Analytic solutions of the time-dependent quasilinear diffusion equation with source and loss terms

Mohamed H. A. Hassan
International Centre for Theoretical Physics, Trieste, Italy

E. A. Hamza
Department of Mathematics and Computing, Sultan Qaboos University, Muscat, Oman
(Received 1 June 1992; revised manuscript received 1 March 1993)

A simplified one-dimensional quasilinear diffusion equation describing the time evolution of collisionless ions in the presence of ion-cyclotron-resonance heating, sources, and losses is solved analytically for all harmonics of the ion cyclotron frequency. Simple time-dependent distribution functions which are initially Maxwellian and vanish at high energies are obtained and calculated numerically for the first four harmonics of resonance heating. It is found that the strongest ion tail of the resulting anisotropic distribution function is driven by heating at the second harmonic followed by heating at the fundamental frequency.

PACS number(s): 52.50.Gj, 52.25.Dg

Ion-cyclotron-resonance heating (ICRH) is currently considered as one of the most promising methods for heating plasmas in large fusion experiments to the required fusion temperature. Minority ions in resonance with the incident radio-frequency wave develop a non-Maxwellian velocity distribution function with larger perpendicular energy which is subsequently transmitted to the background ions and electrons through Coulomb collisions. The analytic nature of this non-Maxwellian distribution function has been the subject of several studies [1–3]. These studies were mainly concerned with obtaining steady-state velocity distribution functions for the minority ions and were largely based on the classic work of Stix [1]. Time-dependent velocity distribution functions have recently been reported [4,5], but these are only confined to heating at the fundamental and second-harmonic frequencies.

In this paper we report explicit time-dependent analytic expressions for the distribution function of ions undergoing radio-frequency heating in the ion cyclotron-frequency range. The results obtained are valid for all harmonics of the ion cyclotron frequency.

The radio-frequency diffusion equation describing the short-time evolution of the resonant ion distribution function, $f(\mathbf{v},t)$, in the presence of an ion source and a heat sink and in the absence of collisional effects has the form

$$\frac{\partial f}{\partial t}=\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}v_\perp D\frac{\partial f}{\partial v_\perp}+\bar{S}(v)-\bar{\lambda}f \ , \tag{1}$$

where D is the quasilinear diffusion coefficient, $\bar{S}$ is a source function representing here neutral beam injection, and $\bar{\lambda}$ is a loss term which can be attributed to charge exchange and radial losses. The exact form of $\bar{\lambda}$, taken here to be a constant, can only be determined by the plasma transport process and does not affect the analytic results presented below in a significant manner. Steady-state solutions of Eq. (1) can only be derived provided that the number of particles created through the source function $\bar{S}$ is balanced by an equal number of particles destroyed through the loss term $\bar{\lambda}$.

Evidently, since (1) does not take into account the effects of Coulomb collisions, it is only applicable in the limit of strong radio-frequency heating and for time scales shorter than the collisional time scale. The results obtained below, therefore, adequately describe the time evolution of the ion distribution function only at its very initial stages of deformation before the onset of collisional effects [2].

To simplify the problem further we neglect the finite Larmor radius effects in which case the diffusion coefficient D takes the simple form

$$D=D_n v_\perp^{2(n-1)} \ . \tag{2}$$

Here n represents the nth harmonic of the heating scheme and D_n is a constant proportional to the square of the wave amplitude.

Since our main concern here is to study the initial development of the ion distribution function in the direction perpendicular to the magnetic field, we may take the dependence of f on $v_\parallel$ to be Maxwellian and assume the separable solution

$$f(v_\parallel,v_\perp,\bar{t})=g(v_\perp,\bar{t})e^{-t_\parallel^2/v_\theta^2} \ . \tag{3}$$

Thus, substituting (3) into (1) and integrating over $v_\parallel$, we obtain the following diffusion-type equation:

$$\frac{\partial g}{\partial t}=\frac{\partial^2 g}{\partial \xi^2}+\frac{(1-2\nu)}{\xi}\frac{\partial g}{\partial \xi}-\lambda g+S(\xi) \ , \tag{4}$$

where we have introduced the nondimensional variables and parameters,

$$\zeta=(v_\perp/v_\theta)^{2-n}\ ,\quad t=\frac{D_n}{v_\theta^{2n-4}}(2-n)^2\bar{t}\ ,\quad \nu=\frac{n-1}{n-2}\ ,$$
$$\lambda=\frac{v_\theta^{2n-4}\bar{\lambda}}{(2-n)^2D_n}\ ,\quad S=\frac{v_\theta^{2n-4}}{(2-n)^2D_n}\int_{-\infty}^{\infty}\bar{S}dv_\parallel\ . \tag{5}$$

The transformed equation (4) is not valid for $n=2$, i.e., the case of heating at the second harmonic. This case will be dealt with separately.

Equation (4) with the initial condition $g(0,\zeta)=F(\zeta)$ and the boundary conditions $g(t,\infty)=0$ and the conditions $g(t,0)$ is bounded can be solved using the method of Laplace transform. The result is

$$g(t,\zeta)=\frac{\zeta^\nu}{2t}e^{-(\zeta^2/4t)-\lambda t}\int_0^\infty x^{1-\nu}e^{-x^2/4t}I_\nu(x\zeta/2t)F(x)dx$$
$$+\zeta^\nu\int_0^\infty x^{1-\nu}S(x)dx\int_0^\infty\frac{y\,dy}{\lambda+y^2}J_\nu(xy)J_\nu(\zeta y)\{1-e^{-(y^2+\lambda)t}\}\ . \tag{6}$$

It is easily checked that (6) satisfies the initial and boundary conditions specified above. It is also readily observed that (6) evolves into the following steady-state solution:

$$g(\infty,\zeta)=\zeta^\nu\int_0^\infty x^{1-\nu}S(x)dx\int_0^\infty\frac{y\,dy}{\lambda+y^2}J_\nu(xy)J_\nu(\zeta y)\ . \tag{7}$$

The case of particular interest in ICRH studies is that of initially Maxwellian ions and neutral beam injection source, i.e.,

$$g(0,v_\perp)=\frac{n_0}{\pi v_\theta^2}e^{-v_\perp^2/v_\theta^2}\Longrightarrow F(\zeta)=\frac{n_0}{\pi v_\theta^2}\exp\{-\zeta^{2/2-n}\}$$

and

$$S(\zeta)=\frac{S_0}{\zeta_0}\delta(\zeta-\zeta_0)\ .$$

The distribution function (6) then takes the form

$$g(t,\zeta)=\frac{\zeta^\nu}{2t}e^{-(\zeta^2/4t)-\lambda t}\frac{n_0}{\pi v_\theta^2}\int_0^\infty x^{1-\nu}e^{-x^2/4t-x^{2/2-n}}$$
$$\times I_\nu(x\zeta/2t)dx$$
$$+S_0\left[\frac{\zeta}{\zeta_0}\right]^\nu\int_0^\infty\frac{y\,dy}{\lambda+y^2}J_\nu(\zeta_0 y)J_\nu(\zeta y)$$
$$\times\{1-e^{-(y^2+\lambda)t}\} \tag{8}$$

and the corresponding steady-state distribution function reduces to the simple form

$$g(\infty,\zeta)=S_0\left[\frac{\zeta}{\zeta_0}\right]^\nu\times\begin{cases}I_\nu(\zeta\lambda^{1/2})K_\nu(\zeta_0\lambda^{1/2})\ , & \text{if } \zeta_0\ge\zeta\\ I_\nu(\zeta_0\lambda^{1/2})K_\nu(\zeta\lambda^{1/2})\ , & \text{if } \zeta_0\le\zeta\ .\end{cases} \tag{9}$$

An approximate analytic expression for the initial development of the distribution function can be obtained in closed form by evaluating the integrals in Eq. (8) for small values of t.

The result can be written in the form

$$g(t,\zeta)\simeq\begin{cases}\dfrac{n_0}{\pi v_\theta^2}e^{-\lambda t-\zeta^{2/2-n}}+\dfrac{S_0}{\zeta_0^2-\zeta^2}\left[\dfrac{\zeta}{\zeta_0}\right]^\nu t^{1/2}\sqrt{1-\lambda t}\,\{\zeta J_\nu(\alpha)J'_\nu(\beta)-\zeta_0J'_\nu(\alpha)J_\nu(\beta)\}\quad(\zeta\neq\zeta_0)\ , & (10)\\ \dfrac{n_0}{\pi v_\theta^2}e^{-\lambda t-\zeta^{2/2-n}}+S_0\dfrac{(1-\lambda t)}{2}\left[\dfrac{\zeta}{\zeta_0}\right]^\nu\left\{\left[1-\dfrac{\nu^2}{\alpha^2}\right]J_\nu^2(\alpha)+J_\nu'^2(\alpha)\right\}\quad(\zeta=\zeta_0), & (11)\end{cases}$$

where

$$\alpha=\zeta_0\left[\frac{1-\lambda t}{t}\right]^{1/2}\ ,\quad \beta=\zeta\left[\frac{1-\lambda t}{t}\right]^{1/2}\ .$$

Explicit expressions for the distribution function $F(t,\zeta)=(\pi v_\theta^2/n_0)g(t,\zeta)$ in the cases of heating at the first, third, and fourth harmonics can be written in the dimensionless forms

$$F_1(t,\zeta)=\frac{e^{-\zeta^2/(4t+1)-\lambda t}}{(4t+1)}+\lambda\int_0^\infty\frac{y\,dy}{\lambda+y^2}J_0(\zeta_0y)J_0(\zeta y)\{1-e^{-(y^2+\lambda)t}\}\ , \tag{12}$$

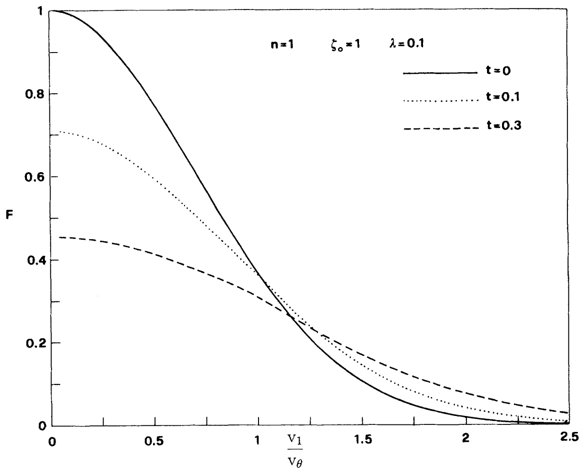

FIG. 1. The profile of the distribution function F_1 [given by Eq. (12) for the case of heating at the fundamental frequency] as a function of perpendicular velocity for different times and for $\zeta_0=1$; $\lambda=0.1$.

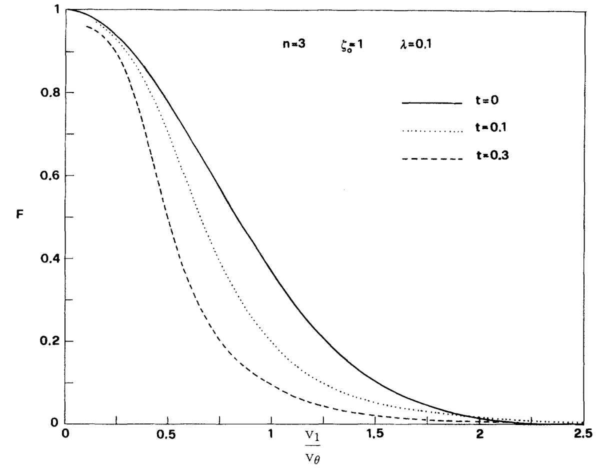

FIG. 2. The profile of the distribution function F_3 [given by Eq. (13) for the case of heating at the third harmonic] as a function of perpendicular velocity for different times and for $\zeta_0=1$; $\lambda=0.1$.

$$F_3(t,\zeta=\frac{\zeta^{-2}}{2t}e^{-\zeta^{-2}/(4t)-\lambda t}\int_0^\infty x^{-1}e^{-x^2/(4t)-x^{-2}}I_2\left[\frac{x}{2t\zeta}\right]dx$$

$$+\lambda\left[\frac{\zeta_0}{\zeta}\right]^2\int_0^\infty\frac{y\,dy}{(\lambda+y^2)^2}J_2\left[\frac{y}{\zeta_0}\right]J_2\left[\frac{y}{\zeta}\right]\{1-e^{-(y^2+\lambda)t}\}\ , \qquad (13)$$

$$F_4(t,\zeta)=\frac{\zeta^{-3}}{2t}e^{-\zeta^{-4}/(4t)-\lambda t}\int_0^\infty x^{-1/2}e^{-x^2/(4t)-x^{-1}}I_{3/2}\left[\frac{x}{2t\zeta^2}\right]2x$$

$$+\lambda\left[\frac{\zeta_0}{\zeta}\right]^3\int_0^\infty\frac{y\,dy}{\lambda+y^2}J_{3/2}\left[\frac{y}{\zeta_0^2}\right]J_{3/2}\left[\frac{y}{\zeta^2}\right]\{1-e^{(y^2+\lambda)t}\}\ , \qquad (14)$$

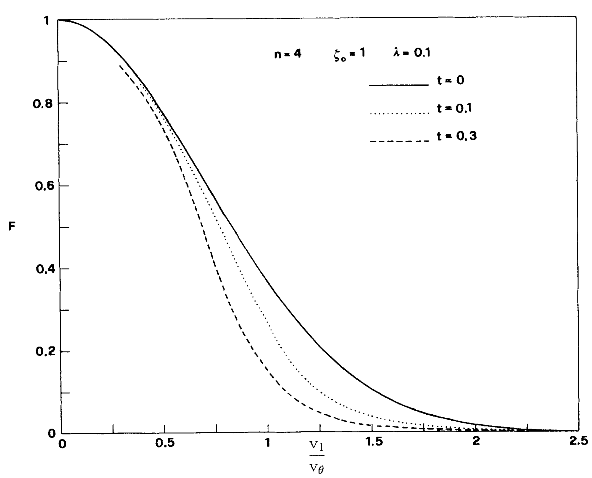

FIG. 3. The profile of the distribution function F_4 [given by Eq. (14) for the case of heating at the fourth harmonic] as a function of perpendicular velocity for different times and for $\zeta_0=1$; $\lambda=0.1$.

where we have used the balance relation $(\pi v_\theta^2/n_0)S_0=\lambda$.

In order to illustrate the nature of these solutions we evaluate the expressions F_1, F_3, and F_4 numerically for various values of the parameters involved. The results are summarized in Figs. 1–3.

HEATING AT THE SECOND HARMONIC

In the case of heating at the second harmonic ($n=2$) we have, instead of Eq. (4), the following diffusion equation:

$$\frac{\partial g}{\partial t}=\frac{1}{\zeta}\frac{\partial}{\partial\zeta}\left[\zeta^3\frac{\partial g}{\partial\zeta}\right]+S(\zeta)-\lambda g\ , \qquad (15)$$

where $\zeta=v_\perp/v_\theta$, $t=D_2\bar{t}$, and $\lambda=\bar{\lambda}/D_2$. In terms of the new variable $y=2t+\ln\zeta$, Eq. (15) can be transformed into the simpler form

$$\frac{\partial g}{\partial t}=\frac{\partial^2 g}{\partial y^2}+S(y,t)-\lambda g(y)\ , \qquad (16)$$

which can readily be solved by the method of Fourier transform. The result is

$$g(t,\zeta)=\frac{1}{2\sqrt{\pi}}\frac{e^{-\lambda t}}{t^{1/2}}\int_{-\infty}^{\infty} g(0,x)\times\exp\{-(2t+\ln\zeta-x)^2/4t\}dx\,.$$

$$+\frac{1}{2\sqrt{\pi}}\int_0^\infty \frac{S(z)}{z}dz\int_0^t \frac{dx}{x^{1/2}}\exp-\left\{\lambda x+\left[2x+\ln\frac{\zeta}{z}\right]^2\Big/4x\right\}\,. \quad (17)$$

To illustrate the nature of the solution (17) we consider the case of initially Maxwellian ions and perpendicular beam injection, i.e.,

$$g(0,x)=\frac{n_0}{\pi v_\theta^2}\exp\{-e^{2x}\}\ ,\quad S(z)=\frac{S_0}{\zeta_0}\delta(z-\zeta_0)\ .$$

Equation (17) then reduces to

$$g(t,\zeta)=\frac{n_0}{2\pi^{3/2}v_\theta^2}\frac{e^{-\lambda t}}{t^{1/2}}\int_{-\infty}^{\infty} dx\exp-\left\{\frac{(2t+\ln\zeta-x)^2}{4t}+e^{2x}\right\}$$
$$+\frac{S_0}{2\sqrt{\pi}\zeta_0^2}\int_0^t\frac{dx}{x^{1/2}}\exp-\left\{\lambda x+\left[2x+\ln\frac{\zeta}{\zeta_0}\right]^2\Big/4x\right\}\,. \quad (18)$$

The corresponding steady-state solution can easily be derived from (18). The result is

$$g(\infty,\zeta)=\frac{S_0}{\sqrt{\pi}\zeta_0^2}\int_0^\infty\frac{dx}{x^{1/2}}\exp-\left\{\lambda x+\left[2x+\ln\frac{\zeta}{\zeta_0}\right]^2\Big/4x\right\}$$
$$=\frac{S_0}{2\zeta_0^2(\lambda+1)^{1/2}}\left[\frac{\zeta_0}{x}\right]^{(1+\sqrt{\lambda+1})}\,. \quad (19)$$

It should be noted that in the case $\lambda=0$ the first and second parts of Eq. (18) reduce to the expressions given in Ref. [2] [Eqs. (14) and (29), respectively].

Finally, as in the previous cases, we can rewrite Eq. (18) in the dimensionless form

$$F_2(t,\zeta)=\frac{e^{-\lambda t}}{2\sqrt{\pi}t^{1/2}}\int_{-\infty}^{\infty}dx\exp\left\{\frac{(2t+\ln\zeta-x)^2}{4t}+e^{2x}\right\}$$
$$+\frac{\lambda}{2\sqrt{\pi}\zeta_0^2}\int_0^t\frac{dx}{x^{1/2}}\exp-\left\{\lambda x+\left[2x+\ln\frac{\zeta}{\zeta_0}\right]^2\Big/4x\right\}\,. \quad (20)$$

The function $F_2(t,\zeta)$ is evaluated numerically and the results are summarized in Fig. 4.

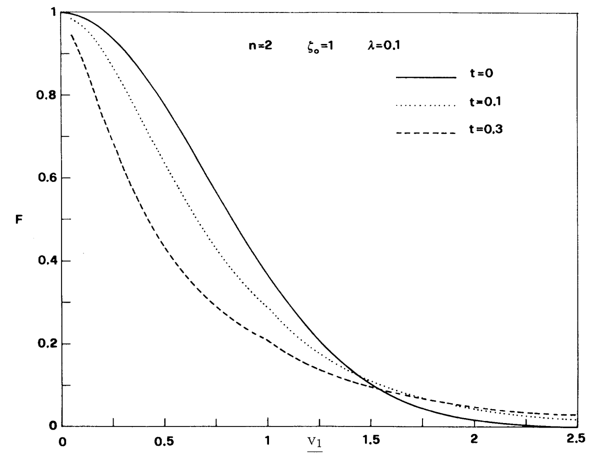

FIG. 4. The profile of the distribution function F_2 [given by Eq. (20) for the case of heating at the second harmonic] as a function of perpendicular velocity for different times and for $\zeta_0=1$; $\lambda=0.1$.

GENERAL DISCUSSION

The main results of this paper are Eqs. (8) and (18) which give analytic expressions for the time evolution of an initially Maxwellian distribution function of collisionless ions in the presence of combined ion-cyclotron-resonance heating, neutral beam injection and losses which may be due to charge exchange. These expressions are valid for all harmonics of the ion cyclotron frequency.

The heated distribution function for the first, second, third, and fourth harmonic is evaluated numerically and the results are presented in Figs. 1–4. In each of the four cases the initially Maxwellian distribution function is shown to develop a tail of non-Maxwellian energetic ions within a fraction of the characteristic time scale $\bar{t}\sim v_\theta^{2n-4}/D_n$ $(n=1-4)$.

In Fig. 5 we have plotted the distribution functions of

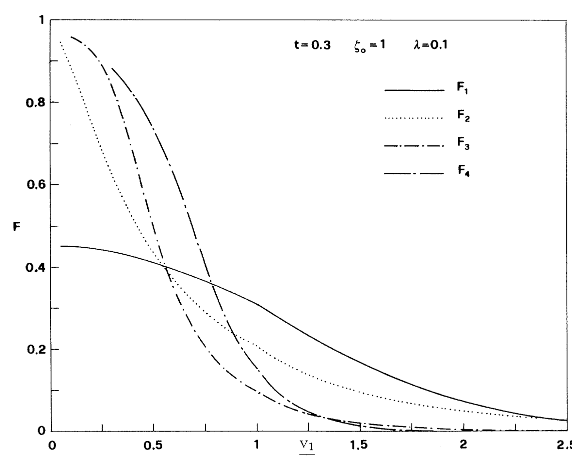

FIG. 5. The profiles of the functions F_1, F_2, F_3, and F_4 for the case $t=0.3$, $\zeta_0=1$, and $\lambda=0.1$.

the first four harmonics for the case $t=0.3$, $\zeta_0=1$, and $\lambda=0.1$. The graphs reveal that second harmonic heating results in the initial development of the most energetic tail, followed by fundamental heating. The tail is less energetic in the case of heating at the third and fourth harmonics, due to the weak absorption of the rf power by the ions at these harmonics.

Recent rf-heating experiments of JT-60 [6] have confirmed the occurrence of an ion temperature tail in the case of third and fourth harmonic heating. In further agreement with the results of this paper, these experiments have also shown that the temperature tail becomes weaker at the third, fourth, and higher harmonics.

Although the validity of this model is limited to time scales less than the collisional time scale [2], the simple exact analytic results reported here may be relevant to experiments in which knowledge of the initial deformation of the ion distribution function due to the combined effect of ion-cyclotron-resonance heating and neutral beam injection is of importance.

[1] T. H. Stix, Nucl. Fusion **15**, 137 (1975).

[2] M. Cox and D. F. H. Start, Nucl. Fusion **24**, 399 (1984).

[3] L.-G. Eriksson *et al.*, Nucl. Fusion **29**, 87 (1989).

[4] D. Anderson, L.-G. Eriksson, and M. Lisak, J. Plasma Phys. **36**, 357 (1986).

[5] M. H. A. Hassan, Phys. Fluids **31**, 596 (1988).

[6] K. Uehara and H. Kimura, in *Radio-Frequency Power in Plasma,* Proceedings of the Eighth Topical Conference on Radio-Frequency Power in Plasmas, Irvine, CA, 1989, edited by Roger McWilliams, AIP Conf. Proc. No. 190 (AIP, New York, 1989).