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Self-consistent approach to the Kardar-Parisi-Zhang equation

J. P. Bouchaud
Cavendish Laboratory, Madingley Road, Cambridge CB3 0HE, United Kingdom
and Service de Physique de l'Etat Condensé, Direction des Recherches sur l'Etat Condensé, les Atomes et les Molecules, Commisariat à l'Energie Atomique, Orme des Merisiers, 91191 Gif-sur-Yvette CEDEX, France

M. E. Cates
Cavendish Laboratory, Madingley Road, Cambridge CB3 0HE, United Kingdom
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We propose a self-consistent treatment of the Kardar-Parisi-Zhang equation in d dimensions, in order to calculate the dynamical exponent z and the roughness exponent χ, and also amplitude ratios and subleading corrections. We assume that the dynamics of each mode is purely exponential, and find agreement with known results in $d=1$ and 2. For $d>d^*\simeq 2.85$, however, none of our solutions is compatible with this assumption. Our method is distinct from, but akin to, the one recently proposed by M. Schwartz and S. F. Edwards [Europhys. Lett. **20**, 301 (1992)].

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The Kardar-Parisi-Zhang equation [1] has attracted quite a lot of attention in recent years, not only because of its connection to a variety of important physical problems (growth, interface dynamics, polymers in random media, and many others), but also because this equation is thought to retain—in a simpler form—some of the features of notoriously difficult problems such as turbulence or spin glasses. In its native version, the KPZ equation [see Eq. (1) below] is a nonlinear diffusion equation driven by an external white noise. The main point is to understand how the short-wavelength noise interacts with nonlinearity to give rise to anomalous statistics at long wavelengths.

As usual, the dimension d of space plays a crucial role. From perturbation theory, it is easy to see that when $d\leq 2$, nonlinearity is "relevant" in the sense that the statistics at large times and length scales must strongly depart from that of the linear equation [1,2]. In fact, in $d=1$, many exact results are available and numerical results quite easy to obtain, so that most aspects of this problem are well understood [3–6]. In $d>2$, perturbation theory is well-behaved, which would normally mean that the linearized "mean-field" results should hold. Numerical simulations, however, suggest that for sufficiently large noise or nonlinearity one enters a "strong-coupling" regime again characterized by anomalous exponents [7]. One major problem is to compute these exponents, and to estimate the upper critical dimensionality d_c (if it exists), above which mean-field values are recovered. Numerical simulations [7] and some scaling arguments [8] suggest that $d_c=\infty$ (where the situation is again well understood [9]), while other works, based on the functional renormalization group [10], Flory arguments [11], or variational replica calculations [12] claim $d_c=4$ or $d_c=2$. The existence of a finite d_c is supported by a $1/d$ expansion [13] (see, however, [14]), and an argument based on a comparison with directed percolation suggests that $d_c \geq 4$ [15].

The aim of this Rapid Communication is to propose a self-consistent treatment of the perturbation expansion (much in the same spirit as the "self-consistent screening approximation" to critical phenomena [16,17]), which suggests the existence of a finite critical dimension. Although by no means exact, this procedure at least provides an interesting scenario of what *could* happen in reality. For $d=1$, we recover some of the known exact results. For $d=2$, we find nontrivial solutions to our equations, which compare satisfactorily with other estimates. For $d \geq 2.85$, however, we find there is no solution compatible with our simplifying assumption that each mode decays exponentially [see Eq. (4) below].

The validity of our approach is difficult to assess, but certainly the method would be totally unreliable if the conclusions were found to heavily depend on the details of the self-consistent closure scheme. An alternative path has been very recently followed by Schwartz and Edwards [18]: they propose a "variational" treatment of the Fokker-Planck equation (see, e.g., [19]), which leads to equations very similar to, but distinct from, our own, which they investigate in $d=2$. We have solved their equation in higher dimensions and the conclusions of both approaches are very similar—although in fact both approaches ultimately rely on the same assumption for the mode dynamics.

We shall consider the KPZ equation, describing, e.g., the evolution of the height $h(\mathbf{x},t)$ of a growing surface under the influence of surface tension, noise, and nonlinearity

$$\frac{\partial h(\mathbf{x},t)}{\partial t}=\nu_0\nabla^2 h(\mathbf{x},t)+\frac{\lambda}{2}[\nabla h(\mathbf{x},t)]^2+\eta(\mathbf{x},t)\ , \qquad (1)$$

where $\mathbf{x}$ is a d-dimensional vector and $\eta(\mathbf{x},t)$ a Gaussian white noise. Here ν_0 is the "bare" surface tension and λ measures the strength of the nonlinearity. The perturbation theory (in λ) has been fully worked out in Refs. [20,2]: one can set up diagrammatic rules for obtaining three important quantities, which are the renormalized propagator G, the renormalized noise correlator D, and the renormalized nonlinear interaction. Defining (in Fourier space) $G(\mathbf{k},\omega)\equiv\langle \partial h(\mathbf{k},\omega)/\partial\eta(\mathbf{k},\omega)\rangle$ (where $\langle\ \rangle$ means averaging over η) and $\langle h(\mathbf{k},\omega)h(-\mathbf{k},-\omega)\rangle \equiv 2D(\mathbf{k},\omega)G(\mathbf{k},\omega)G(-\mathbf{k},-\omega)$, and denoting by G_0 and D_0 the corresponding "bare" quantities, one obtains the following expansion to lowest order (see [20,2] for full details):

$$\Sigma(\mathbf{k},\omega)\equiv G_0^{-1}(\mathbf{k},\omega)-G^{-1}(\mathbf{k},\omega)$$
$$=-2\lambda^2\int\frac{d\Omega}{2\pi}\int\frac{d^d\mathbf{q}}{(2\pi)^d}[\mathbf{q}\cdot(\mathbf{k}-\mathbf{q})][\mathbf{q}\cdot\mathbf{k}]G_0(\mathbf{k}-\mathbf{q},\omega-\Omega)G_0(\mathbf{q},\Omega)G_0(-\mathbf{q},-\Omega)D_0(\mathbf{q},\Omega) \qquad (2)$$

and

$$D(\mathbf{k},\omega)=D_0(\mathbf{k},\omega)+\lambda^2\int\frac{d\Omega}{2\pi}\int\frac{d^d\mathbf{q}}{(2\pi)^d}[\mathbf{q}\cdot(\mathbf{k}-\mathbf{q})]^2G_0(\mathbf{k}-\mathbf{q},\omega-\Omega)$$
$$\times G_0(-\mathbf{k}+\mathbf{q},-\omega+\Omega)G_0(\mathbf{q},\Omega)G_0(-\mathbf{q},-\Omega)D_0(\mathbf{q},\Omega)D_0(\mathbf{k}-\mathbf{q},\omega-\Omega)\ . \qquad (3)$$

The "vertex" correction to λ can be shown to be zero at long wavelengths, due to a "Galilean" invariance of the KPZ equation [20,2] [i.e., that an additional convection term in Eq. (1) can be removed by *tilting the interface*].

A natural self-consistent closure of this expansion is to retain only the above lowest-order terms, but replacing everywhere the "bare" quantities by the renormalized quantities [4,5]. We shall furthermore assume that, in the small-ω, small-k region,

$$G^{-1}(\mathbf{k},\omega)=\mathcal{G}^{-1}k^{\gamma-z}(\nu k^z-i\omega)\ , \qquad (4)$$

$$D(\mathbf{k},\omega)=\mathcal{D}k^{-\mu}\ , \qquad (5)$$

where $\mathcal{G},\mathcal{D}$ are constants and νk^{z-2} is an effective, scale-dependent surface tension. Equation (4) assumes that the relaxation of each mode is exponential. Directly from Eq. (1), one can show that $\int(d\omega/2\pi)G(\mathbf{k}=0,\omega)=1$, which immediately gives $\gamma=z$ and $\mathcal{G}=1$. The exponent z is the usual dynamical exponent, while μ is related to the "roughness" exponent χ through

$$\overline{[h(\mathbf{x},t)-h(\mathbf{y},t)]^2}=4\int\frac{d\Omega}{2\pi}\int\frac{d^d\mathbf{q}}{(2\pi)^d}[1-\cos\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})]|G(\mathbf{q},\Omega)|^2D(\mathbf{q},\Omega)\equiv A|x-y|^{2\chi}\ . \qquad (6)$$

Hence $\mu\equiv d+2\chi-z$. Similarly, one finds that the time evolution of the height is described by $\overline{[h(\mathbf{x},t)-h(\mathbf{x},0)]^2}\equiv Bt^{2\chi/z}$.

Let us focus on the zero frequency limit of Eqs. (2) and (3) and examine the small-k behavior of $\Sigma(\mathbf{k},\omega=0)$ and $D(\mathbf{k},\omega=0)$ (we shall now drop the frequency argument). To leading order, one finds

$$\Sigma(\mathbf{k})=\frac{\lambda^2\mathcal{D}}{\nu^2}k^{d+4-3\gamma-\mu}I\ , \qquad (7)$$

where I is given by

$$I=\int \frac{d^d\mathbf{u}}{(2\pi)^d}\frac{(\frac{1}{4}-u^2)(\frac{1}{2}+\mathbf{u}\cdot\mathbf{1})|\frac{1}{2}\mathbf{1}+\mathbf{u}|^{-z-\mu}}{|\frac{1}{2}\mathbf{1}-\mathbf{u}|^z+|\frac{1}{2}\mathbf{1}+\mathbf{u}|^z} \tag{8}$$

($\mathbf{1}$ is the unit vector along $\mathbf{k}$). Equations (7) and (8) implicitly assume that $d+2-2z-\mu<0$ (for I to be convergent) and hence that $\Sigma(k)/k^2$ diverges for $k\to 0$. We may therefore identify $\Sigma(k)$ with the right-hand side of Eq. (4) (setting $\omega=0$) from which we obtain $3z=d+4-\mu$, and $\nu^3=\lambda^2\mathcal{D}I$. Using the definition of χ above, we then determine μ as

$$\mu=d+z+4\chi-4\ , \tag{9}$$

leading to the exponent relation $z=2-\chi$, which has also been found from more general arguments [3,2]. The condition for the convergence of I then reads $z<2$ or $\chi>0$.

Turning now to the equation for $D(\mathbf{k})$, we find, provided $\mu>0$ [21], the following results: $\mathcal{D}=(\lambda^2\mathcal{D}^2/\nu^3)J$, where J is given by

$$J=\frac{1}{2}\int \frac{d^d\mathbf{u}}{(2\pi)^d}\frac{(\frac{1}{4}-u^2)^2|\frac{1}{2}\mathbf{1}-\mathbf{u}|^{-z-\mu}|\frac{1}{2}\mathbf{1}+\mathbf{u}|^{-z-\mu}}{[|\frac{1}{2}\mathbf{1}-\mathbf{u}|^z+|\frac{1}{2}\mathbf{1}+\mathbf{u}|^z]} \tag{10}$$

and

$$\mu=d+4+z-4z\ . \tag{9'}$$

The exponent equation (9′) is exactly the same as the one obtained from the equation on Σ, Eq. (9). Hence it is the *prefactors* which constrain the possible values of z and χ through the following compatibility condition:

$$I=J\ . \tag{11}$$

It is interesting to note that the same scenario of duplicated exponent equations occurs in the "self-consistent screening approximation" in critical phenomena [16,17], where it is also the prefactor equation which fixes the value of the critical exponents. It turns out that the inequalities needed for our equations to hold are always satisfied in the relevant region, $d\geq 2$ and $z<2$. They are also satisfied in $d=1$, since one may check that $z=\frac{3}{2}$ and $\chi=\frac{1}{2}$ solve Eq. (11): one thus recovers the exact result in this case [4,5,18]. [In fact, although Eqs. (4) and (5) are only approximate, the self-consistent versions of Eqs. (2) and (3) are exact in $d=1$ [5].]

In fact, our final equation (11), together with the sum rule $z+\chi=2$, is extremely close to the one obtained by Schwartz and Edwards [18]; the only difference is in the denominators of Eqs. (8) and (10), which read in their case: $[1+|\frac{1}{2}\mathbf{1}-\mathbf{u}|^z+|\frac{1}{2}\mathbf{1}-\mathbf{u}|^z]$, instead of simply $[|\frac{1}{2}\mathbf{1}-\mathbf{u}|^z+|\frac{1}{2}\mathbf{1}+\mathbf{u}|^z]$. One should note that precisely the same difference exists between Edwards's theory of turbulence and the "direct interaction approximation" of Kraichnan [19]. The present approach to the KPZ equation is perhaps slightly more direct than that of Schwartz and Edwards and the appearance of the compatibility condition [Eq. (11)] is (to our eyes) more transparent.

A numerical solution of Eq. (11) in $d=2$ leads to $z=1.74$, very close to the result found in Ref. [18], while the numerical simulations or other approaches suggest $z\simeq 1.6-1.7$ [7,10,11]. However, when d is increased from 2, z increases and reaches its mean-field value $z_0=2$ for $d=d^*\simeq 2.85$ (or $d^*\simeq 2.78$ for the Schwartz-Edwards equation). Beyond this dimension, no solution satisfying $z<2$ is found within this scheme.

A more stringent test of the theory lies in the value of the prefactors A and B. It is however clear that the value of (say) ν [defined in Eq. (4)] cannot be fixed within the present scheme [22]: this would require one to solve the self-consistent equation in the full k range, rather than just for $k\to 0$. But, as noticed in Refs. [5,6], the combination $R\equiv B/A^{2/z}\lambda^{2\chi/z}$ is a pure number, independent of the value of ν. In $d=1$, we find $A=4.69(\nu/\lambda)^2$ and $R\simeq 0.52$, while simulations give $R\simeq 0.71$ [6], and the exact solution of Eqs. (2) and (3) gives $R\simeq 0.69$. This shows that our simplifying assumptions [Eqs. (4) and (5)] are not particularly accurate. For $d=2$, we find $R\simeq 0.81$ and $A=13.7(\nu/\lambda)^2$.

An interesting question for which our theory provides an answer lies in the nature of the subleading terms. We find that

$$\overline{[h(\mathbf{x},t)-h(\mathbf{y},t)]^2}\simeq A|x-y|^{2\chi}[1+C|x-y|^{-\Delta}+\cdots]\ ,$$

where $\Delta=\chi$, and C can be calculated if ν is known [23]. This form of the subleading correction is not the one assumed previously in the analysis of numerical results [7,6], where $\Delta=2\chi$ was used.

We have thus developed a self-consistent approach to the strong-coupling fixed point of the KPZ equation, which is very similar in spirit to other self-consistent approaches to turbulence or critical phenomena. Our "compatibility" equation turns out to be very close, but not identical, to the one recently obtained by Schwartz and Edwards [18] and the numerical values of the exponents are also very similar. We find satisfactory agreement between these values and known results in $d=1$ and 2. In dimensions greater than $d^*\simeq 2.8$, the nontrivial solution ($z\neq 2$) disappears, suggesting the existence of a finite critical dimension. However, our results (and those of Ref. [18]) rely on our very questionable assumption of a simple pole structure of the response function $G(\mathbf{k},\omega)$, i.e., the purely exponential decay of each mode. It is possible that in reality this decay is violently nonexponential; e.g., a power law, in which case the analytic structure of $G(\mathbf{k},\omega)$ will be very different [24]. It would be extremely interesting to solve (numerically) the full *frequency-dependent* self-consistency equations with the only assumption that $G^{-1}(\mathbf{k},\omega)=k^z g(\omega/k^z)$ and $D(\mathbf{k},\omega)=k^{-\mu}d(\omega/k^z)$, as was done in the one-dimensional case by Hwa and Frey [5]. This would give information on the mode dynamics and thereby corroborate (or dismiss) the idea of a finite critical dimension. It is not clear how the corresponding calculation could be done within the Schwartz-Edwards formalism: this is perhaps another advantage of the present approach.

Note added. Once this work was completed, we learned that J. Doherty, M. A. Moore, and A. Bray were working along similar lines [24].

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[21] In the case of long-range correlated noise such that $D_0(\mathbf{k},\omega)\sim k^{-2\rho}$, this condition becomes $\mu > 2\rho$. In the opposite case, we find $3z = d+4-2\rho$, in agreement with the renormalization-group result [2].

[22] On dimension grounds, one may write that $\nu=\nu_0/\xi^{\chi}$, where ξ is the "correlation" length. Assuming that the short-length cutoff is a and the short-time cutoff is a^2/D_0, one finds that $\xi=af(D_0\lambda^2/\nu_0^3 a^{d-2})$. In $d=1$, the continuum limit exists and $f(u)\simeq f_0/u$ for small u, leading to $\xi=f_0\nu_0^3/D_0\lambda^2$ [3]. In $d>2$, one expects that $f(u)$ diverges for $u<u_c$, where u_c is the transition point separating weak disorder and strong disorder.

[23] Similar subleading terms are obtained in the Schwartz-Edwards approach: J. Ravi Prakash (unpublished).

[24] J. Doherty, M. A. Moore, and A. Bray (unpublished).