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Systems near a critical point under multiplicative noise and the concept of effective potential

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This paper presents a general approach to and elucidates the main features of the effective potential, friction, and diffusion exerted by systems near a critical point due to nonlinear influence of noise. The model is that of a general many-dimensional system of coupled nonlinear oscillators of finite damping under frequently alternating influences, multiplicative or additive, and arbitrary form of the power spectrum, provided the time scales of the system's drift due to noise are large compared to the scales of unperturbed relaxation behavior. The conventional statistical approach and the widespread deterministic effective potential concept use the assumptions about a small parameter which are particular cases of the considered. We show close correspondence between the asymptotic methods of these approaches and base the analysis on this. The results include an analytical treatment of the system's long-time behavior as a function of the noise covering all the range of its table- and bell-shaped spectra, from the monochromatic limit to white noise. The trend is considered both in the coordinate momentum and in the coordinate system's space. Particular attention is paid to the stabilization behavior forced by multiplicative noise. An intermittency, in a broad area of the control parameter space, is shown to be an intrinsic feature of these phenomena.

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I. INTRODUCTION

There is a great deal of interest in the behavior of systems near instabilities such as lasers, hydrodynamic convection, crystal growth, chemical reactions, etc., subjected to chaotic modulation of the control parameters. A nontrivial general phenomenon one observes is that of stabilization, i.e., the noise-induced shift of the instability threshold, while the transition remains often sharp. Such effects can influence or alter the trend of pattern formation qualitatively (see, e.g., Brandt [1] and Kai [2]). Essential advances in the theory are due to Suzuki [3] and Graham and Schenzle [4,5]. As an archetypal model, a modulated oscillator

$$m\ddot{x}+h\dot{x}=-\nabla V(x)+g(x)\xi(t) \tag{1}$$

usually with $g(x)=x$ and $\xi(t)$, a rapidly alternating function treated like a random function, and with a quartic potential $V(x)$, $\nabla V(x)=-ax+bx^3$, near the point $a=0$ serves as a generic representation of a variety of bistable nonlinear systems. It has been of historical and continuing interest in various aspects (see, e.g., Sancho and San Miguel [6], Lefever and Turner [7], Ebeling *et al.* [8], Fronzoni *et al.* [9], Mackey, Longtin, and Lasota [10], Horshemke and Kondepudi [11], Horshemke and Lefevre [12], Lindenberg and West [13], three volumes of reviews edited by Moss and McClintock [14], and a review by Dykman and Lindenberg [15]). Though the phenomena have been considered at length, principal questions still remain unclear. For example, one can find in the reviews by Horsthemke and Lefever [16] and by Lucke [17] formulas derived by different methods, yielding shifts of the instability point which differ even in sign for the linearized model (1).

The multiplicative noise action near a critical point resembles, at first sight, the effects associated with a pendulum under frequent harmonic vibration of its suspension that leads, in a certain range of parameters, to stabilization of the upper vertical state of the pendulum. Such systems, under harmonic modulation of the control parameter, have been investigated at length previously and an essential role has been played by the well-known concept of effective (or high-frequency) potential as described in Landau and Lifshitz [18], especially for complex multidimensional and continuous systems. According to this concept, models such as (1), as well as the coupled oscillators [many dimensional and continuous generalizations of (1)], under the action of fields frequently oscillating in t but smoothly varying in x, have on the average such a behavior as if they were acted upon by an effective potential which is equal to the kinetic energy of the frequent motions of the system generated by the oscillating fields. This concept covers the fields containing a number of high-frequency components as well. It penetrates widely into different physical realms and its advantage is that it allows us to recognize at once a trend, characteristic features and scales of the smoothed behavior and of the stabilization effect in particular, without consideration of details of explicit solutions.

But the multiplicative action of rapidly alternating influences that can be regarded as a high-frequency field can differ critically from that of a random function of short correlation time. The simplest one-dimensional model

$$\dot{x}=ax+\xi(t)x \tag{2}$$

under a given initial condition $x(0)=x_0$ demonstrates this. Averaging the solution of (2) for the case of modulation by Gaussian white noise with $\langle \xi(t) \rangle = 0$ and $\langle \xi(t_1)\xi(t_2) \rangle = 2D\delta(t_1 - t_2)$ one obtains

$$\langle x(t) \rangle = x_0 \left\langle \exp\left[\int_0^t [a + \xi(t')]dt'\right]\right\rangle = x_0 e^{(a+D)t} . \quad (3)$$

In contrast, for the case of modulation by the $\xi(t)$ regarded as a high-frequency field, i.e., containing a number of high-frequency components, the integral in the exponent in (3) is a function of the same high frequencies and, thus, for sufficiently large t, it gives no contribution to $\langle x(t) \rangle$ monotonic in t at all. This difference exists not only for a white-noise model of $\xi(t)$. For example, taking for $\xi(t)$ a Gaussian zero-mean random function of arbitrary covariance, one arrives at (3) provided D is replaced by $D_{\rm eff}(t)$, where

$$D_{\rm eff}(t) = \frac{1}{t}\int_0^t \int_0^t \langle \xi(t')\xi(t'') \rangle dt'dt'' .$$

In the limit $t \to \infty$

$$D_{\rm eff} \to \pi \bar{S}(0)$$

where

$$\bar{S}(0) = \lim_{t \to \infty} \frac{1}{t} \int_0^t S(t',0)dt' ,$$

$$S(t',\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \langle \xi(t')\xi(t'-t'') \rangle e^{i\omega t''} dt'' .$$

These formulas show that the cumulative effect of the random influences at large t's appears only due to the low-frequency, close to zero, area of the noise power spectrum. To deal with such effects is completely beyond the capability of the effective potential concept mentioned above and the effect of a harmonic modulation in particular. The same situation holds true for models such as (1) and other more common ones (see further) near a critical point.

Though the low-frequency spectrum of noise gives rise to a trend quite different from that of a harmonic influence, there exist general features inherent for this trend associated with a small parameter which allow the development of a general concept in the same spirit as the effective potential concept. The first steps towards such a generalization, using a rather simplified analysis and comparing it with exactly solvable models of the parametrically driven linear oscillators, have been undertaken by Shapiro and Loginov (e.g., [19]). At that time it was not recognized that application of the theory in its linearized form to the nonlinear case can lead to incorrect results, e.g., wrong estimation of the instability point of the oscillator model (1) with the quartic potential $V(x)$.

That the stationary regime and transient behavior near a critical point under noise influence differ essentially from the linearized processes was elucidated thoroughly on a particular model (1) by Graham and Schenzle [4,5] but without any reference and discussion of the effective potential concept. The same point applies to the other cited literature on the topic. But the analysis of the type undertaken in [4,5] exploits profound but complex specific mathematical tools and such relevant analytical results are known only for the one-dimensional model (1) with the particular form of $V(x), g(x)$ and only for the case of particular models of random $\xi(t)$. In such circumstances the development of a more generalized concept, such as the effective potential, seems of essential importance.

These points motivated the present work. Most of all we were triggered by a strange character of the stabilization caused by multiplicative noise: the increments of the infinitesimal perturbations do not change their signs at the critical point at all. This touches on a general aspect since it seems to contradict the method of linearization used conventionally to analyze the thresholds of instability. We dwell on it in the end of our treatment, in Sec. VI. As will be elucidated, the stabilization by multiplicative noise resembles an intermittency rather than what one normally thinks of as strict physical stabilization.

The method developed below refers to a broad class of many-dimensional systems near a critical point under chaotic influences, multiplicative and additive, both treated randomly, with finite correlation time and arbitrary power spectrum, and treated deterministically. In Secs. II and III we present the main assumptions and the basis of our approach to the problem. It involves both the conventional statistical as well as the deterministic approaches to the problem. The parameter we assume to be small is shown to include the ones of both the conventional statistical approach and the deterministic effective potential concept as particular cases; also we show the essential similarity between these approaches by means of presenting the procedure of asymptotic method of nonlinear (deterministic) mechanics in a form close to the perturbation theory based on the cumulant expansion. The generalization and extension of the effective potential concept, which is undertaken in Secs. IV–VI, represents, in fact, an analytical treatment of the trend of the critical behavior under nonlinear influence of noise as a function of its spectral characteristics covering all the range of bell-shaped spectra, from the monochromatic limit to white noise.

II. MODEL

We are going to consider the long-time (i.e., smoothed) behavior of the model (1) and its generalization of the form

$$m_{ik}\ddot{x}_k + h_{ik}\dot{x}_k = -\frac{\partial V(x)}{\partial x_i} + g_{ir}(x)\xi_r(t) \quad (4)$$

for a number of variables $x(t) = x_1(t), \ldots, x_n(t)$ near a critical point of a potential function $V(x)$ for the case of frequently alternating $\xi(t) = \xi_1(t), \ldots, \xi_s(t)$. Summation over repeated indexes is implied. The matrices m, h of inertia and damping are assumed to be positive-definite matrices; the overdamped ($m \to 0$) and underdamped ($h \to 0$) systems are also admitted. "Frequently" will mean that the influence of the forces $g(x)\xi(t)$ on the motion of $x(t)$ for a characteristic time scale T_c of the alternations in $\xi(t)$ is small compared with that for time scales of the smoothed motions. In the deterministic approach, with a deterministically given $\xi(t)$, the corresponding dimensionless small parameter is usually associated with the scale $|l^{-1}x_{\sim}|$, which is of the order of magnitude of

$$\epsilon=|l^{-1}(m+hT_c)^{-1}g\xi|T_c^2 \ . \qquad (5)$$

Here l characterizes scales in x associated with the shapes of $\nabla V(x)$ and $g(x)$; $|\ |$ denotes a norm; and $x_{\sim}$ is of the order of fluctuation x during time T_c. It is a similar condition with $\epsilon \ll 1$ (for the underdamped systems) that was assumed by Landau and Lifshitz [18] when deriving the conventional effective potential concept.

In the statistical approach, when the $\xi(t)$ is treated as a random finite-valued function of a short correlation time τ_c, almost all the realizations of $\xi(t)$ alternate near zero with frequencies $\gg \tau_c^{-1}$, so it is natural to associate τ_c with T_c and to take the scale (5) with T_c replaced by τ_c as a small parameter.

For the stochastic systems (4) which are underdamped and linear, with $\nabla V(x)$ and $g(x)$ linear in x, the dimensionless parameter was introduced (Brissaud and Frisch [20])

$$\epsilon_K=|m^{-1}q\xi|\tau_c^2 \qquad (6)$$

where $q=\nabla g(x)$ is a constant matrix. The scale (6) is called the Kubo number and it is frequently used as a small parameter in a more general case (e.g., [16,17]). The Kubo parameter for the nonlinear systems is introduced rather formally, via a differential operator function figuring in the continuity equation associated with the system under consideration.

What is the physical sense of (6) and its generalization for the nonlinear systems? How do we juxtapose (6) and (5)? What then about the scale l for a system linear in x? Our way of looking at these questions is as follows: The quantity (6) is entirely due to multiplicative non-δ-correlated noise, otherwise $q=0$. Such a noise in linear systems gives rise to a drift of $x(t)$, i.e., growth of the mean $\langle x(t)\rangle$, exponential in time. Thus there appears a time scale t_d associated with the increments of the exponential drift. To the order of magnitude

$$t_d^{-1}=|m^{-1}q\xi|\tau_c \ . \qquad (7)$$

Just the ratio τ_c/t_d is the scale given by (6).

In the same way we may interpret the parameter (5): For the overdamped systems the quantity $|l^{-1}(m+hT_c)^{-1}g\xi|T_c$ is a drift time scale, a modified t_d^{-1} as compared with (7). For the overdamped systems such a scale is $|l^{-1}(m+hT_c)^{-1}g\xi|^2T_c^3$. Thus we can compare the use of (5) with that of (6). Moreover, t_d may be easier to determine than l.

Graham and Schenzle introduced a different parameter and assumed it to be small in their analysis [5] of the model (1) with the quartic $V(x)$, $g(x)=x$, and $\xi(t)$ modeled by an Ornstein-Uhlenbeck process. In our notation the parameter reads

$$\epsilon_G=|h^{-1}q\xi|\sqrt{\tau_c/|m^{-1}h|} \ . \qquad (8)$$

For the overdamped limit they dealt with, the scale of t_d^{-1} is of the order of $|h^{-1}q\xi|^2\tau_c$. So ϵ_G is simply the ratio $\sqrt{t_h/t_d}$, where

$$t_h^{-1}=m^{-1}h$$

is the relaxation frequency of their oscillator model. Note that the ratio between (6) and (8) depends on the magnitude of τ_c/t_h:

$$\frac{\epsilon_G}{\epsilon_K}\sim\left[\frac{t_h}{\tau_c}\right]^{3/2} .$$

The scale τ_c appears naturally for the covariance $\langle\xi(t)\xi(0)\rangle$ exponential in t. Estimating t_d via τ_c we used this form of the covariance. In this case τ_c and T_c are of the same order. However, the conditions of smallness of parameters such as (6) and (8) become too restrictive for the case $T_c \gg \tau_c$, e.g., when $\langle\xi(t)\xi(0)\rangle=\langle\xi^2\rangle e^{-\nu|t|}\cos\Omega t$ with $\nu \ll \Omega$.

We shall consider the systems (4) with frequently alternating $\xi(t)$ of a large class, with the only restriction being that the drift time scale t_d is sufficiently large compared with T_c or t_h so that the values of (6) or (8) with τ_c replaced by T_c are much less than 1. This class includes random $\xi(t)$ with τ_c small and not small and even with $\tau_c \to \infty$, i.e., deterministically alternating $\xi(t)$, as well as a combination of deterministic and stochastic influences. We will not specify the functions $V(x)$ and $g(x)\xi(t)$ in the general part of our treatment. In particular, both additive and multiplicative frequently alternating influences are admitted.

III. CLOSE CORRESPONDENCE BETWEEN THE ASYMPTOTIC METHODS IN STATISTICAL AND DETERMINISTIC APPROACHES

Our goal, in the statistical approach, is to investigate the behavior of the probability density $P(x,p,t)$ to find the state $x(t),p(t)\equiv m\dot{x}(t)$ of the system (4) at a moment t near x,p provided an initial state of the system is given. We exploit the conventional device and proceed from the continuity equation

$$\left[\frac{\partial}{\partial t}+m^{-1}p\frac{\partial}{\partial x}-\frac{\partial}{\partial p}[hm^{-1}p+\nabla V(x)]\right]\rho(x,p,t) = -g(x)\xi(t)\frac{\partial}{\partial p}\rho(x,p,t) \qquad (9)$$

for the coarse-grained density distribution $\rho(x,p,t)$ in the space x,p under given initial condition, say a profile $\rho(x,p,0)$. In the interaction representation

$$\wp(x,p,t)=e^{Lt}\rho(x,p,t)$$

where

$$L=m^{-1}p\frac{\partial}{\partial x}-\frac{\partial}{\partial p}[hm^{-1}p+\nabla V(x)] \ ,$$

Eq. (9) is reduced to the form

$$\frac{\partial\wp}{\partial t}=-e^{Lt}g(x)\xi(t)\frac{\partial}{\partial p}e^{-Lt}\rho_i(x,p,t) \equiv \epsilon R(x,p,t)\wp(x,p,t) \ , \qquad (10)$$

which is standard for a systematic expansion, since the operator on the right frequently alternates near zero value; the corresponding small parameter ϵ appears when dealing with scaled variables. A standard and effective device to solve (to average) such type of stochastic equations is the well-known cumulant expansion method (e.g., Van Kampen [21]), the cumulant expansion of the ordered operator exponent

$$\left\langle \exp_t \left[\epsilon \int_0^t R(x,p,t')dt' \right] \right\rangle$$

appearing in the formal solution of (10). This is an asymptotic expansion in powers of ϵ the first terms of which give the main contribution for $t \gg \tau_c$, provided $\epsilon \ll 1$. In the main approximation, neglecting terms of the order $O(\epsilon^3)$, one arrives at the following kinetic equation:

$$\frac{\partial}{\partial t}\langle \wp(x,p,t)\rangle = \epsilon^2 \int_0^t \langle R(x,p,t)R(x,p,t')\rangle \times \langle \wp(x,p,t')\rangle dt' \ . \qquad (11)$$

With the $\xi(t)$ modeled by a Gaussian white noise [and with the time derivatives in (4), (9), and (10) treated in the Stratonovich sense) the result (11) is exact. It is also exact for the scalar $\xi(t)$ modeled by dichotomic Markov process of arbitrary correlation scale τ_c. Note that for arbitrary $\xi(t)$, provided $\epsilon \ll 1$, Eq. (11) describes correctly the evolution of the probability density of the system (4) not only at sufficiently large t's, but in the limit $t \to 0$ as well.

In the original representation, for the probability density $P(x,p,t) = \langle \rho(x,p,t)\rangle$, Eq. (9) reads

$$\left[\frac{\partial}{\partial t} + m^{-1}p\frac{\partial}{\partial x} - \frac{\partial}{\partial p}[hm^{-1}p + \nabla V(x)] \right] P(x,p,t) = g(x)\frac{\partial}{\partial p}\int_0^t C(t,t')e^{-L(t-t')}g^{+}(x)\frac{\partial}{\partial p}P(x,p,t')dt' \qquad (12)$$

where $[g^{+}(x)]_{ir} = [g(x)]_{ri}$ and $C(t,t')$ is the covariance matrix, with the components

$$C_{rq}(t,t') = \langle \xi_r(t)\xi_q(t')\rangle \ .$$

For $t \gg \tau_c$ within the same asymptotic approximation the argument t' in the $\langle \wp(x,p,t')\rangle$ in the integrand in (11) can be replaced by t and the upper limit of the integration taken, ∞. This yields

$$\frac{\partial}{\partial t}\langle \wp(x,p,t)\rangle = \epsilon^2 \int_0^\infty \langle R(x,p,t)R(x,p,t-t')\rangle dt' \times \langle \wp(x,p,t)\rangle \ . \qquad (13)$$

Correspondingly, in the original representation

$$\left[\frac{\partial}{\partial t} + m^{-1}p\frac{\partial}{\partial x} - \frac{\partial}{\partial p}[hm^{-1}p + \nabla V(x)] \right] P(x,p,t) = I(x,p,t)P(x,p,t) \qquad (14)$$

where

$$I(x,p,t) = g(x)\frac{\partial}{\partial p}\int_0^\infty C(t,t-\tau)e^{-\tau L}g^{+}(x)\frac{\partial}{\partial p}e^{\tau L}d\tau \ .$$

Taking for the $\xi(t)$ a δ-correlated covariance $C(t,t') = 2D\delta(t-t')$ results in Eqs. (12) and (14) reducing to the standard Fokker-Planck form, with the diffusion operator

$$I(x,p) = g(x)\frac{\partial}{\partial p}Dg^{+}(x)\frac{\partial}{\partial p} \equiv D_{rq}g_{ir}(x)g_{jq}(x)\frac{\partial^2}{\partial p_i \partial p_j} \ . \qquad (15)$$

The kinetic equation (14) represents a relevant approximation for the investigation of the systems (4) near a critical point under chaotic influences, both additive and multiplicative, when $\xi(t)$ can be modeled by a random function of sufficiently short correlation time decay τ_c. However, the existence of the scale τ_c imposes a strong restriction: All the correlations (the covariance and the higher cumulants) between $\xi(t_1)$ and $\xi(t_2)$ must be negligibly small for arbitrary t_1, t_2 provided $|t_1 - t_2|$ is large compared with τ_c. This restriction on all the cumulants is essentially exploited in the perturbation theory method of the cumulant expansion. Besides, the validity of such a restriction cannot be ascertained in practice. Another essential point is that the restriction does not allow the incorporation of a frequently alternating $\xi(t)$ with larger τ_c. Different perturbation theory methods have been developed (e.g., Hanggi [22], Jung and Hanggi [23]) to cover all the range of τ_c, but they are rather specific, for particular statistics and spectra of the noise and for particular models of the dynamical systems.

All this makes it of importance to turn to the deterministic rather than the statistical approach. Let us consider a deterministic system (4), with a nonrandom function $\xi(t)$ with

$$\langle \xi(t)\rangle_T \equiv \lim_{T\to\infty} \frac{1}{T}\int_0^T \xi(t')dt' = 0 \ , \qquad (16)$$

which frequently alternates in t near zero, so that the deterministic asymptotic method, i.e., the time smoothing method of nonlinear mechanics (e.g., Bogoliubov and Mitropolsky [24]), is applicable. The class of functions $\xi(t)$ for which the effectiveness of this method can be rigorously justified [according to Krylov and Bogoliubov theorems (see [24])] includes quasiperiodical type functions, i.e., having the spectral representation

$$\xi(t) = \mathrm{Re}\left[\sum_k e^{i\omega_k t}\xi_k \right] \qquad (17)$$

with arbitrary real ω_k the number of which is not limited. With this the part of the sum (17) at relatively high frequencies, of a scale T_c^{-1}, is assumed to give the main contribution into $\xi(t)$ so that the parameters (5) and (6) with τ_c replaced by T_c are much less than 1.

The form of the continuity equation (10) is standard for the application of the deterministic asymptotic method and in the main approximation second order in ϵ one gets

$$\frac{\partial}{\partial t}\langle \rho_i(x,p,t)\rangle_T = \epsilon^2 \langle R(t)\tilde{R}(t)\rangle_T \langle \rho_i(x,p,t)\rangle_T \ . \qquad (18)$$

The brackets $\langle\ \rangle_T$ denote the operation of time smoothing [as in (16)] and the tilde denotes the operator which when acting on a function $Z(t)$ represented by the sum

$$Z(t)=\sum_k e^{i\omega_k t}Z_k$$

results in

$$\tilde{Z}(t)=\sum_{k\neq 0}\frac{e^{i\omega_k t}}{i\omega_k}Z_k \ .$$

Let us juxtapose Eqs. (18) and (13). We present the tilde operator in the form

$$\tilde{Z}(t)=\lim_{\mu\to 0+}\int_0^\infty e^{-\mu\tau}[Z(t-\tau)-\langle Z(t-\tau)\rangle_T]d\tau \ , \quad (19)$$

which is readily verified. Multiplying both parts of (19) by $Z(t)$ and averaging we arrive at the identity

$$\langle Z(t)\tilde{Z}(t)\rangle_T = \lim_{\mu\to 0+}\int_0^\infty e^{-\mu\tau}[\langle Z(t)Z(t-\tau)\rangle_T - \langle Z(t)\rangle_T\langle Z(t-\tau)\rangle_T]d\tau \ . \quad (20)$$

The expression in the square brackets looks like a "temporal covariance." This quantity indeed becomes the covariance function of a random process, provided $Z(t)$ is its (representative) realization and the random process is ergodic, i.e., its statistical and temporal averages coincide.

So, it follows from (18) in view of (20) and since $\langle R(x,p,t)\rangle_T=0$

$$\frac{\partial}{\partial t}\langle \wp(x,p,t)\rangle_T = \epsilon^2 \lim_{\mu\to 0+}\int_0^\infty e^{-\mu\tau}\langle R(x,p,t)R(x,p,t-\tau)\rangle_T d\tau \times \langle \wp(x,p,t)\rangle_T \ . \quad (21)$$

Comparing (21) and (13) we see that they practically coincide, up to the accuracy of replacing the time smoothing by the averaging over statistics and the time scale τ_c by T_c. Thus, for a wide class of systems (4) under typical assumptions about the character of the frequently alternating influences one can arrive at the results of the purely statistical method of averaging.

This result is essential in view of the restriction of the statistical asymptotic method mentioned, requiring existence and smallness of the scale τ_c, and in view of the probabilistic character of all statements of a statistical approach. We wish to emphasize that the restrictions on applicability of the perturbation theory in both approaches are the same up to the accuracy of replacement of τ_c by T_c, since the reduced equations (and, as a rule, initial and boundary conditions) are the same.

So, provided the parameters (5) and (6) with τ_c replaced by T_c are small compared with 1, we may rely on the results which follow from the kinetic equations (12) or (14) beyond the assumption of existence (finiteness) and smallness of τ_c and take these equations as the basis of a general problem statement, covering both the deterministic and the statistical approaches.

IV. EFFECTIVE POTENTIAL, FRICTION, AND DIFFUSION IN x,p SPACE AS A FUNCTION OF THE NOISE SPECTRUM

Let us consider general features of the noise action near a critical point within the model considered, i.e., as it follows from the kinetic equation (14). The total effect of noise is represented by the right-hand side of (14) and the only characteristic of noise entering it is the covariance $C(t,t-\tau)$ in the operator $I(x,p,t)$. The dependence on the $C(t,t-\tau)$ is smoothed by the integration, so only a low-frequency part of the power spectrum of the noise determines the evolution of $P(x,p,t)$. To investigate the general features we exploit again the assumption that $\xi(t)$ alternates frequently near zero. This results in the shift operators $\exp[\pm\tau L]$ producing in time $\tau=T_c$ a relatively small influence on the probability function, while the covariance $C(t,t-\tau)$ as a function of τ varies considerably in this time scale. In the limit of a δ-correlated $\xi(t)$ the shift effect does not manifest itself at all and the trend of changes, when going from this limit, can be elucidated by taking a few terms in the expansion

$$I(x,p,t)=I_0+I_1+I_2+\cdots = \sum_{n=0}^{\infty} g(x)\frac{\partial}{\partial p}C^{(n)}(t)L^{(n)}(x,p) \quad (22)$$

where

$$L^{(0)}=g^+(x)\frac{\partial}{\partial p} \ , \quad L^{(n+1)}=[L^{(n)},L]$$

(the brackets [,] denote the commutator $[A,B]=AB-BA$) and

$$C^{(n)}=\frac{1}{n!}\int_0^\infty \tau^n C(t,t-\tau)d\tau$$

are matrix coefficients. To the order of magnitude $|C^{(n+1)}/C^{(n)}|\sim O(T_c)$. The expansion arises by integrating the integral on the right-hand side of (12) by parts repeatedly, provided $t\gg T_c$ and neglecting the terms of the order $<O(\epsilon^3)$. One arrives at the same expansion by using the repeated integration by parts of Eq. (14).

Holding only the term I_0 results in a pure diffusion approximation, of the form of (15) with $D=D_{\text{eff}}$, where

$$D_{\text{eff}}=C^{(0)}=\pi S(0) \quad (23)$$

and $S(\omega)$ is the power spectrum matrix of the noise [for simplicity we consider the case of stationary processes, i.e., when $C(t,t')=C(|t-t'|)$]

$$S(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} C(\tau)e^{-i\omega\tau}d\tau \ .$$

In particular, for the noise spectra with $S(0)=0$ the term I_0 disappears.

The next term in (22), I_1, has a structure that gives a contribution to diffusion [now, unlike (15), both in p and in x space] and contains, unlike I_0, a term contributing to drift. The drift part arises due to the summand $m^{-1}p\partial_x$ in the L in I_1. It acts purely like that of a potential force:

$$(I_1)_{\text{drift}}=\nabla V_{\text{eff}}^{(1)}(x)\frac{\partial}{\partial p}$$

provided $g(x)$ is vortexless, i.e., $\nabla\times g(x)=0$. This is easy to verify by noting that $C^{(1)}$ and m^{-1} are symmetric matrices. The effective potential has the form

$$V_{\rm eff}^{(1)}(x)=-\tfrac{1}{2}C_{rq}^{(1)}m_{ik}^{-1}g_{ir}(x)g_{kq}(x)\ . \tag{24}$$

Remarkably, this function differs from the effective potential caused by a deterministic frequently alternating influence, with $\xi(t)$ of a quasiperiodic type (16) and (17), only by the replacement of the $C^{(1)}$ in (24) by the $C^{(\rm det)}$,

$$C_{rq}^{(\rm det)}=-\sum_s \frac{\xi_{rs}\xi_{qs}^*}{\omega_s^2} \tag{25}$$

where ξ_{rs} denotes the amplitude of the harmonic of $\xi_r(t)$ of frequency ω_s. $C^{(1)}$ exactly reduces to this $C^{(\rm det)}$ for the case of such influences. Indeed, the deterministic limit (remember the discussion of Sec. III) is equivalent to the replacement of the covariance $C(t,t-\tau)$ by the limit of its time-average analogue, i.e.,

$$C_{rq}(\tau)\Longrightarrow \lim_{\mu\to 0+}\langle \xi_r(t)\xi_q(t-\tau)\rangle_T e^{-\mu|\tau|}\ . \tag{26}$$

So, in this limit

$$\begin{aligned} C_{rq}^{(1)}=\int_0^\infty \tau C_{rq}(\tau)d\tau &= \lim_{\mu\to 0+}\int_0^\infty \tau e^{i\omega_s\tau-\mu\tau}d\tau\sum_s \xi_{rs}\xi_{qs}^* \\ &=-\sum_s \frac{\xi_{rs}\xi_{qs}^*}{\omega_s^2} \\ &=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{S_{rq}(\omega)}{\omega^2}d\omega\ . \end{aligned}$$

Here $S(\omega)$ is the Fourier transform of the limit covariance matrix (26).

Thus the first two terms of the expansion (22) produce the main effects associated with the two opposite limits, the diffusional and the deterministic.

The next approximation, given by I_2, has a novel feature, as compared to I_0 and I_1: it also gives a contribution to drift, but this contribution, for a vortexless $g(x)$, contains both a potential and a friction force. The effective friction matrix has the structure

$$(h_{\rm eff})_{ij}=C_{rq}^{(2)}m_{kn}^{-1}\frac{\partial g_{kr}(x)}{\partial x_i}\frac{\partial g_{nq}(x)}{\partial x_j}\ . \tag{27}$$

The contribution to the effective potential becomes considerable for the overdamped systems, provided τ_c is finite; it reads

$$V_{\rm eff}^{(2)}(x)=C_{rq}^{(2)}h_{ik}m_{kn}^{-1}m_{ij}^{-1}g_{kr}(x)g_{jq}(x)\ . \tag{28}$$

It is easy to prove that the matrix formed by the right-hand side of (27) with $C_{rq}^{(2)}\equiv 1$ is non-negative definite for arbitrary $g(x)$. It follows that the effective friction, i.e., any eigenvalue of the matrix $h_{\rm eff}$, is non-negative or non-positive, provided the matrix $C^{(2)}$ is non-negative or non-positive definite. Analogously, the sums on the right-hand side of (28) and (24) with $C_{rq}^{(2)}\equiv 1$ and $C_{rq}^{(1)}\equiv -1$ are non-negative. It follows, e.g., for a multiplicative noise, when $g(x)$ is a linear or a nonlinear function in x with $g(0)=0$, that the effective potentials $V_{\rm eff}^{(1,2)}(x)$ at $x=0$ form humps when the matrices $C^{(2)}, -C^{(1)}$ are non-negative definite and form pits when they are nonpositive definite. We wish to emphasize that while $C^{(0)}$ is a non-negative definite matrix, the matrices $C^{(1)}$ and $C^{(2)}$ are not: the signs of their eigenvalues depend on the form of the noise spectrum $S(\omega)$ at low frequencies. Thus the behavior of the effective potential and the friction depend critically on the low-frequency noise spectrum. Note that $C^{(0)}$ and $C^{(2)}$ and all $C^{(n)}$'s with even n are directly related to this spectrum by

$$C^{(2k)}=\frac{(-1)^k\pi}{(2k)!}\left[\frac{d^{2k}S(\omega)}{d\omega^{2k}}\right]_{\omega=0}\ .$$

Let us observe the effect of the influences that have the covariance function $C(\tau)$ of the form

$$C(\tau)=Qe^{-\nu|\tau|}\cos(\Omega\tau)\ . \tag{29}$$

Then the power spectrum $S(\omega)$ is of the Lorentzian form

$$S(\omega)=\frac{Q}{2\pi}\left[\frac{\nu}{\nu^2+(\omega+\Omega)^2}+\frac{\nu}{\nu^2+(\omega-\Omega)^2}\right]\ .$$

Varying the parameters Q, ν, and Ω allows us to consider a continuous change from white noise to the monochromatic limit. Obviously, Q is a constant non-negative definite matrix [since $Q_{rq}=\langle\xi_r(t)\xi_q(t)\rangle$], τ_c is of the order of ν^{-1}, and T_c is of the order of $(\nu^2+\Omega^2)^{-1/2}$.

The parameters $C^{(n)}$ are given by

$$C^{(n)}=Q\ \mathrm{Re}\left[\frac{1}{(\nu+i\Omega)^{n+1}}\right]\ . \tag{30}$$

Correspondingly the amplitudes of the effective diffusion, potential, and friction take the form

$$C^{(0)}=Q\frac{\nu}{\nu^2+\Omega^2}\ , \tag{31}$$

$$C^{(1)}=Q\frac{\nu^2-\Omega^2}{(\nu^2+\Omega^2)^2}\ , \tag{32}$$

$$C^{(2)}=Q\frac{\nu(\nu^2-3\Omega^2)}{(\nu^2+\Omega^2)^3}\ . \tag{33}$$

These quantities, scaled to their maximum values, are plotted as functions of ν/Ω in Fig. 1. The amplitude of the diffusion is scaled to its magnitude at the limit $\Omega=0$ and of the effective potential at $\nu=0$.

In the limit $\nu\to 0$ all the $C^{(2k)}\to 0$, so the diffusion I_0 and the friction $h_{\rm eff}$ go to zero. While at $\nu\to 0$ the amplitude of the effective potential, given by (31), takes its maximum value and $V_{\rm eff}^{(1)}(x)$ reduces exactly to the result of the effective potential concept [18] for the case of a deterministic $\xi(t)$ alternating harmonically at frequency Ω. With the growth of ν/Ω the $C^{(1)}$ and $C^{(2)}$ change their signs, so one arrives at the trend of $V_{\rm eff}(x)$ opposite the conventional effective potential concept: the humps become pits and vice versa. Parallel with the change of $V_{\rm eff}(x)$, both $D_{\rm eff}$ and $h_{\rm eff}$ become significant, so in general the effective potential, the friction, and the diffusion have to be taken into account together. The limit of white noise corresponds to $\nu\to\infty$ and $|Q|\to\infty$, provided the norm of $C^{(0)}$ remains finite and nonzero. In this limit

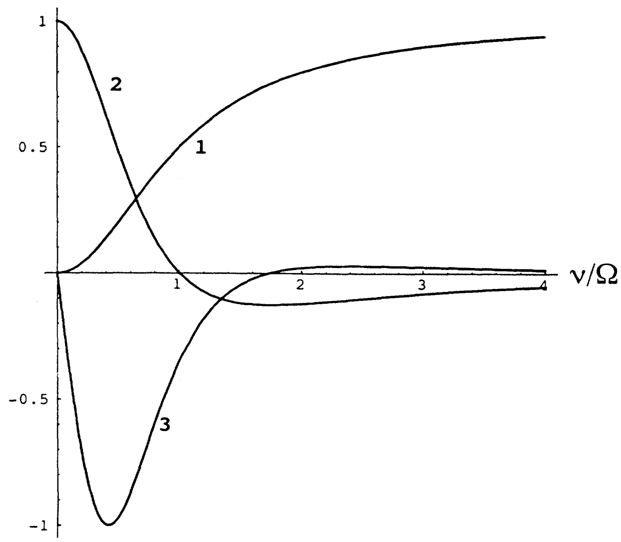

FIG. 1. Behavior of the effective potential (1), diffusion (2), and friction (3) caused by noise vs its power spectrum characteristics.

all the other $C^{(k)} \to 0$ and thus the effective potential and the friction disappear.

Concerning the case of $\xi(t)$ of a more general form of the power spectrum, note that physically it is natural to consider frequently fluctuating influences as a number of random sources that are statistically independent, each with the power spectrum of the Lorentzian form, with particular parameters ν, Ω, and $Q = Q(\nu,\Omega)$. With such a conventional modeling one arrives at the formulas cited, provided the expressions on the right-hand side of (31)–(33) are replaced by the corresponding sums of such expressions with different values of ν, Ω, and $Q(\nu,\Omega)$. So, the observed frequency dependences D_{eff}, V_{eff}, and h_{eff} allow us to understand the general character of the effective diffusion, potential, and friction as functions of the noise spectrum.

Our splitting of the noise action into the three kinds of action, as any classification scheme, is effective only to some extent, i.e., when the joined action roughly resembles a superposition of each action estimated separately rather than a strongly nonlinear mixture. The scheme works, e.g., when considering a period of the linear regime of transient behavior, in particular increments of a linearized system (4) near a critical point. Then the estimation of the frequency dependences and the scales of D_{eff}, V_{eff}, and h_{eff} by means of the formulas cited above represents a simple device for the analysis and understanding of the general trend of the noise effect.

However, when analyzing the steady-state regime, the shift of the critical point and the transient behavior at large times, the interplay of the three effects is particularly strong and nontrivial. Also validity of the termination of the expansion (22) may become questionable. Dealing with the probability density $P(x,t)$ rather than with the $P(x,p,t)$, i.e., at the expense of further contraction of the description, allows us to simplify the analysis and to develop further the concept of the effective potential. We shall address this now.

V. EFFECTIVE POTENTIAL AND DIFFUSION IN TERMS OF x SPACE

We shall restrict ourselves here to the systems (4) with finite damping, so that the time scale t_0 of the drift caused by the influences $\xi(t)$ is large compared with t_h, where t_h^{-1} is of the order of the eigenvalues of the matrix $m^{-1}h$, and as a small parameter ϵ we take here

$$\epsilon = \left[\frac{t_h}{t_0} \right]^{1/2} \ll 1 \; . \tag{34}$$

Let us analyze the behavior of the probability density $P(x,t)$,

$$P(x,t) = \int P(x,p,t)\,dp$$

where the integration is over all the p subspace of the system (4), in the same fashion as in Graham and Schenzle [5]—by extension of the well-known adiabatic expansion method developed by Wilemski [25]. The approach [5] was developed for the stochastic oscillator equation (1) with $\xi(t)$ modeled by an Ornstein-Uhlenbeck process. Here we present a more general treatment, for many-dimensional models (4) with arbitrary frequently alternating $\xi(t)$, provided the parameter $\epsilon \ll 1$. For the particular model of [5] the scale ϵ reduces to the ϵ_G, Eq. (8). For concreteness and to juxtapose with the results of [5], let us first develop the method for the same model (1), with $g(x) = x$ and $\nabla V(x) = -ax + bx^3$, but assuming $\xi(t)$ to be an arbitrary random zero mean function with the covariance of the form (29). Then by order of magnitude

$$t_0^{-1} = \frac{2Q}{h^2 \nu_0} \tag{35}$$

and

$$\epsilon^2 = \frac{2Qm}{h^3 \nu_0} \tag{36}$$

where $\nu_0 = \nu + \Omega^2/\nu$.

Introducing the scaled variables

$$\bar{t} = \frac{1}{t_0} t \; , \quad \bar{x} = \left[\frac{b t_0}{h} \right]^{1/2} x \; , \tag{37}$$

the model is then given by the scaled equations

$$\frac{d\bar{x}}{d\bar{t}} = \frac{1}{\epsilon} z \; , \quad \frac{dz}{d\bar{t}} = \frac{1}{\epsilon} k(\bar{x}) - \frac{1}{\epsilon^2} z + \frac{t_0}{\epsilon h} g(\bar{x}) \xi(t_0 \bar{t})$$

with

$$k(\bar{x}) = a_0 \bar{x} - \bar{x}^3 \; , \quad g(\bar{x}) = \bar{x} \; .$$

The parameter $a_0 = a t_0 / h$ may be arbitrary, $a_0 \sim O(1)$. The kinetic equation (14) in such terms may be written in the form

$$\left[\frac{\partial}{\partial \bar{t}} + \frac{1}{\epsilon^2}(L_0 + \epsilon L_1) \right] P(\bar{x}, z, \bar{t}) = \bar{I}(\bar{x}, z) P(\bar{x}, z, \bar{t}) \tag{38}$$

where

$$L_0=-\frac{\partial}{\partial z}z\ ,\quad L_1=z\frac{\partial}{\partial \bar{x}}+k(\bar{x})\frac{\partial}{\partial z}$$

and

$$\bar{I}(\bar{x},z)=\frac{\bar{\nu}_0}{2\epsilon^2}g(\bar{x})\frac{\partial}{\partial z}\int_0^\infty e^{-\bar{\nu}s}\cos(\bar{\Omega}s)e^{-(L_0+\epsilon L_1)s}\times g(\bar{x})\frac{\partial}{\partial z}e^{(L_0+\epsilon L_1)s}ds\ .$$

Here

$$\bar{\nu}=\nu t_h\ ,\quad \bar{\Omega}=\Omega t_h\ ,\quad \bar{\nu}_0=\nu_0 t_h\ .$$

These parameters may be assumed arbitrary, of the order $O(1)$, provided the condition (34).

It follows from the Eq. (38) for the probability density $P(\bar{x},\bar{t})$

$$\frac{\partial P(\bar{x},\bar{t})}{\partial \bar{t}}=-\frac{1}{\epsilon}\frac{\partial}{\partial \bar{x}}P_1(\bar{x},\bar{t})\ , \tag{39}$$

where the moments $P_n(\bar{x},\bar{t})$ are defined as

$$P_n(\bar{x},\bar{t})=\int z^n P(\bar{x},z,\bar{t})dz\ ,$$

the integration is over all z space. All these moments are related by (38) and one can derive a closed equation for $P(\bar{x},\bar{t})$ in the form of a perturbation expansion in the parameter ϵ. Now the expansion (22) reads

$$\bar{I}(\bar{x},z)=\frac{\bar{\nu}_0}{2\epsilon^2}g(\bar{x})\frac{\partial}{\partial z}\sum_{n=0}^{\infty}\mathrm{Re}\left[\frac{1}{(\bar{\nu}+i\bar{\Omega})^{n+1}}\right]l^{(n)}(\bar{x},z)$$

with

$$l^{(0)}=g(\bar{x})\frac{\partial}{\partial z}\ ,\quad l^{(n+1)}=[l^{(n)},L_0+\epsilon L_1]\ .$$

Noting that

$$l^{(1)}=-l^{(0)}+\epsilon[l^{(0)},L_1]\ ,\quad [[l^{(0)},L_1],L_0]=0\ ,$$

one finds the following ϵ expansion of $\bar{I}(\bar{x},z)$:

$$\bar{I}(\bar{x},z)=\frac{\bar{\nu}_0}{2\epsilon^2}l^{(0)}[\Delta_1 l^{(0)}+\epsilon(\Delta_0-\Delta_1)[l^{(0)},L_1]+\epsilon^2\cdots]$$

where

$$\Delta_n=\frac{\bar{\nu}+n}{(\bar{\nu}+n)^2+\bar{\Omega}^2}\ .$$

As a result, we arrive at the set of equations for $P_n(\bar{x},\bar{t})$ of the form

$$\epsilon^2\frac{\partial}{\partial \bar{t}}P_n+nP_n+\epsilon\left[\frac{\partial}{\partial \bar{x}}P_{n+1}-nk(\bar{x})P_{n-1}\right]=\frac{n(n-1)}{2}\bar{\nu}_0\Delta_1 g^2(\bar{x})P_{n-2}+\frac{\epsilon n}{2}\bar{\nu}_0(\Delta_1-\Delta_0)\left[g(\bar{x})\frac{\partial}{\partial \bar{x}}g(\bar{x})P_{n-1}+(n-1)g(\bar{x})g'(\bar{x})P_{n-1}\right]+O(\epsilon^2)\ . \tag{40}$$

Here $g'(\bar{x})=dg(\bar{x})/d\bar{x}$.

In the limit $\epsilon\rightarrow 0$ the set (40) describes a rapidly damped time evolution of the moments P_n for $n\neq 0$ on a time scale n/ϵ^2 as indicated by the diagonal term on the left-hand side of (40). Provided a finite damping and validity of the condition (34), this is easy to understand physically, taking into consideration that the systems in the vicinity of a critical point behave as if they are overdamped. Taking the steady state as a basic approximation, this finally produces a series for the moment $P_1(\bar{x},\bar{t})$ of the structure

$$P_1(\bar{x},\bar{t})=\sum_{l=1}^{\infty}\epsilon^l\mathcal{L}^{(l)}\left[\bar{x},\frac{\partial}{\partial \bar{x}}\right]P(\bar{x},\bar{t})\ .$$

Restricting this to the first order in ϵ, the equation for $P_1(\bar{x},\bar{t})$ takes the form

$$P_1=\epsilon\left[k(\bar{x})P-\frac{\partial}{\partial \bar{x}}P_2+\frac{\bar{\nu}_0}{2}(\Delta_1-\Delta_0)g(\bar{x})\frac{\partial}{\partial \bar{x}}g(\bar{x})P\right]\ . \tag{41}$$

Here one needs to evaluate the P_2 from (40) in zeroth order in ϵ, which yields

$$P_2=\frac{\bar{\nu}_0}{2}\Delta_1 g^2(\bar{x})P(\bar{x},\bar{t})\ .$$

We finally obtain

$$\frac{\partial}{\partial \bar{t}}P(\bar{x},\bar{t})=-\frac{\partial}{\partial \bar{x}}k(\bar{x})P+\frac{1}{2}\frac{\Delta_1-\Delta_0}{\Delta_0}\frac{\partial}{\partial \bar{x}}g(\bar{x})g'(\bar{x})P+\frac{1}{2}\frac{\partial^2}{\partial \bar{x}^2}g^2(\bar{x})P\ . \tag{42}$$

For the particular case of the $\xi(t)$ modeled by an Ornstein-Uhlenbeck process the covariance (29) is exponential, $\Omega=0$, and Eq. (42) reduces exactly to the result in [5].

After returning to unscaled variables via Eq. (37) we arrive at the following Fokker-Planck equation for the desired probability density $P(x,t)$:

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\frac{\nabla V(x)+\nabla V_{\mathrm{eff}}(x)}{h}\right]P(x,t)=\frac{\partial^2}{\partial x^2}\mathcal{D}_{\mathrm{eff}}(x)P(x,t) \tag{43}$$

where

$$\mathcal{D}_{\text{eff}}(x)=\frac{Q}{h^2}\frac{\nu}{\Omega^2+\nu^2}g^2(x) \tag{44}$$

and

$$V_{\text{eff}}(x)=\frac{Q}{2m}\frac{\Omega^2-\nu(\nu+\gamma)}{(\Omega^2+\nu^2)[\Omega^2+(\nu+\gamma)^2]}g^2(x) \tag{45}$$

with $\gamma=m^{-1}h$.

For the case under consideration the diffusion term $\mathcal{D}_{\text{eff}}(x)$ is nothing but the diffusion term in the kinetic equation in x,p space presented in Sec. IV scaled by h^2. The potential $V_{\text{eff}}(x)$ of the "spurious drift" force, in fact, also coincides with the effective potential $V_{\text{eff}}^{(1)}(x)$ given by Eqs. (24) and (32) for the case considered: both have the same dependence in x and both practically coincide at relatively large as well as at small values of Ω. The difference decreases still more when taking into account the correction to the $V_{\text{eff}}^{(1)}(x)$ given by the $V_{\text{eff}}^{(2)}(x)$, Eqs. (28) and (33) for the case.

One may suggest a similar situation for the random influences of an arbitrary power spectrum and for $V(x),g(x)$ of more general form, provided the assumption (34) holds. And, indeed, modeling $\xi(t)$ by a sum of zero mean functions that fluctuate statistically independently and each with a covariance of the form (29), one can easily find appropriate scales for $\bar{x},\bar{t}$ in (37) and repeat all the reasoning and calculations. As a result, we then arrive at an equation that differs from (43) in that in (44) and (45) the parameter Q is replaced by a function $Q(\nu,\Omega)$ and a summation over ν,Ω appears. Furthermore, the derivation can be readily extended to many-dimensional systems (4), so that one arrives at a many-dimensional analogue of the corresponding Fokker-Planck equation (43).

The diffusion equation under consideration is the same as describing the system governed by the stochastic Langevin equation

$$h\frac{dx}{dt}=-\nabla V(x)-\nabla U_{\text{eff}}(x)+g(x)\eta(t) \tag{46}$$

where $\eta(t)$ is Gaussian white noise, with

$$\langle\eta\rangle=0\ ,\quad \langle\eta(t)\eta(0)\rangle=Q(\nu,\Omega)\frac{\nu}{\Omega^2+\nu^2}\delta(t)\ .$$

The summation over ν,Ω is implied and the time derivative is treated in the Stratonovich sense. The effective potential $U_{\text{eff}}(x)$ differs from $V_{\text{eff}}(x)$ and reads (for simplicity we cite the formulas for one-dimensional case)

$$\begin{aligned}U_{\text{eff}}(x)&=V_{\text{eff}}(x)+\tfrac{1}{2}\mathcal{D}_{\text{eff}}(x)\\&=\frac{Q(\nu,\Omega)}{2m}\frac{\nu+\gamma}{\gamma[\Omega^2+(\nu+\gamma)^2]}g^2(x)\ .\end{aligned} \tag{47}$$

Thus the problem for random influences of arbitrary power spectrum is reduced to a much more simple and standard problem.

For the particular choice $\nabla V(x)=-ax+bx^3, g(x)=x$ one arrives at the well-studied Landau equation with a white-noise-dependent control parameter. Then it immediately follows from (46) and (47) that the original critical point $a=a_c^0=0$ is shifted by the value

$$\delta a_c=\Delta U_{\text{eff}}(x)|_{x=0}=\frac{Q(\nu,\Omega)}{m}\frac{\nu+\gamma}{\gamma[\Omega^2+(\nu+\gamma)^2]}\ . \tag{48}$$

At $a<a_c^0+\delta a_c$ the stationary distribution reads $P_{\text{st}}(x)=\delta(x)$. This is the only steady-state solution of (43) at $a<a_c^0+\delta a_c$, so it is stable since the system is ergodic. At $a>a_c+\delta a_c$ this steady state is unstable while the stable state takes the form

$$P_{\text{st}}=Nx^{(a-a_c^0-\delta a_c)2ht_0}e^{-ht_0bx^2}$$

where N is the constant of normalization.

Thus the potential $U_{\text{eff}}(x)$, and not $V_{\text{eff}}(x)$, determines the steady state and the shift of the critical point. The same formula (48) holds in a more general case, when $x=0$ is an unperturbed critical point [where $\nabla V(x)=0$] and the influences are multiplicative.

Strikingly, the potential $U_{\text{eff}}(x)$ has quite different frequency dependences than $V_{\text{eff}}(x)$. It does not change the sign and decays monotonically with the growth of Ω. As evident from the Eq. (47), the difference $U_{\text{eff}}(x)-V_{\text{eff}}(x)$ that causes such a trend originates from the inhomogeneous diffusion [in terms of the Eq. (43)]. So, similar to the diffusion, the difference is entirely due to the power spectrum of the noise $\xi(t)$ at zero frequency. Measuring the noise effect by its diffusion coefficients, Eq. (47) takes the form

$$U_{\text{eff}}(x)=\frac{\nu+\gamma}{\nu\gamma}\frac{\nu^2+\Omega^2}{\Omega^2+(\nu+\gamma)^2}\mathcal{D}_{\text{eff}}(x,\nu,\Omega)\ . \tag{49}$$

The frequency-dependent factors standing before the $\mathcal{D}_{\text{eff}}$ on the right-hand side are always positive and practically do not depend on Ω. For an arbitrary nonlinear function $g(x)$ with $g(0)=0$ or with $\Delta g=0$ at $x=0$ there always occurs, similar to the result (48), the stabilization effect, i.e., a positive shift δa_c of the critical point. Note that for the overdamped systems, of finite h and $m\to 0$, $U_{\text{eff}}(x)\to 0$ while $V_{\text{eff}}(x)\to -\mathcal{D}_{\text{eff}}(x)/2\neq 0$ as follows from (45) and (49).

One may infer from this consideration that the effective potential U_{eff}, appearing in the equivalent Langevin equation (46), plays the main role in the determination of the steady-state regime near a critical point and its stability, rather than the effective potential $V_{\text{eff}}(x)$, figuring when dealing in terms of the (x,p) space. However, this is only a part of the truth as will be seen from the following discussion.

VI. CRITERIA OF INSTABILITY AND INTERMITTENCY UNDER MULTIPLICATIVE NOISE

The estimation of instability thresholds of a system via analysis of the eigenvalues of a linearized system associated with it is a conventional method. However, this method fails for systems near a critical point under multiplicative noise. This failure is not that associated with the pathological cases of zero eigenvalues of the linearized systems.

The simplest one-dimensional stochastic Landau equation

$$h\frac{dx}{dt}=ax-bx^3+x\xi(t) \tag{50}$$

demonstrates this paradoxical feature. The feature is essential for understanding the character of the stabilization action caused by chaotic modulation. Nevertheless, to the present author's knowledge, it has scarcely been discussed in the literature. Let, e.g., $\xi(t)$ in (50) be δ correlated, $\langle\xi(t)\rangle=0, \langle\xi(t)\xi(0)\rangle=D\delta(t)$. Then, as mentioned in Sec. V, the probability density $P(x,t)$ for this system evolves from a given initial state so that

$$\lim_{t\to\infty} P(x,t)=\delta(x)\ , \quad a<0\ .$$

The state $x=0$ is stable at $a<0$, while at $a>0$ this steady state is unstable and the stable distribution of P is given by

$$P_{st}=Nx^{ah/D-1}e^{-x^2bh/2D}\ .$$

These results, which are exact, show that the critical point is at

$$a=a_c=0\ .$$

The transition is sharp and the relaxation behavior at large t's is determined at $a\approx 0$ by the time scale h^2/D rather than by the magnitude of the large scale $h/|a|$ (see [4,5]).

Now let us consider the behavior of this system linearized in the close vicinity of the stable equilibrium state $x=0$. In the linear approximation the moment $\langle x(t)\rangle$ obeys the equation

$$h\frac{d}{dt}\langle x\rangle=\left[a+\frac{D}{h}\right]\langle x\rangle\ , \tag{51}$$

which shows that the $\langle x(t)\rangle$ becomes unstable not at $a=0$ but earlier, at $a=a_1$,

$$a_1=-\frac{D}{h}\ . \tag{52}$$

This result is also exact. So, when the control parameter is inside the interval $a_1<a<a_c$, our system, being ultimately stable, nevertheless is linearly unstable. Furthermore, the interval of linear instability spreads, in fact, below the threshold a_1. Indeed, the threshold of instability of $\langle x^n(t)\rangle$ of the linearized at $x=0$ system (50) for a Gaussian δ-correlated $\xi(t)$ is

$$a_n=-n\frac{D}{h}\ ,$$

i.e., for large n a_n ranges much below $a_c=0$ [26].

Strictly speaking there is no contradiction: though the system is linearly unstable, after a perturbation arises the system will finally relax to the state $x=0$ and stay there infinitely long, resulting in $P_{st}(x)=\delta(x)$. This phenomenon is not due to the specifics of the statistical approach and it should take place also in the deterministic systems of the same kind, since (as far as we are interested in ergodic systems) the deterministic systems corresponding to almost all realizations of the random process, after temporal smoothing, should behave the same. There is no contradiction with the asymptotic time smoothing method of deterministic nonlinear mechanics in that the threshold of instability of a nonlinear deterministic system is not determined by the criterion estimated from the linear approximation near equilibrium. This method (see the Krylov-Bogoliubov theorems in [24]) guarantees an adequate approximation at $t\to\infty$ when the linear approximation of the averaged equations near equilibrium is stable. However, the opposite, i.e., that the instability of the linearized system implies the instability of the nonlinear system, does not follow from anything.

Let us look at our system more closely. Being at $x=0$, it reacts to *any* infinitesimal perturbation with exponential growth in t of $\langle x(t)\rangle$, i.e., it explodes until the nonlinear terms come into the play. According to our estimations (details will be published elsewhere) the magnitude of $x(t)$ achieves at $a<0, |a|\ll D$, the values

$$x_m\sim\max(\sqrt{\langle x^2\rangle})\sim\sqrt{D/bh} \tag{53}$$

while the characteristic time scale T of the transient process is of the order

$$T\sim\frac{h^2}{D}\ln\left[\frac{D}{x_0^2bh}\right] \tag{54}$$

provided $x_0^2bh/D\ll 1$, where x_0 is the initial value of x induced by a perturbation. The infinitesimal perturbations, of course, are unavoidable as long as we are dealing with physical systems and not with pure mathematics. This means that our system will explode from time to time, i.e., a regime of intermittency takes place. In the case when these explosions are rather rare, and correspondingly the external influences causing them are rare, the steady-state regime will differ from the $P_{st}(x)$ infinitesimally small.

Let us turn to the systems governed by Eq. (43) or (46). In the case of multiplicative noise, of the form $g(x)=x$, we obtain for the corresponding threshold a_1 of the linear instability

$$a_1=\Delta[V(x)+V_{eff}(x)]|_{x=0}=a_c^0+\delta a_1$$

with

$$\delta a_1=\frac{Q}{m}\frac{\Omega^2-\nu(\nu+\gamma)}{(\Omega^2+\nu^2)[\Omega^2+(\nu+\gamma)^2]}\ . \tag{55}$$

At $a<a_1$ the system may be considered as "really" stable. (The border of the instability is rather eroded and spread into the region $a<a_1$ as evident from Ref. [26].) Thus the effective potential $V_{eff}(x)$ rather than $U_{eff}(x)$ determines the "real" instability threshold.

The interval of intermittency as follows from (55) and (48) is given by the difference

$$\delta a_1-\delta a_c=\Delta[U_{eff}(x)-V_{eff}(x)]|_{x=0}$$
$$=\tfrac{1}{2}\Delta D_{eff}(x)|_{x=0}=\frac{Q(\nu,\Omega)}{m}\frac{\nu/\gamma}{\nu^2+\Omega^2}\ . \tag{56}$$

So, the larger the diffusion, the larger the interval of intermittency. Such is the price of the stabilization effect (estimated via the shift of a_c) resulting from the low-

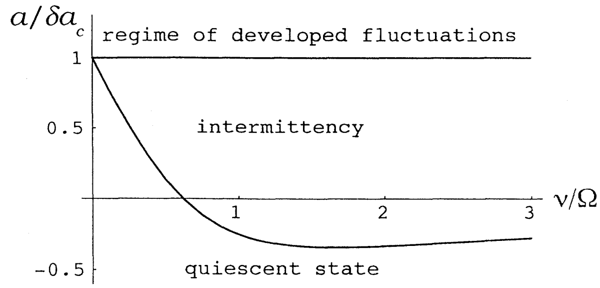

FIG. 2. Character of the steady-state regime for the case of multiplicative noise. The interval of intermittency is given by (56). The noise covariance is taken from the form (26) with $\Omega=\gamma$.

frequency part of the power spectrum of the noise. The trend in the plane $(a/\delta_c, \nu/\Omega)$ is illustrated in Fig. 2. In the interval of intermittency the system can respond strongly to a small perturbation. The formulas (53) and (54) may serve as a rough estimate of the scales of the time period of the enhancement and of the maximum value that $x(t)$ takes. A detailed analysis is beyond the scope of this paper.

VII. CONCLUSION

There were two approaches used, the conventional statistical (dealing with the "colored" noise, i.e., table-shaped spectra) and deterministic effective potential concept (valid for the case of no low-frequency component in the spectra). The theory represented here is more general and contains both limits. A small parameter is introduced which is related to the time scale τ_d of the system's drift and the frequency scale T_c^{-1} of alternations of the chaotic influences rather than their correlation decay scale τ_c. This and the close correspondence demonstrated between the asymptotic methods of both approaches have made the method rather general and simple at the same time.

The nonlinear influence of the noise was split, in the coordinate momentum picture, into the three contributions—effective potential, friction, and diffusion. The general structure of these quantities was shown to possess features allowing determination of the sign, the trend, and the contributions and their dependences on the spectral characteristics of the noise without going into the details of actual analysis. We wish to emphasize that these results apply to multidimensional systems, of arbitrary number of modes (waves) of motion, and the analytical treatment covers all the range of table- and bell-shaped spectra of noise, from the monochromatic limit to white noise. They show essential nonmonotonous dependences when going from one limit to the other and this trend gives a clear notion about the modifications and extensions of the conventional, deterministic, effective potential concept.

In view of the essential interplay between the effective potential, friction, and diffusion that takes place when considering the transient behavior at large times, it appeared at first that the characteristics introduced were not useful, e.g., for the analysis of the noise-induced shift of the critical point and the steady-state regime. However, the treatment given here shows that this is not so and that the characteristics determine, in fact, the trend and scales in a standard and constructive way—at the expense of dealing with the system's drift and diffusion in the x space rather than the (x,p) space. An essential feature is that the effective potential in these pictures differ drastically. The difference originates purely from the low-frequency components of the noise spectra and is related to the diffusion coefficients in a straightforward manner.

Finally, the result should be emphasized that the stabilization by multiplicative noise should be always followed by an intermittency, as far as it concerns the noise with nonzero intensity at low frequencies. The intermittency, as elucidated, occurs over a rather broad interval of the controlling parameters. For the case of a colored or white noise the area of intermittency covers all the region of stabilization. This feature, though intrinsic for the stabilization behavior, is hidden, and may not show itself when making conclusions only on the basis of the stationary probability distribution.

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[26] The reason for $\lim a_n \to -\infty$ when $n \to \infty$ is due to the possibility of arbitrarily large amplitudes of rare realizations of the Gaussian (δ-function and non-δ-function correlated) processes with component smooth in t. This feature does not occur, e.g., for $\xi(t)$ modeled by a Dichotomic Markov process of finite correlation time. Then all the a_n are given by

$$a_n = -\left[Q + \frac{\nu^2 h^2}{4n^2}\right]^{1/2} + \frac{\nu h}{2n}$$

and range in the finite interval $[-\sqrt{Q}, a_1]$, where ν is the mean frequency of alternations in $\xi(t)$ and $Q = \langle \xi^2 \rangle$. Note that for $Q = \nu D$ [then the power spectrum $S(0)$ of the process corresponds to that of the white noise] the parameter ϵ, Eq. (34), is equal to $\sqrt{Q}/\nu h$. Further, to be specific, we shall take the quantity $-D/h$, which is equal to a_1, Eq. (52), as the threshold of linear stability.