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Nonuniversal diffusion in a chain with deterministic local drifts

Amos Maritan
Dipartimento di Fisica, Università Padova, Padova, Italy
and Sezione Istituto Nazionale di Fisica Nucleare di Padova, Padova, Italy

Jayanth R. Banavar
Department of Physics and Materials Research Laboratory, Pennsylvania State University, 104 Davey Laboratory, University Park, Pennsylvania 16802

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A renormalization-group analysis is carried out of the long-time behavior of random walks in a one-dimensional lattice consisting of three kinds of bonds generated by a deterministic inflation rule. The random walker is subject to a local drift force, depending on the kind of bond it traverses. The mean-square displacement after time t is found to scale as $t^{2\nu}$ where the exponent ν is nonuniversal and depends continuously on the magnitude of the drift force. The model studied is one of a general class of systems that exhibits similar behavior.

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A wide range of problems in such diverse fields as physics, chemistry, metallurgy, and ecology can be mapped onto a random walk. In the physical sciences alone, problems involving reactions, noise, fluctuations, relaxation, and transport may all be described by means of random-walk processes [1]. In this paper, we show that nonuniversal anomalous diffusion can occur on a one-dimensional chain with *deterministic* position-dependent local drift forces. Previous studies [2–11] have considered classical diffusion on a *random* chain and have found a variety of behaviors for the scaling of the mean-square displacement $\langle x^2(t)\rangle \sim t^{2\nu}$ ranging from $\nu=0$ [$\langle x^2(t)\rangle \sim (\ln t)^4$] to $\nu=\frac{1}{2}$ and also nonuniversal behavior, where ν depended continuously on the probability distribution of the local drift forces [8]. In contrast, our model has *no* randomness and is a simple example of a *general* class of deterministic models that exhibits such behavior. It corresponds to diffusion in a landscape that is neither smooth (yielding normal diffusion) nor randomly rugged [3], but in between. Our conclusions are based on a renormalization-group analysis.

The master equation for diffusion is

$$P_x(t+1)=P_x(t)+W_{x,x+1}P_{x+1}(t)+W_{x,x-1}P_{x-1}(t) -(W_{x+1,x}+W_{x-1,x})P_x(t)\ , \quad (1)$$

where $P_x(t)$ denotes the probability of the walker being at site x at time t and $W_{x,y}$ is the hopping probability for going from site y to site x. We define the initial condition as $P_x(t=0)=\delta_{x,x_0}$ and using the transform $\tilde{P}_x(\omega)=\sum_{t=0}^{\infty}(1+\omega)^{-1-t}P_x(t)$ we rewrite Eq. (1) as

$$(\alpha_x\omega+W_{x+1,x}+W_{x-1,x})\tilde{P}_x =W_{x,x-1}\tilde{P}_{x+1}+W_{x,x-1}\tilde{P}_{x-1}+\delta x,x_0\ , \quad (2)$$

where α_x is a (possibly site-dependent) coefficient that is initially equal to 1. The long-time behavior of the random walk is obtained in the $\omega\to 0$ limit. In our subsequent analysis, we will neglect higher-order corrections in ω that contribute to corrections to the leading scaling behavior.

To specify the model, we need to assign $W_{x,y}$. Our

model has three kinds of bonds denoted by +, −, and 0. When the bond connecting x and $x+1$ is a + bond, we let $W_{x,x+1}=W_-$; $W_{x+1,x}=W_+$. For a − bond, $W_{x,x+1}=\overline{W}_+$ and $W_{x+1,x}=\overline{W}_-$, whereas for the 0 bond, $W_{x,x+1}=W_{x+1,x}=W_0$. We choose $W_0=\frac{1}{2}$, without any loss of generality. Note that the + and − bonds have asymmetric hopping probabilities. We choose the ratios W_+/W_- and $\overline{W}_+/\overline{W}_-$ to be equal and denote the ratio by r. r is a measure of the local drift force. Finally, the arrangement of +'s, −'s, and 0's is determined by a deterministic inflation scheme. A chain of length $2L$ is obtained from a chain containing L bonds by the following substitutions: $+\rightarrow+0$, $-\rightarrow-0$, and $0\rightarrow+-$. For example, the $+\rightarrow+0$ denotes that each + bond is replaced by a + bond followed by a 0 bond. Thus starting from one bond 0, one gets +−, +0−0, +0+−−0+−, etc. after successive applications.

We now envisage carrying out a decimation of the one-dimensional chain. The decimation entails reversing the inflation scheme by eliminating every other site and collapsing pairs of bonds into a renormalized bond. On renormalization, the ratio r is preserved so that, for a given r, only W_+ and $\overline{W}_+$ are independent variables. Further, since there are three kinds of bonds, there are nine possible combinations of local environments of two successive bonds that one needs to consider. However, the inflation scheme does not lead to any ++ or 00 nearest-neighbor bonds, so that there remain only seven relevant local environments making α_x in Eq. (2) a seven-component vector $\boldsymbol{\alpha}$. The decimation can be carried out explicitly leading to nine renormalization-group recursion relations as follows:

$$W'_+=\frac{W_++\overline{W}_+}{W_++r}\frac{r}{\overline{W}_+}\ , \tag{3}$$

$$\overline{W}'_+=\frac{W_++\overline{W}_+}{\overline{W}_++1}\frac{1}{W_+}\ , \tag{4}$$

$$\boldsymbol{\alpha}'=T(W_+,\overline{W}_+)\boldsymbol{\alpha}\ , \tag{5}$$

where $\boldsymbol{\alpha}$ is a seven-component vector and T is a 7×7 matrix. There is no $\boldsymbol{\alpha}$ dependence in Eqs. (3) and (4), since we are interested in the leading behavior in the $\omega\rightarrow0$ limit. Even if one chooses $W_\pm=\overline{W}_\pm$ initially, they become unequal after renormalization. After decimation, the chain initially labeled by sites $1,2,\ldots,2x,2x+1,\ldots,2N$ now has N sites. The xth site after decimation corresponds to the $(2x)$th site before decimation. We find after one decimation that there is a scaling form for P given by

$$P_x(\boldsymbol{\alpha}'\omega,W'_+\overline{W}'_+)=A(W_+,\overline{W}_+)P_{2x}(\boldsymbol{\alpha}\omega,W_+,\overline{W}_+)\ , \tag{6}$$

where $A(W_+,\overline{W}_+)=W_+\overline{W}_+/(W_++\overline{W}_+)$. Letting $\alpha_1,\alpha_2,\ldots,\alpha_7$ represent the pairs −0, 0+, +0, 0−, +−, −+, and −−, respectively, the nonzero matrix elements of T are (dropping the subscript + in W_+ and $\overline{W}_+$)

$$T_{11}=\frac{W+\overline{W}}{W\overline{W}}\frac{1}{1+\overline{W}}\ ,\quad T_{12}=\frac{W+\overline{W}}{W\overline{W}}\ ,\quad T_{15}=r/\overline{W}\ ,$$

$$T_{23}=\frac{W+\overline{W}}{r+\overline{W}}\frac{r}{\overline{W}}\ ,\quad T_{25}=\frac{r}{W}\ ,\quad T_{41}=\frac{W+\overline{W}}{1+\overline{W}}\frac{1}{rW}\ ,$$

$$T_{71}=\frac{r+\overline{W}}{r\overline{W}W}\frac{W+\overline{W}}{1+\overline{W}}\ ,$$

$$T_{62}=T_{26}=T_{32}=T_{54}=T_{47}=T_{74}=T_{12}\ ,$$

$$T_{33}=\frac{T_{23}}{W}\ ,\quad T_{35}=T_{15}\ ,\quad T_{45}=T_{25}\ ,\quad T_{51}=T_{41}\ ,$$

$$T_{53}=\frac{T_{23}}{W}\ ,\quad T_{63}=T_{23}\ ,\quad \text{and}\quad T_{61}=T_{41}\frac{r}{\overline{W}}\ .$$

It is straightforward to evaluate the fixed points of (3) and (4). The fixed-point values of W^*_+ and $\overline{W}^*_+$ are $W^*_+=r\overline{W}^*_+$ and $\overline{W}^*_+=\frac{1}{2}(\sqrt{5+4/r}-1)$. Furthermore, this fixed pont is attractive. Thus the exponent ν can be determined by diagonalizing the matrix T evaluated at W^*_+ and $\overline{W}^*_+$. Indeed, following the standard definition [12] ν is related to the largest eigenvalue λ by $\nu=\ln2/\ln\lambda$ and is found to be

$$\nu(r)=\{\log_2[(\sqrt{5+4/r}+1)(\sqrt{5+4r}+1)/4]\}^{-1}\ . \tag{7}$$

We note that $\nu(r)=\nu(1/r)$. The Laplace transform of $\langle x^2(t)\rangle$, the mean-square displacement, $\omega^{-1}R^2(\alpha\omega,W_+,\overline{W}_+)$, scales as in Eq. (6) with P_x replaced by R and $A(W_+,\overline{W}_+)$ by $\frac{1}{4}$ (to leading order). Thus $R\sim\omega^{-\nu}$ or $\langle x^2(t)\rangle^{1/2}\sim t^\nu$, where ν is given in Eq. (7). Following the treatment in Ref. [13], it is straightforward to show that the probability of return to the origin, $P_{x_0}(t)$, averaged over all initial positions x_0, scales as $1/t^\nu$ with the same ν as in the above equation.

We now turn to a special case. We visualize starting the inflation scheme with a + or − bond and obtaining a semi-infinite lattice. We start a walker on the first site—we assume no sites are present on the left and thus the walker can initially move only to the right or stay where it is. Equations (3)–(6) still hold with one significant simplification: the functions P_x and P_{2x} in Eq. (6) become one and the same function P_0, when $x=0$, which physically denotes the probability that the walker returns to the origin. As before, we consider Eq. (6) at the fixed-point values of W^*_+ and $\overline{W}^*_+$. Iterating the equation many times so that the leading eigenvalue dominates, we find that the probability of returning to the origin scales as $t^{-\nu'}$, where

$$\nu'=1+\nu(r)\ln A^*/\ln2$$
$$=1-\nu(r)\log_2\frac{\sqrt{5+4/r}+1}{2} \tag{8}$$

and $A^*=A(W^*_+,\overline{W}^*_+)$. Substituting the value of ν from Eq. (7), one finds the simple result that $\nu'(r)+\nu'(1/r)=1$.

It is interesting to note that if the +, −, and 0 bonds in our deterministic chain were distributed randomly, the behavior of the random walker would be qualitatively different and be given by the Sinai result [3] $\langle x^2(t)\rangle\sim(\ln t)^4$. Physically [6], the origin of the Sinai result stems from the fact that the imbalance between the

number of + bonds and − bonds over a length scale L scales as $\sqrt{L}$ from simple random-walk arguments. Because of the opposing local drift in the + and − bonds, this corresponds to a barrier that scales as $\sqrt{L}$ leading to an activated result, $\tau \sim e^{\sqrt{L}}$, or equivalently that a characteristic length traversed scales as $(\ln t)^2$. In a similar vein, our results may be understood simply by noting that the arrangement of the +, −, and 0 bonds resulting from the inflation rule is such that the maximum imbalance [14] between the number of + and − bonds over a length scale L scales as $\ln L$ corresponding to a barrier scaling as $(1/\nu)\ln L$, with $\nu < \frac{1}{2}$ and depending on the degree of asymmetry r. This, in turn, leads to the result $\tau \sim \exp[(\ln L)/\nu]$ or $\langle x^2(t) \rangle \sim t^{2\nu(r)}$. (We have been unsuccessful in deriving the expression for $\nu(r)$ [Eq. (7)] with similar hand-waving arguments.) Thus our deterministic model is merely one in a class of models that exhibit nonuniversal anomalous diffusion. Indeed, we have verified with computer simulations that another model (that is not amenable to exact renormalization-group analysis) with the +, −, and 0 on the sites instead of on the bonds also exhibits qualitatively similar behavior.

There is an infinite class of models that one may readily construct with the generic nonuniversal behavior. Specifically, all models obtained by replacing a given bond of our model with a sequence of 1_0 bonds such that a $\pm(0)$ bond's replacement has an imbalance of $\pm n_0(0)$ with $n_0 \geq 1$ can be readily mapped into our model with a "prefacing" transformation. Such a transformation would consist of an exact replacement of the 1_0 bonds by an effective bond using a decimation technique. Other deterministic inflation rules such as $+ \rightarrow + - +$, $- \rightarrow - + -$, $0 \rightarrow + 0 -$ would also result in novel behavior. It would be interesting to determine whether a random arrangement of the +, −, and 0 bonds but with a constraint limiting the growth of the imbalance between + and − bonds to $\ln L$ would also exhibit anomalous behavior or merely that of a regular random walk. It has been suggested that the case where the +, −, and 0 bonds are distributed randomly corresponding to a random walk in a random environment may be applicable to the diffusion of particles in the presence of flow in a porous medium. It would be interesting to find experimental realizations of our deterministic model.

We conclude by noting that Golden and co-workers [15] have studied diffusion in a quasiperiodic one-dimensional system and find that for a dense set of incommensurate potentials, the velocity-autocorrelation function decays slower than $1/t^{1+\epsilon}$ for any $\epsilon > 0$.

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[14] The logarithmic dependence of the imbalance on the length is obtained from the following recursive argument: let the maximum imbalance over a length 2^n starting from a $\pm(0)$ after n iterations be denoted by $I_{\pm(0)}(n)$. Note that at the end of the chain, the net imbalance is $\pm 1(0)$ in the three cases. Starting from a + after $(n+1)$ iterations, the first half of the chain is that produced by a + after n iterations whereas the second half originates from a 0 after n interations. Thus $I_+(n+1) = \max[I_+(n), I_0(n)+1]$, where the two terms correspond to the maximum occurring in the first half or second half of the chain and the addition of 1 arises from the net imbalance in the first half of the chain. Similar equations are obtained for $I_-(n+1)$ and $I_0(n+1)$, which when combined yields $I_+(n+1) = \max[I_+(n), I_+(n-1)+1]$. Choosing $I_+(0) = I_+(1) = 1$, this equation can be solved to yield $I_+(n) = 1 + \text{int}(n/2)$, which is the desired logarithmic dependence, where $\text{int}(n/2)$ is the smallest integer at least as large as $n/2$.

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