , \tag{1}$$

where z is the propagation distance and t is the so-called reduced time [1,2]. It is also well known that the NLS equation is exactly integrable by means of the inverse scattering transform [3], so allowing construction of a number of exact solutions in explicit form. Nevertheless, since not all the physically interesting solutions can be found explicitly, less rigorous methods have been developed to describe the required solutions approximately. Among these methods, the simplest and most elegant is based upon the Lagrangian representation of the NLS equation [4]. The technique is particularly useful since it can be applied to perturbations of Eq.(1) which are not exactly integrable, such as arise for inhomogeneous fibers.

The general idea underlying this approximation is well known: one presumes a certain ansatz for the shape of solution sought, but leaves in the ansatz a set of free parameters which may evolve with z. Next one evaluates the full Lagrangian of the NLS equation when the ansatz is inserted. When doing this, all z differentiations which appear in the Lagrangian are applied to the above-mentioned free parameters. Eventually, one finds the Lagrangian as a function of the free parameters and their first derivatives. The resulting variational equations for this effective Lagrangian then become a system of ordinary differential equations (ODE).

The most important exact particular solution to the NLS equation (1) is the soliton

$$u_{\rm sol}(z,t)=a^{-1}{\rm sech}(t/a)\exp(\tfrac{1}{2}ia^{-2}z)\ , \tag{2}$$

where the arbitrary parameter a measures the soliton "width." Expression (2) shows that the amplitude a^{-1} and wave number $\frac{1}{2}a^{-2}$ of the soliton are determined by a. An interesting and practically important question is, how will a solitary pulse initially having a sech profile as in Eq. (2) evolve if its initial amplitude A and width a do not satisfy $aA=1$? From inverse scattering theory the answer is well known: as $z\to\infty$, the pulse will evolve into a soliton with different amplitude and into quasilinear dispersive waves (radiation). The relevant "scattering data" were first given in Ref. [5]. However, in reality, the asymptotic stage of the evolution may arise only at very large z, while in an intermediate region the actual dynamic behavior may be quite different. To describe these intermediate dynamics approximately, the variational technique was applied in Ref. [6] with trial wave form (ansatz) [cf. Eq. (2)]

$$u(z,t)=A(z){\rm sech}\{t/a(z)\}\exp\{i[\varphi(z)+b(z)t^2]\}\ . \tag{3}$$

The meaning of the parameters $A(z)$, $a(z)$, and $\varphi(z)$ is obvious. The parameter $b(z)$, the so-called chirp, characterizes the dependence of the carrier frequency on position within the soliton. The set of evolution equations

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arising when the ansatz (3) is inserted into the full Lagrangian for (1) may be reduced [6] to a single second-order equation for $a(z)$. Formally, that resulting equation may be regarded as the equation of motion for a mechanical particle in a smooth potential well with an infinitely high wall at $a=0$ and vanishing potential as $a \rightarrow \infty$. If the "particle" initially has positive energy, a escapes to infinity, corresponding to complete decay of the pulse into radiation [although the radiation degrees of freedom are not included in the ansatz (3)]. Alternatively, if the initial energy is negative, the "particle" is trapped within the potential well and oscillates. These oscillations correspond to periodic oscillations of the shape of the pulse (3). Of course, within the framework of the full NLS equation (1), the oscillations will be gradually damped due to radiative losses so that eventually the particle falls to the bottom of the well (which itself becomes shallower due to radiation). The rest state of the "particle" corresponds to the exact soliton solution (2).

In the present work, we use the above technique to study the evolution of a sech-shaped pulse in an inhomogeneous fiber. Note that usually nonlinear fibers operate in a spectral region near the zero of the dispersion coefficient [the coefficient of $\frac{1}{2}u_{tt}$ which has been scaled to unity in Eq. (1)], to allow competition between dispersion and the weak Kerr nonlinearity of silica [1,2]. Since the total dispersion coefficient, being the sum of material dispersion of the silica and of geometric dispersion, is rather small, it can be quite sensitive to small changes due to inhomogeneity. Recent works [7–9] have analyzed some effects arising near a point where fiber inhomogeneity causes the dispersion coefficient to change sign. These effects include the decay of a soliton passing from an anomalous-dispersion domain [such as described by Eq. (1)] to a normal-dispersion domain (with negative coefficient of $\frac{1}{2}u_{tt}$) where solitons cannot exist [7]; development of modulational instability of a continuous wave traveling in the opposite direction in either a monomode or bimodal fiber [8]; and formation of a soliton from an initial localized pulse passing from a domain of normal dispersion to one of anomalous dispersion [9]. In this paper, we concentrate on the case of a dispersion coefficient which is periodically inhomogeneous but always positive, so that the fiber exhibits anomalous dispersion everywhere.

The underlying physical idea to be developed is that oscillations of the soliton shape may resonate with the periodic inhomogeneity of the fiber. We show that, when the soliton is close to resonance, a relatively small amplitude of the inhomogeneity is sufficient to completely destroy a soliton which otherwise would be stable. [For the particular inhomogeneity profile given by Eq. (11) below, the threshold amplitude ϵ for fiber inhomogeneity above which the soliton is completely destroyed is typically below $\frac{1}{4}$.] At smaller values of the inhomogeneity amplitude, a resonantly driven soliton exhibits, in most cases, chaotic shape oscillations.

Results in Ref. [6] show that a typical length scale for the natural oscillations of soliton shape in a homogeneous fiber are comparable with the so-called "soliton period," which is inversely proportional to the square of the soliton width. This period may be hundreds of kilometers for picosecond solitons [1,2], but up to eight orders of magnitude smaller for femtosecond pulses. Since this is comparable with typical inhomogeneity periods, the resonance effect can be practically important in the femtosecond range.

The problem outlined and the results obtained, besides having considerable methodological interest, may find two practical applications: (i) in determining how possible inhomogeneities within the fiber can affect propagation conditions for optical solitons: and (ii) in exploiting the strong interaction of a soliton with an artificially formed periodic inhomogeneity within soliton-based optical logic elements.

With regard to artificial periodic inhomogeneity, much work exists concerning solitons in fibers with periodically inhomogeneous refractive index (see, e.g., the recent papers [10], and references therein). The model to be considered here differs from those considered previously, since it includes dispersion, which is usually ignored in those models. However, our model omits Bragg coupling between counterpropagating waves, which was the central point in Refs. [10]. Nevertheless, periodic modulation of the refractive index is apt to generate, at least as a byproduct, a parallel modulation of the dispersion coefficient, so that the effects to be analyzed here are relevant to that work.

The paper is organized as follows. In Sec. II we outline the derivation of the evolution equations for the parameters in (3), omitting some details since the derivation is essentially similar to that developed for a homogeneous fiber in [6]. As in Ref. [6], the final form of the effective evolution equation is equivalent to the equation of motion for a particle in a potential field, the coordinate of the "particle" being the soliton width a. However, the potential changes periodically in "time" and, additionally, the "particle" is subject to a "friction force," with "friction coefficient" also periodic in "time." In Sec. III we analyze the three most interesting cases in which resonance between the shape oscillations and the inhomogeneity might be expected. This analysis follows classical nonlinear resonance theory [11], being based upon expansion in powers of the perturbation in a and small detuning from resonance. Last, in Sec. IV we display results of direct numerical simulations of the above-mentioned effective equation of motion. We first consider small oscillations under near-resonant conditions, showing that oscillations may become chaotic (quasiharmonic locally but irregularly modulated) at fairly small values of the inhomogeneity amplitude. We then consider in some details the way in which inhomogeneity can completely destroy a soliton—corresponding in the "particle" analogy to escape to infinity from the potential well. Using simulations, we concentrate on finding the threshold characteristics. Concluding remarks are gathered in Sec. V.

II. EVOLUTION EQUATIONS FOR PARAMETERS OF THE SOLITON

In an inhomogeneous fiber with a z-dependent dispersion coefficient $\alpha(z)$, the underlying equation is the variable-coefficient NLS equation:

$$iu_z+\tfrac{1}{2}\alpha(z)u_{tt}+|u|^2u=0\ . \tag{4}$$

The Lagrangian density which gives rise to Eq. (4) is

$$L=\tfrac{1}{2}i(u_zu^*-u_z^*u)-\tfrac{1}{2}\alpha(z)|u_t|^2+\tfrac{1}{2}|u|^4\ , \tag{5}$$

the asterisk denoting a complex conjugate. As in Ref. [5] one inserts the ansatz (3) into the Lagrangian density (5) and then calculates the total Lagrangian $\int_{-\infty}^{\infty}L(z,t)dt$, where z differentiation is applied to the parameters occurring in (3). Performing the integration and then taking variations of the resulting Lagrangian with respect to all the parameters yields a system of evolution which proves to be a straightforward generalization of the equations derived in Ref [6] for the case $\alpha(z)=1$:

$$A^2a=N^2=\text{const}\ , \tag{6}$$

$$b=\frac{1}{2}a'/\alpha a\ , \tag{7}$$

$$\varphi'-\frac{-\alpha}{3a^2}+\frac{5N^2}{6a}\ , \tag{8}$$

$$a''=\left[\frac{\alpha'}{\alpha}\right]a'=\left[\frac{4}{\pi^2}\right]\alpha^2a^{-3}-\left[\frac{4}{\pi^2}\right]N^2\alpha a^{-2}\ . \tag{9}$$

Here, the prime denotes d/dz and N^2 is an integration constant which measures the intensity of the initial pulse. Without loss of generality, the z scale may be chosen so that $\alpha(z)$ has period 2π. Then, writing

$$a=(2\alpha_0/\pi)^{1/2}\tilde{a}(z),\quad N=(\pi\alpha_0/2)^{1/4}\tilde{N}\ ,$$

$$\alpha(z)=\alpha_0\tilde{\alpha}(z)\ ,$$

where α_0 is the mean value of $\alpha(z)$, transforms Eq. (9) to

$$\tilde{a}''-(\tilde{\alpha}'/\tilde{\alpha})\tilde{a}'=\tilde{\alpha}^2\tilde{a}^{-3}-\tilde{N}^2\tilde{\alpha}\tilde{a}^{-2}\ .$$

Dropping the tildes finally yields the equation

$$a''-\left[\frac{\alpha'}{\alpha}\right]a'=-\frac{\partial U}{\partial a},\quad U(a,\alpha)=\tfrac{1}{2}\alpha^2a^{-2}-\alpha N^2a^{-1}, \tag{10}$$

which as mentioned earlier corresponds to a mechanical equation of motion with "time"-dependent potential (see Fig. 1) and periodic friction coefficient.

We assume that the spatial inhomogeneity is weak and, for simplicity, take it in the sinusoidal form,

$$\alpha(z)=1+\epsilon\sin z\ . \tag{11}$$

In the limit $\epsilon=0$, Eqs. (10) show that the "energy" $E=\frac{1}{2}(a')^2+U(a,1)$ is conserved. This allows explicit integration, so giving z as a function a. In particular, if the energy E is negative, the frequency q of anharmonic oscillations of a particle in the potential well is

$$q=(2|E|)^{3/2}/N^2\ . \tag{12}$$

This corresponds to the wave number of the spatial oscillations of soliton shape. The bottom of the well (see Fig. 1) corresponds to $E=-\frac{1}{2}N^4$, so that for small oscillations the limiting (maximum) value of the wave number

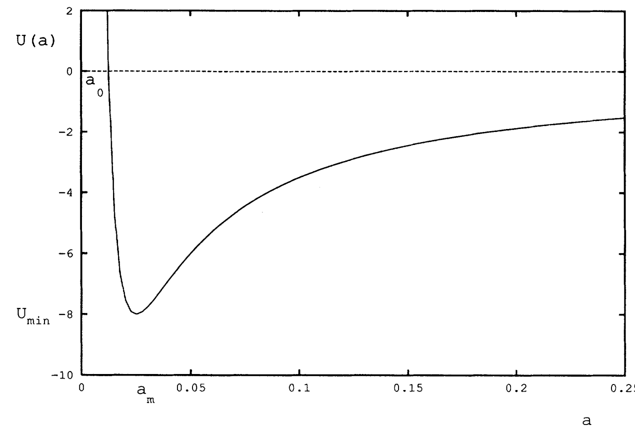

FIG. 1. The shape of the potential (10) when $\alpha\equiv1$. The minimum value $U_{\min}=-\frac{1}{2}N^4$ is attained at $a=a_m\equiv N^{-2}$; $U(a,1)=0$ at $a=a_0\equiv\frac{1}{2}N^{-2}$.

(12) is

$$q_0=N^4\ . \tag{13}$$

Resonance between the free oscillations and the small driving force in Eq. (1) is expected when the wave number (12) is commensurate with the driving wave number [1, for $\alpha(z)$ taken in the form (11)], i.e., when $q=m/n$ for some integers m and n. Of course, only small values of m and n are of practical interest. In the analytical work within Sec. III, we shall concentrate on small-amplitude oscillations for which $q_0=N^4$ and consider the simplest resonances which occur for N^4 close to $\frac{1}{2}$, 1, or 2. This analysis is based on the equation of motion (10) with $\alpha(z)$ in the form (11) and with a expanded in powers of $b(z)\equiv a(z)-a_m$, where $a_m=N^{-2}$ is the location of the bottom of the well for $\epsilon=0$. As is well known from the classical theory of nonlinear resonance [11], in this expansion nonlinear terms should be retained up to order b^3. Consequently, the appropriate approximation obtained after simple algebra is

$$b''+N^8b-(\cos z)b'=N^6\epsilon\sin z+3N^{10}b^2-6N^{12}b^3-4N^8\epsilon(\sin z)b\ . \tag{14}$$

III. ANALYSIS OF RESONANCES FOR SMALL-AMPLITUDE OSCILLATIONS

A. The resonance near $N^4=\frac{1}{2}$

First, we consider the case when the second harmonic of small-amplitude oscillations resonates with the small driving force. According to Eq. (13), this occurs for

$$N^4=\tfrac{1}{2}(1+k)\ ,\quad k\ll1\ , \tag{15}$$

k being a small detuning from resonance. In this case, we seek a solution to Eq. (14) in the form

$$b(z)=b_1\cos(\tfrac{1}{2}z+\delta)+b_{21}\cos(z+2\delta)$$
$$+b_{22}\sin(z+2\delta)+b_0\ , \tag{16}$$

where b_1, b_{21}, b_{22}, b_0, and δ may all vary slowly with z, while b_1 is the dominant coefficient. Inserting Eq. (16) into Eq. (14), we first set the coefficients of the second and zeroth harmonics to zero, to eliminate b_2, b_{22}, and b_0 in favor of b_1 and δ:

$$b_0=3b_1^2\sqrt{2}\ , \tag{17}$$

$$b_{21}=\left[\frac{\sqrt{2}\epsilon}{3}\right]\sin(2\delta)-\left[\frac{b_1^2}{4\sqrt{2}}\right]\ , \tag{18}$$

$$b_{22}=-(\sqrt{2}\epsilon/3)\cos(2\delta)\ . \tag{19}$$

Then, using relations (17)–(19) when setting to zero the coefficients of the fundamental harmonic [the terms with $\cos(\frac{1}{2}z+\delta)$ and $\sin(\frac{1}{2}z+\delta)$] leads us to the following equations for $b_1(z)$ and $\delta(z)$:

$$b_1'=\tfrac{3}{4}\epsilon b_1\cos(2\delta)-\tfrac{9}{8}b_1^3\ , \tag{20a}$$

$$\delta'=\tfrac{1}{2}k-\tfrac{3}{4}\epsilon\sin(2\delta)-\tfrac{15}{32}b_1^2\ . \tag{20b}$$

Finally, by introducing the notation

$$K\equiv 2k/3\epsilon,\quad \Delta\equiv 2\delta,\quad Z\equiv 3\epsilon z/2,\quad B\equiv b_1/2\sqrt{2\epsilon}\ , \tag{21}$$

Eqs. (20) are simplified to

$$\frac{dB}{dZ}=\tfrac{1}{2}B\cos\Delta-6B^3\ , \tag{22a}$$

$$\frac{d\Delta}{dZ}=K-\sin\Delta-5B^2\ . \tag{22b}$$

The system (22) possesses stationary points (Δ_0,B_0) of two types, the first being the obvious one,

$$B_0=0,\quad \sin\Delta_0=K\ , \tag{23}$$

which exists for $|K|<1$. Expressions (17)–(19) show that each such stationary point corresponds to driven oscillations at the second harmonic, with negligible excitation of free oscillations at the fundamental one. Elementary analysis of the stability of these stationary points for (22) shows that there are two instability growth rates

$$\gamma_1=\tfrac{1}{2}\cos\Delta_0,\quad \gamma_2=-\cos\Delta_0\ . \tag{24}$$

Accordingly, each stationary point satisfying (23) is unstable, being a saddle point in the phase plane (Δ,B).

The second type of stationary solution to Eqs. (22) is given by

$$B_0^2=\tfrac{1}{12}\cos\Delta_0\ , \tag{25a}$$

$$\sin\Delta_0+\tfrac{5}{12}\cos\Delta_0=K\ . \tag{25b}$$

Real solutions exist provided that Eq. (25b) has solutions with $\cos\Delta_0>0$, which requires that K lies in the interval

$$-1<K<\tfrac{13}{12}\ . \tag{26}$$

The eigenvalues γ for the instability growth rates associated with the roots (25) satisfy

$$\gamma^2+2\gamma\cos\Delta_0+\tfrac{169}{144}\cos^2\Delta_0-\tfrac{5}{12}K\cos\Delta_0=0\ . \tag{27}$$

Further analysis shows that, for each K lying within the interval (26), there is a single stable stationary point (Δ_0,B_0) with $0\le\Delta_0<2\pi$ and that this is given by

$$\tfrac{13}{12}\cos\Delta_0=\tfrac{5}{13}K+\sqrt{1-(12K/13)^2}\ . \tag{28}$$

In the narrow subregion $1<K<\frac{13}{12}$, Eqs. (25) possess only a solution with negative sign in front of the radical in (28), but this solution is unstable. Further analysis of Eq. (27) shows that the stable stationary points (Δ_0,B_0) are spirals for $-1<K<\frac{5}{12}$ and nodes for $\frac{5}{12}<K<\frac{13}{12}$. For exact resonance ($K=0$), for which results of numerical simulations are displayed in Sec. IV, the stationary points are always stable spirals, as is seen in the phase plane (Δ,B) given in Fig. 2. It may be noted that $|B_0|$ is maximum just at $K=\frac{5}{12}$, the value separating spirals from nodes.

Expressions (17)–(19) and (21) show that the stable stationary points (Δ_0,B_0) represent a "nonlinear mixture" of the free oscillations at the fundamental and forced oscillations at the second harmonic. The free oscillations have amplitude $0(\epsilon^{1/2})$, while the second-harmonic contributions are $0(\epsilon)$, where ϵ is the amplitude of the forcing due to inhomogeneity.

When K lies outside the interval (26), the resonance is strongly detuned and the system (22) has no stationary points. In this case, all trajectories are dragged to the line $B=0$, as shown in Fig. 3. This latter property follows from the fact that, if $dS=d(B^2)d\Delta$ denotes an element of the phase-plane area, the divergence $\mathrm{div}\,\mathbf{j}$ of the phase-plane flow defined by Eqs. (22) is strictly negative:

$$\mathrm{div}\,\mathbf{j}\equiv\frac{\partial}{\partial\Delta}\left\{\frac{d\Delta}{dZ}\right\}+\frac{\partial}{\partial B^2}\left[\frac{dB^2}{dZ}\right]=-24B^2\ .$$

Thus all solutions have asymptotic state $B=0$. Moreover, referring to expressions (16)–(19) it is easy to see

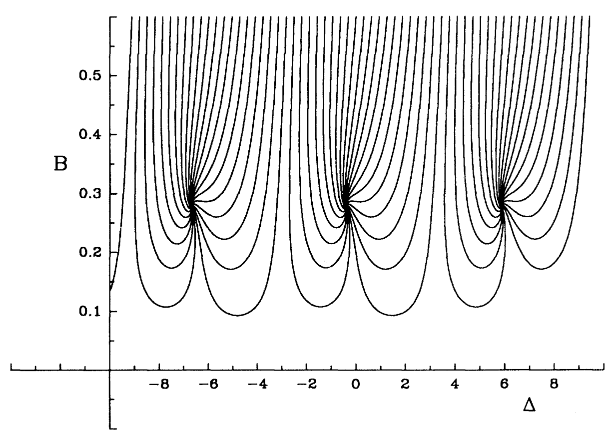

FIG. 2. The phase plane of the dynamical system (22) in the case $K=0$. The picture is periodic in Δ with period 2π. The lower half plane (not shown) is the mirror image of the upper. The points F are the stable spirals given by Eqs. (25a) and (28). The saddles at the points $B=0$, $\Delta=\pi n$ ($n=0,\pm1,\pm2,\dots$) correspond to Eq. (23) (with $K=0$).

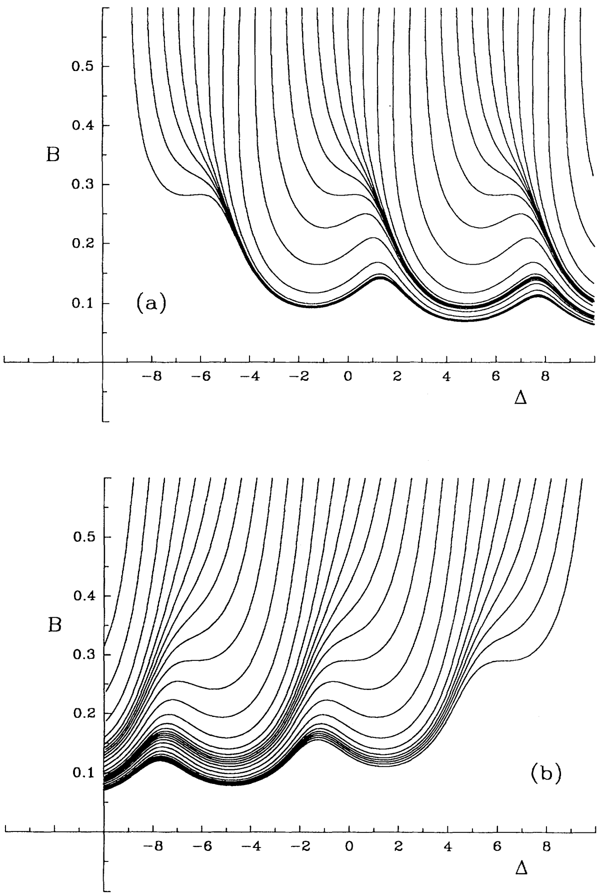

FIG. 3. The phase plane of the system (22) in the case when K lies outside the interval (26): (a) $K > \frac{13}{12}$; (b) $K < -1$.

that, at least to the accuracy of the present analysis, this asymptotic state is simply a second-harmonic oscillation with the same period as the forcing due to inhomogeneity, with negligible contribution from the resonantly coupled fundamental. The final inference is that the resonance (15) arises only in the interval $-1 < K < \frac{13}{12}$ given by Eq. (26). As $K \to -1$, the resonance behavior goes over smoothly into nonresonance, since Eqs. (25a) and (28) show that $B_0 \to 0$. In contrast, as $K \to \frac{13}{12}$ the resonance disappears abruptly, since Eqs. (25a) and (28) give $B_0^2 = \frac{5}{156}$ for $K = \frac{13}{12}$. The reason is that, as mentioned earlier, two stationary solutions to Eqs. (22), the stable and unstable ones, merge and disappear at the boundary $K = \frac{13}{12}$.

B. The resonance near $N^4 = 1$

Direct resonance of small-amplitude oscillations with the periodic forcing occurs [cf. Eq. (15)] for

$$N^4 = 1 + k, \quad |k| \ll 1 \ . \tag{29}$$

In this case, the analysis is simpler than before, since inspection of Eq. (14) shows that the only term proportional to ϵ which contributes in the lowest approximation is the first term on the right-hand side. Neglect of the other terms yields from Eq. (14) the standard equation of the classical nonlinear resonance theory [11]. Explicitly, substitution into Eq. (14) of an expansion [cf. Eq. (16)]

$$b(z) = b_1 \sin z + b_0 + b_2 \cos(2z) \tag{30}$$

for a steady solution allows b_0 and b_2 to be eliminated in favor of b_1 as

$$b_2 = \tfrac{1}{2} b_1^2, \quad b_0 = \tfrac{3}{2} b_1^2 \ . \tag{31}$$

Then, the amplitude b_1 of the fundamental is found to satisfy

$$2kb_1 - 6b_1^3 = \epsilon \ . \tag{32}$$

In the cases $k < 0$, or $k > 0$ with $|\epsilon| > \frac{4}{9} k^{3/2}$, Eq. (32) has a single root which corresponds to stable resonantly driven oscillations. However, for $k > 0$ with $|\epsilon| < \frac{4}{9} k^{3/2}$ there are three roots, two stable and one unstable. It is well known that, if one regards b_1 and a phase shift δ of the fundamental relative to the forcing as slowly varying dynamical variables, the stationary point in the phase plane (δ, b_1) corresponding to a stable solution of Eq. (32) is found to be a center. Therefore a solution of the forced oscillator equation (10) in the vicinity of such a point will appear as oscillations at the fundamental spatial frequency (wave number) modulated by a low-frequency envelope (beats). In Sec. IV such a behavior is found by numerical simulation of Eq. (10) near this resonance.

C. The resonance near $N^4 = 2$

The third resonance to be considered occurs for

$$N^4 = 2(1+k), \quad |k| \ll 1 \ , \tag{33}$$

so that it is the second harmonic of an oscillation at the forcing frequency which is resonant. The two terms from Eq. (14) which were unimportant in case B may again be omitted, thus allowing application of well-known results for nonlinear subharmonic resonance [11]. The steady solution sought as in case B is

$$b(z) = b_2 \cos(2z) + b_1 \cos z + b_0 + b_4 \cos(4z) \ , \tag{34}$$

with amplitudes b_1, b_0, and b_4 found in the form

$$b_1 = 2\sqrt{2}\epsilon/3, \quad b_0 = 3b_2^2/\sqrt{2}, \quad b_4 = -b_2^2/\sqrt{2} \ . \tag{35}$$

The equation determining b_2 is found [cf. Eq. (32)] as

$$kb_2 - 3b_2^3 = 17\sqrt{2\epsilon^2}/24 \ , \tag{36}$$

which has a single root (and exhibits monostability) if $k > 0$, or if $k > 0$ with $\epsilon^2 > 8\sqrt{2k}^{\,3/2}/51$, but exhibits bistability otherwise [with three roots, two of which give rise to stable solutions of type (34)].

Higher resonances for small-amplitude oscillations, arising when N^4 is close to m/n with larger m and n, can be analyzed similarly, but are of less practical interest.

IV. NUMERICAL SIMULATIONS

We computed numerically the ODE (10) with $\alpha(z)$ in the form (11), but without restriction to $\epsilon \ll 1$. The objectives were (i) to see how much the presence of a weak inhomogeneity can affect the shape of a soliton and (ii) to determine the minimum inhomogeneity amplitude ϵ at the chosen inhomogeneity period 2π which will destroy a soliton of specified initial intensity N^2. Both these issues are of obvious practical interest.

A. Near-resonant oscillations

The influence of forcing upon oscillations can be expected to be most dramatic close to a resonance. Therefore to address issue (i) we first simulate the three resonances analyzed in Sec. III.

In Fig. 4 we display the oscillations of soliton width $a(z)$, governed by Eqs. (10) and (11) in the case $N^4=\frac{1}{2}$, at various values of ϵ with initial conditions

$$a(0)=a_m=N^{-2}, \quad a'(0)=0 \; . \tag{37}$$

These initial conditions correspond to a "particle" initially at rest at $z=0$, exactly at the bottom of the potential well of Fig. 1.

The sequence shows how the resonantly driven oscillations change with increase of the forcing amplitude ϵ. Even for $\epsilon=0.05$ [Fig. 4(a)] the oscillations are irregular, although the fundamental period 4π, corresponding to the eigenfrequency $\frac{1}{2}$ [Eq. (13)] can be seen. With increase in ϵ the oscillations become more chaotic [e.g., $\epsilon=0.2$ in Fig. 4(b)] and, finally, around $\epsilon=0.25$ [Fig. 4(c)] the "particle" escapes from the well after several large slow oscillations.

We infer that, for subharmonic resonance corresponding to Eq. (15), the soliton will always be destroyed by an inhomogeneity having amplitude exceeding a threshold value ϵ_{thr} lying between 0.20 and 0.25. If $\epsilon<\epsilon_{\text{thr}}$, the oscillations of soliton shape are irregular (chaotic)—the analytical theory in Sec. III being applicable only for small values of ϵ. Indeed, it follows from Eqs. (21), (25a), and (28) that, for $K=0$, the amplitude of the driven oscillations is $b_1 \simeq \sqrt{2\epsilon/3}$, whereas a characteristic width of the potential well, estimated as $a_m-a_0=\frac{1}{2}N^{-2}$ (Fig. 1), is in the present case $1/\sqrt{2}$. Thus the fact that for $\epsilon=0.05$ the ratio $b_1/(\frac{1}{2}N^{-2})$ exceeds $\frac{1}{4}$ explains why apparently chaotic oscillations [Fig. 4(a)] appear even for a forcing amplitude as small as $\epsilon=0.05$.

Figure 5 displays results of simulations of the resonance (29) with $k=0$ and with initial conditions again of form (37). The forcing amplitude here is $\epsilon=0.0025$ and produces a quasilinear picture—with beats between a forced oscillation at unit spatial frequency and free oscillations with frequency slightly shifted due to the very weak nonlinearity. In terms of the analytical description (Sec. III B), the beats can be interpreted as small oscillations around the center which is the corresponding stationary point in the phase plane (as mentioned already in Sec. III B). A similar picture applies for $\epsilon=0.05$, with beat period reduced from around 38 to 29 but with much increased oscillation in b_1, between 0.85 and 1.27. At large values of ϵ the oscillations become more chaotic and finally lead to the escape of the particle from the potential well (or destruction of the soliton).

In the case of the second-harmonic resonance (33), the criterion for smallness of ϵ is the least restrictive. Thus, at $\epsilon=0.01$, quite regular, although anharmonic, forced oscillations are observed [Fig. 6(a)] [the initial conditions were again taken in the form of Eqs. (37)]. However, for

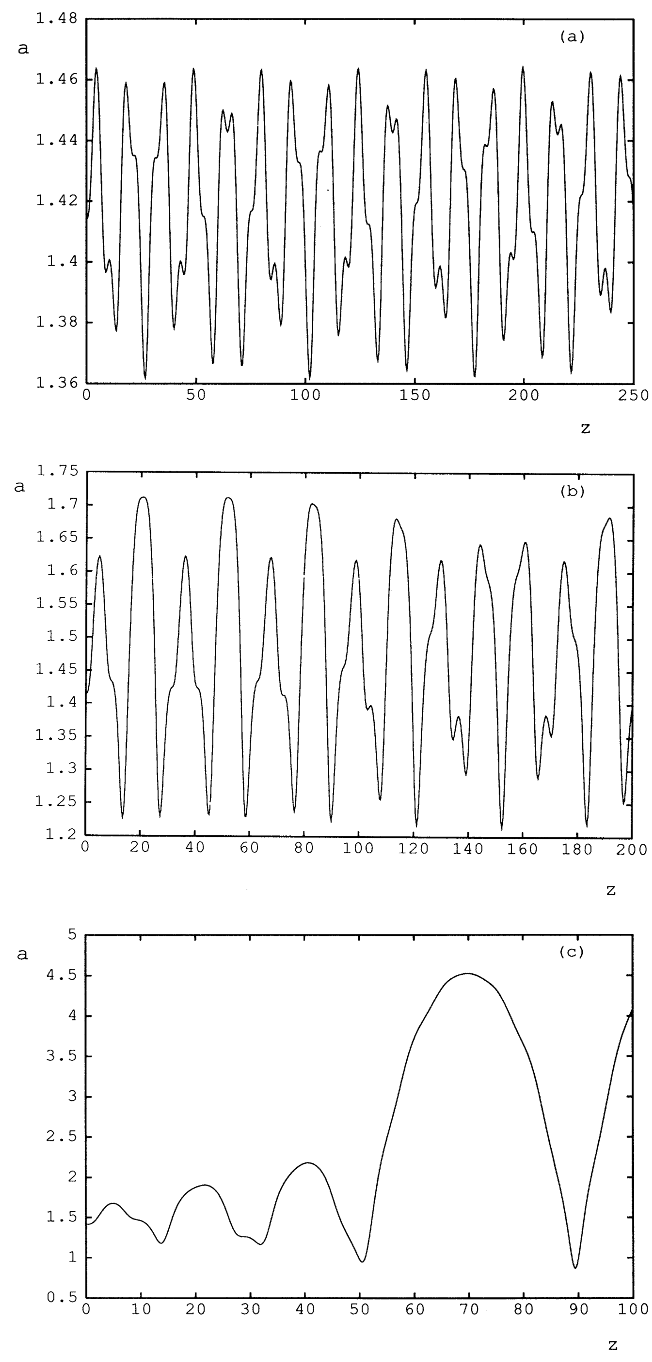

FIG. 4. The results of numerical integration of Eqs. (10) and (11) with the initial conditions (37) for $N^4=\frac{1}{2}$: (a) $\epsilon=0.05$; (b) $\epsilon=0.20$; (c) $\epsilon=0.25$.

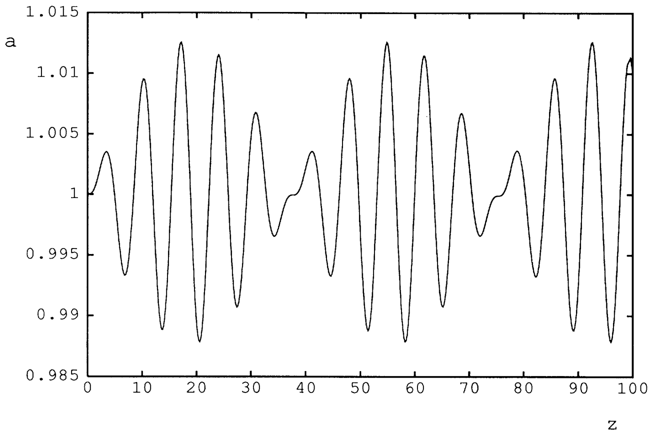

FIG. 5. The same as in Fig. 4 for $N^4=1$, $\epsilon=0.0025$.

$\epsilon=0.05$ "beat oscillations" appear [Fig. 6(b)] and the value $\epsilon=0.25$ seems to lie slightly above the escape threshold [Fig. 6(c)].

B. Further analysis of the escape process

Since soliton destruction, represented in terms of Eq. (10) by $a\rightarrow\infty$ (escape from a potential well), is very important for applications to optical fibers, additional analysis of the escape has been undertaken. We fixed $N^4=2$, $\epsilon=0.01$, and $a'(0)=0$, but gradually decreased the initial value $a(0)$, so that the initial "energy" of the oscillations in the potential well increased. It should be noted from Fig. 1 that, for unforced oscillations (homogeneous fibers), bounded oscillations occur only for $a(0)>a_0=1/2\sqrt{2}\simeq0.3536$.

Recall that the initial value $a(0)=1/\sqrt{2}$ corresponding to Eq. (37) yielded, with $\epsilon=0.01$, the regular oscillations shown in Fig. 6(a). For $a(0)=0.6571$, irregularly modulated oscillations arise [Fig. 7(a)], with relatively small amplitude so that each cycle appears quasiharmonic. For $a(0)=0.4000$, we find strongly anharmonic large-amplitude oscillations which also display irregular modulation [Fig. 7(b)]. The strong anharmonicity is due to the marked asymmetry of the potential well at these amplitudes. At the still smaller value $a(0)=0.3750$, similar irregular oscillations with a maximum of $a\simeq6$ are obtained and with minima of a at z spacings $\simeq40$. Finally, for $a(0)=0.3670$, escape occurs. Thus, for this type of soliton destruction, the threshold value lies in the interval $0.3670<a(0)<0.3750$ and so slightly exceeds the value $a_0\simeq0.3536$ relevant in the absence of inhomogeneity ($\epsilon=0$). We conclude that the weak inhomogeneity being considered here ($\epsilon=0.01$) slightly narrows the range within which the disturbed soliton of Eq. (3) remains stable, but remark that if the amplitude of the inhomogeneity exceeds the threshold value ϵ_{thr} [which for both $N^4=2$ and $N^4=\frac{1}{2}$ lies near 0.25, see Figs. 4(c) and 6(c)] the soliton *completely* loses its stability.

V. CONCLUSION

In this paper, we have demonstrated that a soliton can be quite sensitive to the presence of a relatively weak inhomogeneity. Since the approximation employed completely ignores emission of radiation and possible departure of the soliton shape from the ansatz of Eq. (3), it would be desirable to solve numerically the underlying partial differential equations (PDE) (4) with $\alpha(z)$ in the form (11). Very recently, a few numerical results for a

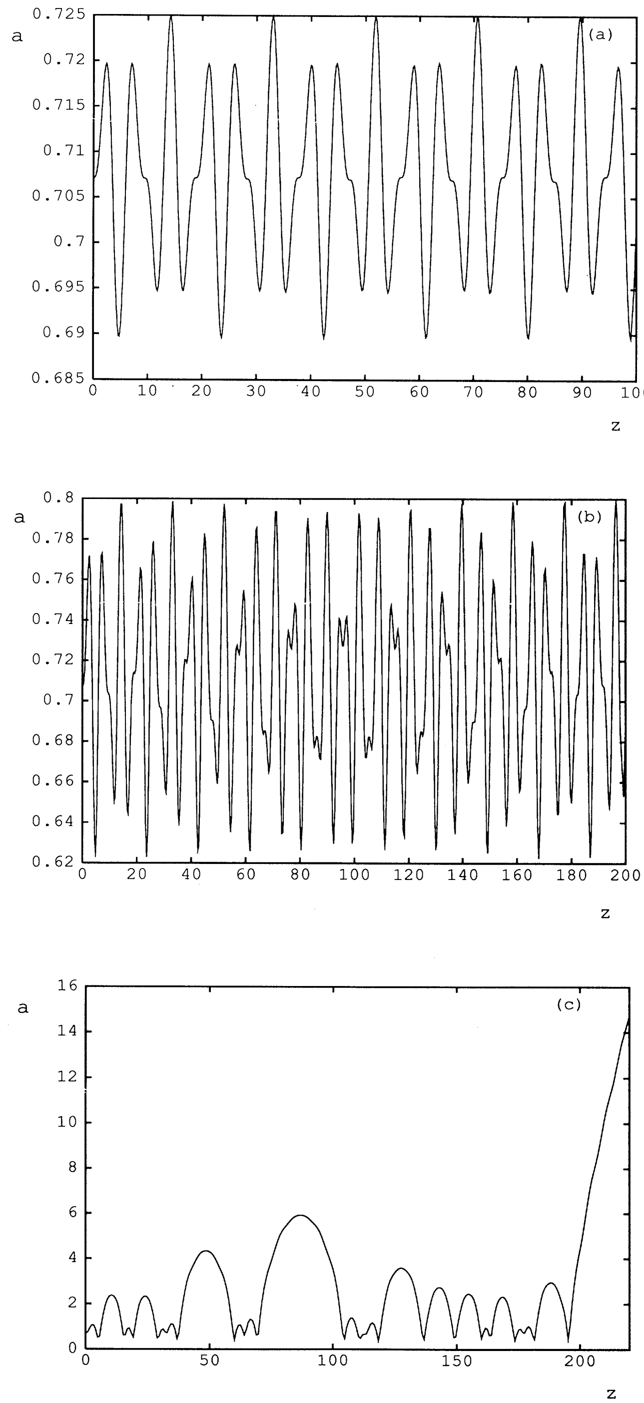

FIG. 6. The same as in Fig. 4 for $N^4=2$: (a) $\epsilon=0.01$; (b) $\epsilon=0.05$; (c) $\epsilon=0.25$.

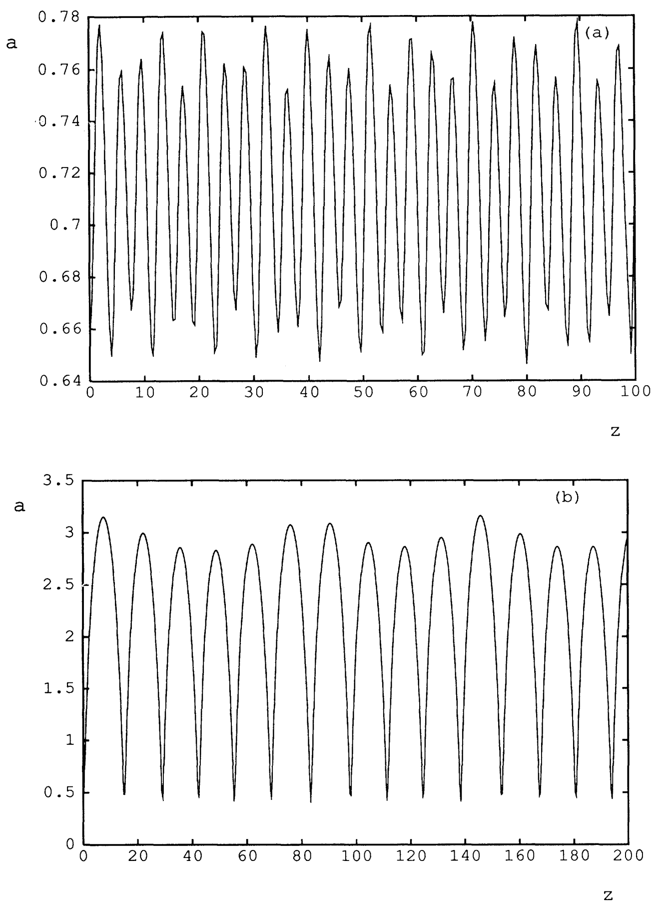

FIG. 7. The results of numerical integration of Eqs. (10) and (11) in the case of $N^4=2$, $\epsilon=0.01$, $a'(0)=0$, with different initial values $a(0)$: (a) $a(0)=0.6571$; (b) $a(0)=0.4000$.

PDE which is equivalent to Eqs. (4) and (11) have been reported in Ref. [12]. For relatively weak inhomogeneity it was found that shape oscillations of the soliton were locally quasiharmonic, but, over longer intervals, irregular. No conspicuous emission of radiation or appreciable departure from a sech profile has been noticed. In general, these observations agree with those found in the present work by means of the variational approximation.

This work can be extended in various directions. For example, it would be interesting to develop a similar analysis for a periodically inhomogeneous *bimodal* nonlinear fiber, for which the model system of equations comprises two coupled NLS equations for two separate polarizations. The variational approach to describing the dynamics of a two-component soliton in a homogeneous bimodal fiber has been developed in Refs. [13]. It shows that two-component solitons possess an additional mode of oscillation associated with relative motion of the two components. Thus such a model contains more possibilities for resonance between internal modes of a soliton and the periodic inhomogeneity. This work is now under way [14]. Among the preliminary results obtained there, certainly noteworthy is the splitting of a two-component soliton into simple solitons due to resonance of the corresponding internal mode with the spatial inhomogeneity.

Another interesting development would be to consider models with a random inhomogeneity, in which internal modes might resonate with a suitable spectral component of the random inhomogeneity.

The present analysis is confined to the case of inhomogeneity amplitude sufficiently small that the dispersion coefficient in Eq. (11) never becomes negative. A challenging problem would be to analyze the dynamics of nonlinear pulses in a medium with periodic change of sign of the dispersion coefficient.

Finally, it is remarked that the results obtained here apply not only to optical fibers, but also to other systems giving nonlinear guided wave propagation, e.g., to nonlinear internal wave channels in geophysical hydrodynamics [7].

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*Author to whom the correspondence should be addressed. Electronic address: malomed@leo.math.tau.ac.il

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