

Kosterlitz-Thouless transition in a dilute Bose gas

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(Received 27 July 1992)

We study the two-dimensional dilute Bose gas using the ladder or T -matrix approximation, which is valid if $\ln(1/na^2) \gg 1$, and show that this approximation can describe also the critical region of the Kosterlitz-Thouless transition accurately. In particular, we consider the superfluid properties of the gas and find that the superfluid density shows the square-root cusp together with the universal jump at the critical temperature. The theory is applied to spin-polarized atomic hydrogen adsorbed on a superfluid helium film, since this appears to be one of the most promising experimental setups to achieve the Kosterlitz-Thouless transition in a weakly interacting Bose gas.

PACS number(s): 64.10.+h, 67.65.+z, 67.40.-w

I. INTRODUCTION

In the last two decades it has become increasingly clear that the physics of a many-body system in two dimensions is fundamentally different from the physics of the same system in a three-dimensional geometry. This can, for example, be seen in semiconductors, where the study of the quantum Hall effect [1] has led to the notion of particles with fractional statistics and in the recently discovered high-temperature superconductors, which have such unusual normal state properties that it has been suggested that the copper-oxide layers in these materials are responsible for a breakdown of Fermi-liquid theory [2].

This difference is in particular illustrated by the celebrated Mermin-Wagner-Hohenberg theorem [3], which forbids the spontaneous breaking of a continuous symmetry in two dimensions due to the enhanced importance of long-wavelength thermal fluctuations. However, a different kind of phase transition to a state with algebraic long-range order can occur in various two-dimensional systems, as first pointed out by Kosterlitz and Thouless [4]. In the case of a two-dimensional superfluid this so-called topological phase transition is associated with the dissociation of a large number of vortex pairs and the immediate destruction of superfluidity caused by the phase-slip processes that occur once unbound vortices are present [5]. Nelson and Kosterlitz predicted by means of a renormalization-group analysis that the discontinuous drop in the superfluid density n_s at the critical temperature T_c is of a universal nature and such that $n_s \Lambda_c^2 = 4$, where $\Lambda = (2\pi\hbar^2/mk_B T)^{1/2}$ is the thermal de Broglie wavelength and m the mass of the particles in the superfluid [6]. This prediction was subsequently verified experimentally for superfluid ^4He films, using third-sound measurements [7] and also a torsional-oscillator technique [8].

Another two-dimensional Bose system, which is expected to show the Kosterlitz-Thouless transition to a superfluid phase and on which we will concentrate in this paper, is spin-polarized atomic hydrogen adsorbed on a superfluid ^4He surface [9]. In contrast with liquid heli-

um, atomic hydrogen has the advantage of being a weakly interacting Bose gas and offers the opportunity to derive its physical behavior from first principles. Moreover, the property of an, in principle, independent control over density and temperature makes this an almost ideal two-dimensional model system for both theory and experiment.

However, there are two important problems. First, at the densities needed to observe the Kosterlitz-Thouless transition at moderate temperatures the adsorbed gas decays quite rapidly because of three-body recombination processes. Fortunately, it seems possible to circumvent this problem by using a relatively large buffer volume to increase the lifetime of the total system [10]. Second, due to the finite lifetime of the gas sample the time scale for the nucleation of the phase transition becomes of the utmost importance. A similar problem in the context of achieving Bose-Einstein condensation in magnetically trapped atomic gases was recently studied by one of us [11]. In that case it was shown that the actual nucleation is a fast process and that two-body elastic collisions act as a bottleneck for the formation of the condensate. Because of the different nature of the phase transition it is, however, far from clear that the same is true for the formation of the quasicondensate in two dimensions. Evidently, we can only address this complicated nonequilibrium question after we have first thoroughly studied the equilibrium situation. This is our aim in the present paper.

The two-dimensional dilute Bose gas was first treated by Popov [12], who introduced the concept of a quasicondensate as the microscopic explanation for the appearance of superfluidity in the gas. However, it was stressed by Fisher and Hohenberg [13] that Popov's treatment is only valid in the extreme limit that $\ln[\ln(1/na^2)] \gg 1$, where n is the density and a the scattering length. Furthermore, the superfluid density does not show the universal jump predicted by Nelson and Kosterlitz. This implies that the theory is not able to describe the critical region below T_c , which notably is of $O(T_c/\ln[\ln(1/na^2)])$ and very small in the limit $\ln[\ln(1/na^2)] \gg 1$ [13]. In the case of atomic hydrogen

the above-mentioned diluteness condition is not fulfilled and we have only the much weaker inequality $\ln(1/na^2) \gg 1$. Hence Popov's theory is not applicable in this case and cannot give an accurate prediction for the critical density of the system, which is experimentally of great importance because the lifetime of the gas is inversely proportional to the density squared.

The paper is organized as follows. In Sec. II we discuss the dilute Bose gas in both two and three dimensions, because we want to point out explicitly the great similarity between the two cases, as well as their essential differences. In Sec. II A we use a saddle-point or loop expansion and straightforwardly rederive Popov's results. We extend this theory in Sec. II B by applying the so-called T -matrix or ladder approximation, which in three dimensions can be rigorously justified if $na^3 \ll 1$ and in two dimensions if $1/\ln(1/na^2) \ll 1$ [14]. In particular, we find in this section the universal jump in the superfluid density, which indicates that with this approach we have indeed obtained an accurate description of a weakly interacting Bose gas. In Sec. III we use this formalism to obtain various quantities of a two-dimensional spin-

polarized atomic hydrogen gas, such as the critical temperature and the superfluid density. Moreover, we also consider the situation of the two-dimensional gas in thermal contact with a three-dimensional buffer gas and calculate the surface density as a function of temperature and volume density. We end with some concluding remarks in Sec. IV.

II. DILUTE BOSE GAS IN TWO AND THREE DIMENSIONS

In the functional approach to the imaginary-time formalism [15] the grand-canonical partition function of the Bose gas is a functional integral

$$Z = \int d[\psi^*] d[\psi] \exp \left\{ -\frac{1}{\hbar} S[\psi^*, \psi] \right\}, \quad (1)$$

over c -number fields $\psi^*(\mathbf{x}, \tau)$ and $\psi(\mathbf{x}, \tau)$ that obey the periodicity conditions $\psi^*(\mathbf{x}, \tau + \hbar\beta) = \psi^*(\mathbf{x}, \tau)$ and $\psi(\mathbf{x}, \tau + \hbar\beta) = \psi(\mathbf{x}, \tau)$ because of the trace involved in the calculation of the partition function. The Euclidean action in the exponent is

$$S[\psi^*, \psi] = \int_0^{\hbar\beta} d\tau \left\{ \int d\mathbf{x} \psi^*(\mathbf{x}, \tau) \left[\hbar \frac{\partial}{\partial t} - \frac{\hbar^2 \nabla^2}{2m} - \mu \right] \psi(\mathbf{x}, \tau) \right. \\ \left. + \frac{1}{2} \int d\mathbf{x} \int d\mathbf{x}' \psi^*(\mathbf{x}, \tau) \psi^*(\mathbf{x}', \tau) V(\mathbf{x} - \mathbf{x}') \psi(\mathbf{x}', \tau) \psi(\mathbf{x}, \tau) \right\}, \quad (2)$$

with $\beta = 1/k_B T$, μ the chemical potential, and $V(\mathbf{x} - \mathbf{x}')$ the interaction potential.

Since we are dealing with a homogeneous system it is convenient to expand the fields into Fourier modes via

$$\psi(\mathbf{x}, \tau) = \frac{1}{(\hbar\beta V)^{1/2}} \sum_{\mathbf{k}, n} a_{\mathbf{k}, n} e^{i(\mathbf{k} \cdot \mathbf{x} - \omega_n \tau)} \quad (3)$$

and the complex conjugate expression for $\psi^*(\mathbf{x}, \tau)$. Here, V is the volume or area of the system and $\omega_n = 2\pi n / \hbar\beta$ are the bosonic Matsubara frequencies, which incorporate the above-mentioned periodicity condition. The action becomes

$$S[a^*, a] = \sum_{\mathbf{k}, n} (-i\hbar\omega_n + \varepsilon_{\mathbf{k}} - \mu) a_{\mathbf{k}, n}^* a_{\mathbf{k}, n} \\ + \frac{1}{2} \frac{1}{\hbar\beta V} \sum_{\substack{\mathbf{k}, \mathbf{k}', \mathbf{q} \\ n, n', m}} V(\mathbf{q}) a_{\mathbf{k} + \mathbf{q}, n + m}^* a_{\mathbf{k}' - \mathbf{q}, n' - m}^* \\ \times a_{\mathbf{k}', n'} a_{\mathbf{k}, n}, \quad (4)$$

introducing the kinetic energy $\varepsilon_{\mathbf{k}} = \hbar^2 \mathbf{k}^2 / 2m$ and the Fourier transform of the interaction potential $V(\mathbf{q}) = \int d\mathbf{x} V(\mathbf{x}) \exp(i\mathbf{q} \cdot \mathbf{x})$.

This action completely determines the thermodynamics of the Bose gas. In the case of atomic hydrogen the envisaged experimental conditions are such that we are dealing with a gas well inside the quantum regime, which means that $a/\Lambda \ll 1$ and consequently that only s -wave scattering is of importance. Hence we can neglect the

momentum dependence of all collision quantities in the following and in particular use $V(\mathbf{q}) \simeq V(0) \equiv V_0$.

A. One-loop approximation

The thermodynamics of the system up to first order in the loop expansion follows from a calculation of the free energy after a quadratic approximation to the action. In the "normal" phase the order parameter $\langle \psi(\mathbf{x}, \tau) \rangle$ of the Bose gas is equal to zero and the action is approximated by

$$S_n[a^*, a] = \sum_{\mathbf{k}, n} (-i\hbar\omega_n + \varepsilon_{\mathbf{k}} - \mu) a_{\mathbf{k}, n}^* a_{\mathbf{k}, n}. \quad (5)$$

Performing the functional integral [15] the free-energy density becomes

$$f_n(\mu) = \frac{k_B T}{V} \sum_{\mathbf{k}} \ln(1 - e^{-\beta(\varepsilon_{\mathbf{k}} - \mu)}) + \mu n, \quad (6)$$

which is minimized with respect to the chemical potential if

$$n = \frac{1}{V} \sum_{\mathbf{k}} \frac{1}{e^{\beta(\varepsilon_{\mathbf{k}} - \mu)} - 1} \equiv \frac{1}{V} \sum_{\mathbf{k}} N(\varepsilon_{\mathbf{k}} - \mu). \quad (7)$$

As expected, this is just the equation of state for the ideal Bose gas.

In the "superfluid" phase we have $\langle \psi(\mathbf{x}, \tau) \rangle \neq 0$ and equal to the square root of the (quasi)condensate density

n_0 . Expanding up to quadratic order the action is given by

$$S_s[a^*, a] = \frac{n_0^2 V_0}{2} - \mu n_0 + \sum'_k \sum_n \{ (-i\hbar\omega_n + \varepsilon_k + 2n_0 V_0 - \mu) a_{k,n}^* a_{k,n} + \frac{1}{2} n_0 V_0 (a_{k,n} a_{-k,-n} + a_{-k,-n}^* a_{k,n}^*) \}, \quad (8)$$

where the prime denotes that the summation is over wave numbers $|\mathbf{k}| > k_0$ only. (In three dimensions a condensate can exist and we must use $k_0 = 0$. However, in two dimensions a single momentum state cannot be macroscopically occupied and the quasicondensate density n_0 corresponds to the density of particles with momenta smaller than $\hbar k_0$. The magnitude of the low-momentum cutoff $\hbar k_0$ does not follow from the one-loop theory but is determined in the next section.) Diagonalizing the above action by means of a Bogoliubov transformation, using the dispersion $\hbar\Omega_{\mathbf{k}}$ defined by $(\hbar\Omega_{\mathbf{k}})^2 = (\varepsilon_{\mathbf{k}} + 2n_0 V_0 - \mu)^2 - (n_0 V_0)^2$ and the usual coherence factors $u_{\mathbf{k}} = \frac{1}{2} [(\hbar\Omega_{\mathbf{k}}/\varepsilon_{\mathbf{k}})^{1/2} + (\varepsilon_{\mathbf{k}}/\hbar\Omega_{\mathbf{k}})^{1/2}]$ and $v_{\mathbf{k}} = \frac{1}{2} [(\hbar\Omega_{\mathbf{k}}/\varepsilon_{\mathbf{k}})^{1/2} - (\varepsilon_{\mathbf{k}}/\hbar\Omega_{\mathbf{k}})^{1/2}]$, we find for the free-energy density

$$f_s(\mu, n_0) = \frac{n_0^2 V_0}{2} - \mu n_0 + \frac{1}{2} \frac{1}{V} \sum'_k (\hbar\Omega_{\mathbf{k}} - \varepsilon_{\mathbf{k}} - 2n_0 V_0 + \mu) + \frac{k_B T}{V} \sum'_k \ln(1 - e^{-\beta\hbar\Omega_{\mathbf{k}}}) + \mu n. \quad (9)$$

Minimizing with respect to the chemical potential now leads to

$$n = n_0 + \frac{1}{V} \sum'_k \left\{ \frac{\varepsilon_{\mathbf{k}} + 2n_0 V_0 - \mu}{\hbar\Omega_{\mathbf{k}}} N(\hbar\Omega_{\mathbf{k}}) + \frac{\varepsilon_{\mathbf{k}} + 2n_0 V_0 - \mu - \hbar\Omega_{\mathbf{k}}}{2\hbar\Omega_{\mathbf{k}}} \right\}, \quad (10a)$$

whereas minimizing with respect to the (quasi)condensate gives

$$\mu = n_0 V_0 + V_0 \frac{1}{V} \sum'_k \left\{ \frac{2\varepsilon_{\mathbf{k}} + 3n_0 V_0 - 2\mu}{\hbar\Omega_{\mathbf{k}}} N(\hbar\Omega_{\mathbf{k}}) + \frac{2\varepsilon_{\mathbf{k}} + 3n_0 V_0 - 2\mu - 2\hbar\Omega_{\mathbf{k}}}{2\hbar\Omega_{\mathbf{k}}} \right\}. \quad (10b)$$

At this point it is important to realize that the loop expansion can formally be seen as an expansion in orders of \hbar [16]. Therefore we can substitute in the second term on the right-hand side of Eqs. (10) the zero-loop result $\mu = n_0 V_0$ because this leads to corrections that are of higher order in \hbar . In particular, this implies that $\hbar\Omega_{\mathbf{k}}$ is replaced by the well-known Bogoliubov dispersion relation $\hbar\omega_{\mathbf{k}} = (\varepsilon_{\mathbf{k}}^2 + 2n_0 V_0 \varepsilon_{\mathbf{k}})^{1/2}$ [17]. Taking the linear behavior of $\hbar\omega_{\mathbf{k}}$ for small $|\mathbf{k}|$ into account we notice that in three dimensions we can take the limit $k_0 \searrow 0$ to find the

following equation of state for the "superfluid" phase:

$$n = n_0 + \frac{1}{V} \sum_{\mathbf{k} (\neq 0)} \left\{ \frac{\varepsilon_{\mathbf{k}} + n_0 V_0}{\hbar\omega_{\mathbf{k}}} N(\hbar\omega_{\mathbf{k}}) + \frac{\varepsilon_{\mathbf{k}} + n_0 V_0 - \hbar\omega_{\mathbf{k}}}{2\hbar\omega_{\mathbf{k}}} \right\}, \quad (11)$$

determining the condensate density as a function of temperature and density.

In two dimensions we are, in first instance, not allowed to take the limit $k_0 \searrow 0$, which is an *ad absurdum* proof of the fact that here a condensate cannot exist. However, solving for n_0 in Eq. (10b) we find that the logarithmic divergence in the expression for n_0 exactly cancels the divergence in Eq. (10a). Therefore the total density

$$n = \frac{\mu}{V_0} - \frac{1}{V} \sum_{\mathbf{k} (\neq 0)} \left\{ \frac{\varepsilon_{\mathbf{k}}}{\hbar\omega_{\mathbf{k}}} N(\hbar\omega_{\mathbf{k}}) + \frac{\varepsilon_{\mathbf{k}} - \hbar\omega_{\mathbf{k}}}{2\hbar\omega_{\mathbf{k}}} \right\} \quad (12)$$

is free of infrared divergences if we use $\hbar\omega_{\mathbf{k}} = (\varepsilon_{\mathbf{k}}^2 + 2\mu\varepsilon_{\mathbf{k}})^{1/2}$, which is again justified since it only gives rise to corrections that are of $O(\hbar^2)$.

Unfortunately, Eq. (12) still has an ultraviolet divergence, due to the neglect of the momentum dependence of the potential $V(\mathbf{q})$. Indeed, the two-body T matrix describing the scattering of two particles with initial relative momentum $\hbar\mathbf{k}'$ and final relative momentum $\hbar\mathbf{k}$ at complex energy z obeys the Lippmann-Schwinger equation [18]

$$T^{2B}(\mathbf{k}, \mathbf{k}'; z) = V(\mathbf{k} - \mathbf{k}') + \frac{1}{V} \sum_{\mathbf{k}''} V(\mathbf{k} - \mathbf{k}'') \frac{1}{z - 2\varepsilon_{\mathbf{k}''}} \times T^{2B}(\mathbf{k}'', \mathbf{k}'; E). \quad (13)$$

Neglecting the momentum dependence we find from this

$$\frac{1}{T^{2B}(\mathbf{0}, \mathbf{0}; z)} = \frac{1}{V_0} + \frac{1}{V} \sum_{\mathbf{k}} \frac{1}{2\varepsilon_{\mathbf{k}} - z}, \quad (14)$$

having for $z = -2\mu$ exactly the same ultraviolet behavior as the right-hand side of Eq. (12). In a very good approximation, with an error of $O(1/\ln(1/na^2))$, we thus find that

$$n = \frac{\mu}{T^{2B}(\mathbf{0}, \mathbf{0}; -2\mu)} - \frac{1}{V} \sum_{\mathbf{k} (\neq 0)} \frac{\varepsilon_{\mathbf{k}}}{\hbar\omega_{\mathbf{k}}} N(\hbar\omega_{\mathbf{k}}), \quad (15)$$

in agreement with the results of Popov [12] and Fisher and Hohenberg [13].

The superfluid density is obtained from an evaluation of the momentum-momentum correlation function and is given in d dimensions by

$$n_s = n - \frac{\Lambda^2}{2\pi d} \frac{1}{V} \sum_{\mathbf{k} (\neq 0)} \mathbf{k}^2 [N^2(\hbar\omega_{\mathbf{k}}) + N(\hbar\omega_{\mathbf{k}})]. \quad (16)$$

As mentioned previously, the superfluid density as a function of temperature goes to zero in a continuous fashion and, in two dimensions, does not describe the expected discontinuity at the critical temperature. Mathematically, this comes about because at a fixed density Eq. (15) can be satisfied for all temperatures.

In addition, comparing the free energies of the “normal” and “superfluid” phases, we note that at zero temperature the free energy of the “normal” phase is zero and smaller than the free energy of the “superfluid” phase, which is equal to $n_0^2 T^{2B}/2$. This inequality can actually be shown to hold at all temperatures. Solely on the basis of the one-loop calculation we are therefore forced to conclude that the “normal” phase is more stable and will be favored in thermal equilibrium. In three dimensions this situation represents no problem, because the ideal Bose gas is indeed a very good first approximation to the weakly interacting case at all temperatures. However, in two dimensions the interactions are essential for the superfluid properties of the system and the approximation of an ideal gas always breaks down at sufficiently low temperatures. We thus have to treat the interactions more carefully. This is done in the following section, where we apply the T -matrix approximation and find that both problems are solved.

B. T -matrix approximation

To extend the previous theory of the extremely dilute Bose gas to higher densities and temperatures, we calculate the relevant self-energies in the ladder or T -matrix approximation and hence take diagrams with an infinite number of loops into account. The procedure to be used is summarized diagrammatically in Fig. 1.

$$S_{\text{eff}}^{(2)}[a^*, a] = -\frac{\hbar}{2} \sum_{\mathbf{k}}' \sum_n [a_{\mathbf{k},n}^* \ a_{-\mathbf{k},-n}] \cdot \left\{ \begin{bmatrix} G_0^{-1}(\mathbf{k}, \omega_n) & 0 \\ 0 & G_0^{-1}(-\mathbf{k}, -\omega_n) \end{bmatrix} \right. \\ \left. - \begin{bmatrix} \Sigma_{11}(\mathbf{k}, \omega_n) & \Sigma_{12}(\mathbf{k}, \omega_n) \\ \Sigma_{12}^*(\mathbf{k}, \omega_n) & \Sigma_{11}(-\mathbf{k}, -\omega_n) \end{bmatrix} \right\} \cdot \begin{bmatrix} a_{\mathbf{k},n} \\ a_{-\mathbf{k},-n}^* \end{bmatrix}, \quad (20)$$

where the zeroth-order Green's function $G_0(\mathbf{k}, \omega_n) = \hbar/(i\hbar\omega_n - \epsilon_{\mathbf{k}} + \mu)$ describes the ideal Bose gas [cf. Eq. (5)]. Using this action and neglecting the momentum and frequency dependence of the self-energies we find for the density of the system

$$n = n_0 + \frac{1}{\hbar\beta V} \sum_{\mathbf{k}}' \sum_n \langle a_{\mathbf{k},n}^* a_{\mathbf{k},n} \rangle \\ = n_0 + \frac{1}{V} \sum_{\mathbf{k}}' \left\{ \frac{\epsilon_{\mathbf{k}} + \hbar\Sigma_{12}}{\hbar\omega_{\mathbf{k}}} N(\hbar\omega_{\mathbf{k}}) + \frac{\epsilon_{\mathbf{k}} + \hbar\Sigma_{12} - \hbar\omega_{\mathbf{k}}}{2\hbar\omega_{\mathbf{k}}} \right\}, \quad (21)$$

with $\hbar\omega_{\mathbf{k}} = (\epsilon_{\mathbf{k}}^2 + 2\hbar\Sigma_{12}\epsilon_{\mathbf{k}})^{1/2}$. To eliminate the chemical potential and Σ_{11} from this result we used the Hugenholtz-Pines theorem [20], which states that

$$\mu = \hbar\Sigma_{11}(\mathbf{0}, 0) - \hbar\Sigma_{12}(\mathbf{0}, 0). \quad (22)$$

Because of its central importance for the theory in this section and to make the paper more self-contained, we give a simple proof of this theorem in the Appendix.

We first consider the normal phase above the critical temperature of Bose-Einstein condensation or the Kosterlitz-Thouless transition. Introducing $\mu' \equiv \mu - \hbar\Sigma$ the self-energy is given by

$$\hbar\Sigma = \frac{2}{V} \sum_{\mathbf{k}} T^{2B}(\mathbf{0}, \mathbf{0}; \epsilon_{\mathbf{k}} - \hbar\Sigma) N(\epsilon_{\mathbf{k}} - \mu'), \quad (17)$$

where μ' is determined from

$$n = \frac{1}{V} \sum_{\mathbf{k}} N(\epsilon_{\mathbf{k}} - \mu'). \quad (18)$$

Note that we have neglected the momentum and frequency dependence of the self-energy, because in the temperature range of interest $a/\Lambda \ll 1$. In three dimensions the two-body T matrix is also independent of energy in this range, which leads to $\hbar\Sigma = 2nT^{2B}(\mathbf{0}, \mathbf{0}; 0)$ or equivalently $\hbar\Sigma = 8\pi n a \hbar^2/m$. However, this is not the case in two dimensions where the T matrix has a logarithmic behavior and we must use

$$T^{2B}(\mathbf{0}, \mathbf{0}; \epsilon_{\mathbf{k}}) = \frac{4\pi\hbar^2/m}{\pi i - \ln(\mathbf{k}^2 a^2/8) - 2\gamma}, \quad (19)$$

with $\gamma \simeq 0.5772$ Euler's constant [19].

The situation in the superfluid phase below the critical temperature is more complicated. From the definition of the normal and anomalous self-energies $\Sigma_{11}(\mathbf{k}, \omega_n)$ and $\Sigma_{12}(\mathbf{k}, \omega_n)$, respectively, the quadratic part of the effective action including all orders of perturbation theory is

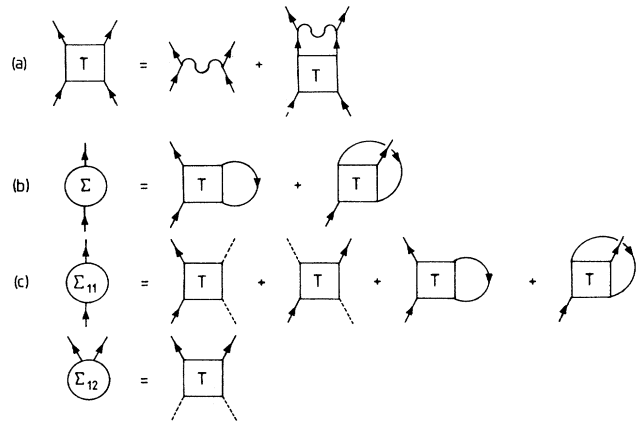


FIG. 1. Diagrammatic representation of the T -matrix approximation. The wavy line corresponds to the interaction V , the straight line to the dressed one-particle propagator, and the dashed line to a factor $(n_0)^{1/2}$. (a) T -matrix equation. (b) Self-energy in the normal phase. (c) Normal and anomalous self-energies in the superfluid phase.

Notice that Eqs. (21) and (22) constitute a complete set of equations for the (quasi)condensate density n_0 and the chemical potential μ at a fixed temperature and density, once the normal and anomalous self-energies are known. As announced, they will here be evaluated in the T -matrix approximation.

To do so we first have to discuss the many-body T matrix $T^{\text{MB}}(\mathbf{k}, \mathbf{k}', \mathbf{K}; z)$, which, due to the surrounding gas, depends also on the center-of-mass momentum $\hbar\mathbf{K}$ of the colliding particles and which should incorporate the influence of the (quasi)condensate on the scattering process. Since we are interested in the matrix element with all momenta equal to zero, we consider only this case. In a good approximation we find first of all that

$$T^{\text{MB}}(\mathbf{k}, \mathbf{0}, \mathbf{0}; z) = V(\mathbf{k}) + \frac{1}{V} \sum_{\mathbf{k}'} V(\mathbf{k}') \frac{[1 + 2N(\hbar\omega_{\mathbf{k}'})]}{z - 2\hbar\omega_{\mathbf{k}'}} \times T^{\text{MB}}(\mathbf{k}', \mathbf{0}, \mathbf{0}; z), \quad (23)$$

having the structure of a Bethe-Salpeter equation for the scattering of Bogoliubov quasiparticles. To arrive at this result we assume that the range of momenta for which the Bogoliubov dispersion $\hbar\omega_{\mathbf{k}}$ has a linear behavior is small compared to $1/a$. Because $\hbar\Sigma_{12}$ is at most of $O(nT^{2B})$ this condition reduces to $na^d \ll 1$, which is consistent with the condition for the validity of the T -matrix approximation in both two and three dimensions. It is important to note that the derivation of Eq. (23) is more involved in three dimensions due to the absence of the low-momentum cutoff $\hbar k_0$ and the divergence of the coherence factors $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ at long wavelengths. However, a rigorous justification that avoids all divergences can be given and is based on the use of the Hartree-Fock-Bogoliubov approximation to the one-particle propagators in the kernel of the T -matrix equation.

From Eq. (23) we conclude that, even in two dimensions, T^{MB} does not have a strong energy dependence and we are allowed to take z also equal to zero. Hence the self-energies are (see Fig. 1)

$$\begin{aligned} \hbar\Sigma_{11} &= 2nT^{\text{MB}}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0), \\ \hbar\Sigma_{12} &= n_0T^{\text{MB}}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0), \end{aligned} \quad (24)$$

and indeed momentum and frequency independent as assumed initially. Moreover, using Eq. (13) and assuming that $\varepsilon_0 \equiv \hbar^2 k_0^2 / 2m \ll \hbar\Sigma_{12}$, which will be verified below, we obtain the convenient relation

$$\frac{1}{T^{\text{MB}}(\mathbf{0}, \mathbf{0}, \mathbf{0}, 0)} = \frac{1}{T^{2B}(\mathbf{0}, \mathbf{0}; -2\hbar\Sigma_{12})} + \frac{1}{V} \sum_{\mathbf{k}}' \frac{N(\hbar\omega_{\mathbf{k}})}{\hbar\omega_{\mathbf{k}}}. \quad (25)$$

In three dimensions Eq. (25) shows that the influence of the surrounding gas on the collision of two particles is very small, and more quantitatively, that $T^{\text{MB}}(\mathbf{0}, \mathbf{0}, \mathbf{0}, 0) = T^{2B}(\mathbf{0}, \mathbf{0}; 0)[1 - O(na\Lambda^2)]$. Therefore the equation of state is given by [cf. Eq. (21)]

$$n = n_0 + \frac{1}{V} \sum_{\mathbf{k} \neq 0} \left\{ \frac{\varepsilon_{\mathbf{k}} + n_0 T^{2B}(\mathbf{0}, \mathbf{0}; 0)}{\hbar\omega_{\mathbf{k}}} N(\hbar\omega_{\mathbf{k}}) + \frac{\varepsilon_{\mathbf{k}} + n_0 T^{2B}(\mathbf{0}, \mathbf{0}; 0) - \hbar\omega_{\mathbf{k}}}{2\hbar\omega_{\mathbf{k}}} \right\}, \quad (26)$$

the chemical potential by $\mu = (2n - n_0)T^{2B}(\mathbf{0}, \mathbf{0}; 0)$, and the dispersion relation of the Bogoliubov quasiparticles is $\hbar\omega_{\mathbf{k}} = [\varepsilon_{\mathbf{k}}^2 + 2n_0 T^{2B}(\mathbf{0}, \mathbf{0}; 0)\varepsilon_{\mathbf{k}}]^{1/2}$ [13]. The superfluid density is, of course, found from Eq. (16). Note that in the limit of vanishing condensate density, we exactly recover the results of the normal phase at $\mu' = 0$. Therefore the critical temperature equals the ideal gas value

$$T_c = \frac{2\pi\hbar^2}{mk_B} \left[\frac{n}{\zeta(\frac{3}{2})} \right]^{2/3} \quad (27)$$

and $\zeta(\frac{3}{2}) \simeq 2.612$.

In the two-dimensional case we eliminate the quasicondensate density n_0 in favor of the chemical potential μ . Introducing $\mu' \equiv \mu - \hbar\Sigma_{11} = -\hbar\Sigma_{12}$ we find

$$n = \frac{-\mu'}{T^{2B}(\mathbf{0}, \mathbf{0}; 2\mu')} \left[1 + 2T^{2B}(\mathbf{0}, \mathbf{0}; 2\mu') \frac{1}{V} \sum_{\mathbf{k}}' \frac{N(\hbar\omega_{\mathbf{k}})}{\hbar\omega_{\mathbf{k}}} \right] + \frac{1}{V} \sum_{\mathbf{k}}' \left\{ \frac{\varepsilon_{\mathbf{k}}}{\hbar\omega_{\mathbf{k}}} N(\hbar\omega_{\mathbf{k}}) + \frac{\varepsilon_{\mathbf{k}} - \mu' - \hbar\omega_{\mathbf{k}}}{2\hbar\omega_{\mathbf{k}}} \right\}, \quad (28)$$

where $\hbar\omega_{\mathbf{k}} = (\varepsilon_{\mathbf{k}}^2 - 2\mu'\varepsilon_{\mathbf{k}})^{1/2}$ and we used Eq. (25) to express the density solely in terms of the two-body scattering length a , as in the three-dimensional case.

Comparing with Eq. (15) we note two fundamental differences between the one-loop result and the result within the T -matrix approximation. First, due to the difference in sign of the quasiparticle contribution, Eq. (28) does not have a solution for all temperatures at a fixed density: As shown in Fig. 2 the right-hand side as a function of μ' has a minimum for nonzero temperatures. Hence the Kosterlitz-Thouless transition occurs at the density and temperature corresponding to this minimum.

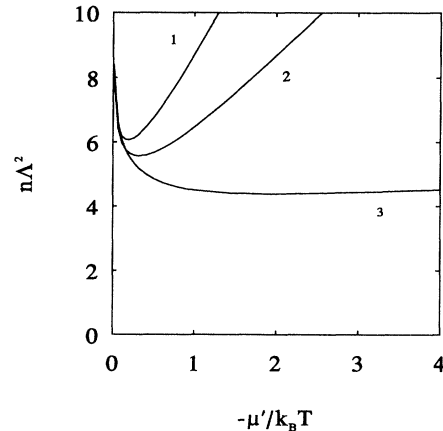


FIG. 2. Density as a function of chemical potential for three different temperatures such that (1) $a/\Lambda = 1 \times 10^{-3}$, (2) $a/\Lambda = 1 \times 10^{-2}$, and (3) $a/\Lambda = 1 \times 10^{-1}$.

Note that in this situation the superfluid density, following from Eq. (16), is nonzero and we immediately obtain a discontinuous behavior at the critical temperature.

Secondly, Eq. (28) explicitly depends on the energy cutoff ε_0 , which on physical grounds is proportional to $-\mu'$ since the correlation length for amplitude fluctuations is proportional to $(mn_0 T^{\text{MB}}/\hbar^2)^{-1/2}$. Furthermore, the proportionality constant c_0 should be such that at the critical temperature the universal jump ($n_s \Lambda_c^2 = 4$) is reproduced. We expect c_0 to depend only slightly on temperature and to be small compared to one for Eq. (25) to be valid and our theory to be self-consistent. This is borne out by a numerical implementation of the above procedure, the result of which is shown in Fig. 3. For convenience we mention that the function $c_0(T)$, within the range of a/Λ displayed in Fig. 3, is very accurately described by the expansion

$$c_0(T) \simeq -A_0 \ln(a/\Lambda) + A_1 + A_2(a/\Lambda),$$

with $A_0 \simeq 4.343 \times 10^{-4}$, $A_1 \simeq 3.123 \times 10^{-2}$, and $A_2 \simeq -9.773 \times 10^{-2}$.

For a better understanding of the validity of this approach it is important to notice the following: From the preceding results we find that at length scales larger than $1/k_0$ the system is described by the effective action

$$S^{\text{eff}}[\psi^*, \psi] = \int_0^{\hbar\beta} d\tau \left\{ \int d\mathbf{x} \psi^*(\mathbf{x}, \tau) \times \left[\hbar \frac{\partial}{\partial \tau} - \frac{\hbar^2 \nabla^2}{2m} + \mu' \right] \psi(\mathbf{x}, \tau) + \int d\mathbf{x} \frac{T^{\text{MB}}}{2} |\psi(\mathbf{x}, \tau)|^4 \right\}, \quad (29)$$

which involves only nontopological phase fluctuations, because the Landau-Ginzburg equation derived from this action does not have vortex solutions due to the fact that the correlation length $(-m\mu'/\hbar^2)^{-1/2}$ is much smaller than $1/k_0$. Therefore all the vortex dynamics has been integrated out and we expect to find the critical behavior

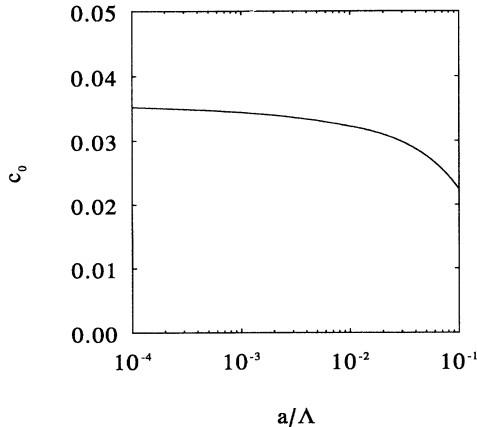


FIG. 3. Proportionality constant c_0 as a function of temperature. (See text for more details.)

predicted by Nelson and Kosterlitz in the superfluid density. Moreover, the phase fluctuations at these length scales lead to a negligible renormalization of the superfluid density, although they are clearly crucial for an account of the algebraic long-range order in the system.

In principle, this still does not justify entirely our procedure for determining the low-momentum cutoff, because an estimate of the higher-order diagrams to the self-energies shows that the T -matrix approximation breaks down close to the critical temperature. This also reveals itself in the right-hand side of Eq. (28) becoming sensitive to the cutoff near its minimum. However, the fact that the right-hand side as a function of μ' has a minimum for nonzero temperatures is anticipated to be an exact property of the two-dimensional Bose gas. Our approach, therefore, not only captures the essential physics but even incorporates in a good approximation the effect of the higher-order corrections near the critical temperature, because the critical region is of $O(T_c/\ln(\Lambda/a))$ and thus small for the temperatures of interest. Herewith, we complete our discussion of the dilute Bose gas in two and three dimensions. In the next section we apply the above theory to spin-polarized atomic hydrogen and calculate various observables, which may be relevant to future experiments.

III. SPIN-POLARIZED ATOMIC HYDROGEN

We consider a gas of doubly spin-polarized atomic hydrogen (the atoms are in the so-called $|b\rangle$ state of the $1s$ hyperfine manifold [9]) adsorbed on a superfluid helium film, with an adsorption energy ε_a of approximately 1 K. The interaction between the adsorbed atoms is in a very good approximation given by the weighted average of the volume triplet potential, using the square of the wave function for the motion perpendicular to the surface as a weighting factor [21]. The two-dimensional scattering length found in this manner is $2.40a_0$ [19].

In Fig. 4(a) we show the critical density of the Kosterlitz-Thouless transition in a temperature interval that covers the experimentally most relevant range of temperatures. As expected, the critical density changes almost linearly with temperature. The deviations from this linear behavior are more clearly displayed if we plot the critical degeneracy parameter versus temperature. This is shown in Fig. 4(b). Note that the $\ln[\ln(\Lambda^2/a^2)]$ behavior, predicted by Fisher and Hohenberg [13], is only found at temperatures much lower than presented here. (The diluteness parameter $\ln[\ln(\Lambda^2/a^2)]$ is about five at 1×10^{-50} K.) We also calculate the superfluid density as a function of temperature for a surface coverage of 1×10^{13} cm^{-2} and give the result in Fig. 5. The superfluid density clearly shows the universal jump at the critical temperature. Moreover, the behavior near T_c is

$$n_s(T) \underset{T \nearrow T_c}{\sim} n_s(T_c) \left\{ 1 + \eta \left[1 - \frac{T}{T_c} \right]^{1/2} \right\} \quad (30)$$

and in agreement with Nelson and Kosterlitz, who find this square-root cusp from a numerical integration of the renormalization-group equations [6]. The value of the

superfluid density at the critical temperature is typically about 75% of the total density, which implies that replacing the superfluid density by the total density in the Nelson-Kosterlitz criterion gives a rather poor estimate for the critical temperature.

In the experiments proposed the adsorbed gas is in contact with a three-dimensional atomic hydrogen gas that is also doubly spin polarized. Therefore we consider this situation and determine the adsorption isotherm in the following manner. In thermal equilibrium we have $\mu_3 = \mu_2 - \varepsilon_a$, with μ_3 and μ_2 the chemical potentials of the volume and surface gases, respectively. For the three-dimensional gas we use $\mu_3 = \mu' + 8\pi n_3 a \hbar^2 / m$, where μ' is obtained from Eq. (18) and we take the value $1.34a_0$ for the scattering length of the triplet potential [22]. Below the critical density of the Kosterlitz-Thouless transition, the two-dimensional chemical potential is also obtained from Eqs. (17) and (18). However, above this density we must use $\mu_2 = \mu' + 2n_2 T^{\text{MB}}(\mathbf{0}, \mathbf{0}, \mathbf{0}; 0)$ together with Eqs. (25) and (28). The results of this procedure are shown in Fig. 6 for temperatures of 50, 100, and 200 mK.

The most prominent feature of the isotherms is the

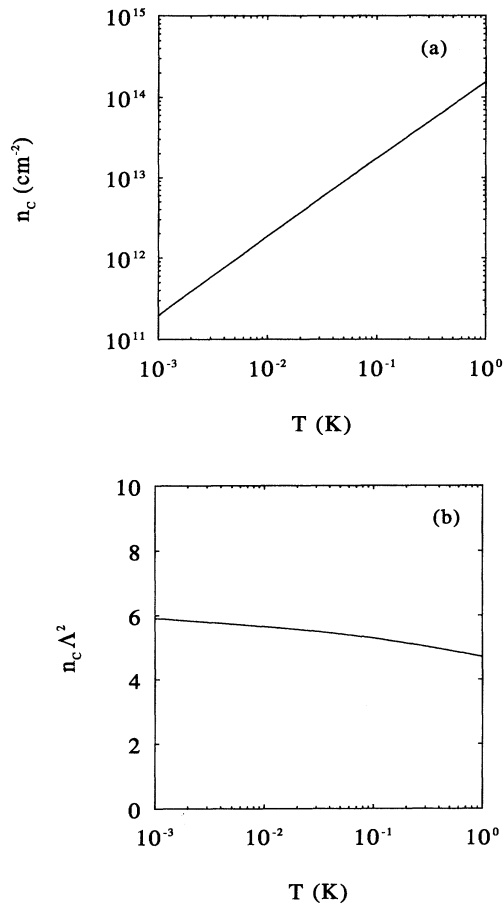


FIG. 4. Two-dimensional criticality condition for the Kosterlitz-Thouless transition in spin-polarized atomic hydrogen. (a) Critical density n_c as a function of temperature. (b) Critical degeneracy parameter $n_c \Lambda^2$ as a function of temperature.

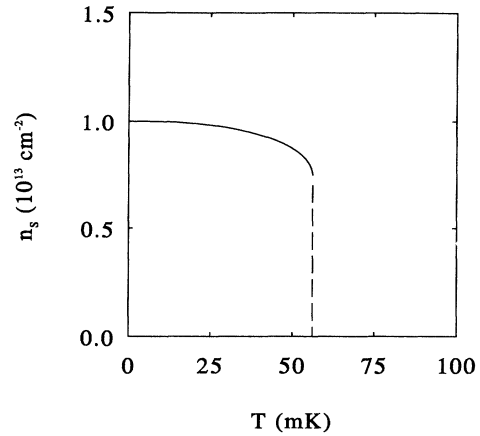


FIG. 5. Superfluid density vs temperature for a surface coverage of adsorbed hydrogen atoms of $1 \times 10^{13} \text{ cm}^{-2}$.

discontinuity in the two-dimensional density as the phase transition takes place. Indeed, the increase in the density is approximately a factor of 3. The importance of this effect for the experimental observation of the Kosterlitz-Thouless transition was already pointed out by Svistunov *et al.* [10]. However, the effect is even larger than anticipated by these authors: They find an increase in the density by a factor that is always smaller than 2, which follows from the fact that they simply use $\mu_2 = \mu' + 2n_2 V_0$ below and $\mu_2 = n_2 V_0$ above the critical density of the Kosterlitz-Thouless transition, respectively. In our approach we take the renormalization of the interaction into account, which for temperatures not too low is more effective in the superfluid phase. This leads to a smaller interaction strength in the superfluid phase as compared to the normal phase and at 100 mK to an increase in density by a factor of more than 3. At higher temperatures the magnitude of the effect is reduced due to the enhanced quasiparticle contribution to the chemical potential.

Finally, we deduce from these isotherms the critical three-dimensional density as a function of temperature. The results are given in Fig. 7. In these figures the criti-

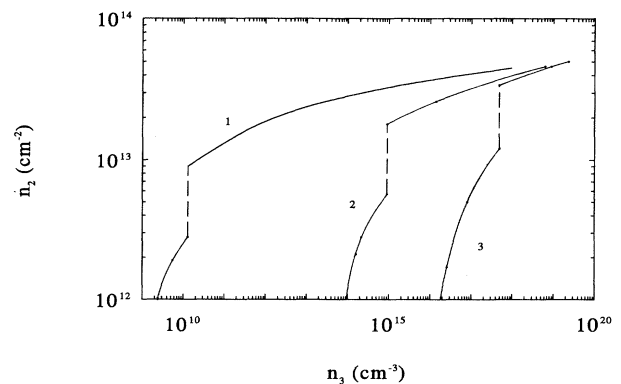


FIG. 6. Adsorption isotherms for atomic hydrogen on superfluid helium. The temperatures are (1) $T=50$ mK, (2) $T=100$ mK, and (3) $T=200$ mK, respectively.

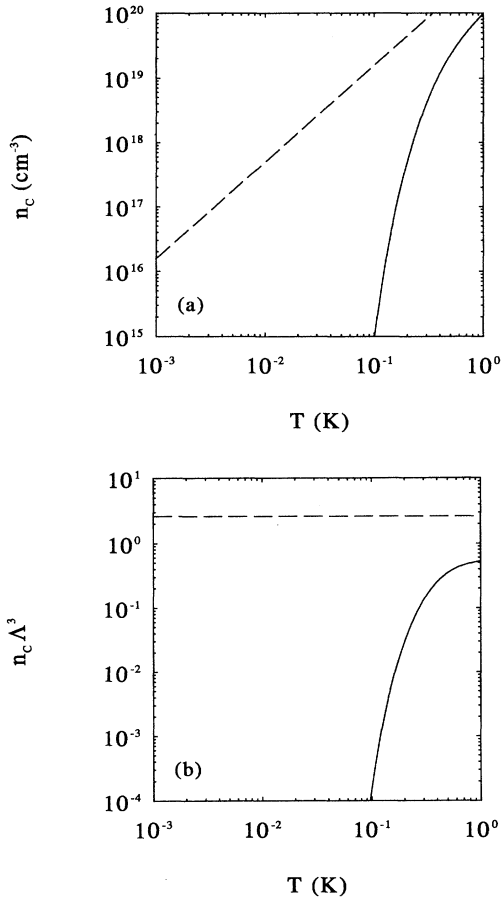


FIG. 7. Three-dimensional criticality condition for the Kosterlitz-Thouless transition in spin-polarized atomic hydrogen. (a) Critical density n_c as a function of temperature. (b) Critical degeneracy parameter $n_c \Lambda^3$ as a function of temperature. In both figures the dashed line corresponds to the condition for the onset of Bose-Einstein condensation.

cal density for Bose-Einstein condensation is also shown. Clearly, in the temperature range of interest, the Kosterlitz-Thouless transition always takes place before condensation in the buffer volume can occur, which *a posteriori* justifies our use of Eqs. (17) and (18) to calculate the chemical potential μ_3 . Of course, a discussion of the condensed regime is possible in principle, and should make use of Eq. (26) to determine the condensate density and subsequently the chemical potential. However, we do not consider this case here since it is not relevant to experiments aiming at the achievement of the Kosterlitz-Thouless transition in spin-polarized atomic hydrogen.

IV. CONCLUSIONS

We have studied the dilute Bose gas in two dimensions and have shown that Popov's one-loop theory is unable to describe a realistic weakly interacting Bose gas, such as atomic hydrogen adsorbed on superfluid helium, at experimentally obtainable temperatures. Therefore we have

formulated a theory that is based on the T -matrix approximation and only requires that the condition $\ln(1/na^2) \gg 1$, instead of the much stronger condition $\ln[\ln(1/na^2)] \gg 1$, is satisfied. In addition, this theory can give an accurate description of the critical region of the gas and, in particular, of the universal jump and the square-root cusp in the superfluid density.

We have applied this theory to spin-polarized atomic hydrogen and have calculated various thermodynamical properties relevant to experiment, such as the superfluid density, the isotherms of the adsorbed surface gas in contact with a buffer volume, and the critical densities of the two- and three-dimensional gases. The latter are of great importance for experiments aiming at the achievement of the Kosterlitz-Thouless transition, because two-body relaxation and especially three-body recombination processes limit the lifetime of the atomic gases. However, on the basis of our results in combination with the analysis of Ref. [10], it seems that successful experiments are feasible and may be expected in the next few years.

ACKNOWLEDGMENTS

We thank Jook Walraven for various enlightening discussions and for his continued interest in the progress of our work. We also acknowledge helpful conversations with Steve Girvin, Kieran Mullen, and Mats Wallin.

APPENDIX

The Hugenholtz-Pines theorem [20] can be derived in the following manner. The effective action of the Bose gas after integration over all Fourier modes with momenta and frequencies unequal to zero is

$$S_{\text{eff}}[\psi_0^*, \psi_0] = \hbar\beta V \sum_{n=1}^{\infty} \frac{\Gamma^{(2n)}}{(n!)^2} |\psi_0|^{2n}, \quad (\text{A1})$$

where $\Gamma^{(2n)}$ denotes the $2n$ -point vertex function [15,16] with all $2n-1$ momentum and frequency arguments equal to zero. The free energy of the system acquires a minimum in a stationary point of the effective action, i.e., a solution of $\delta S_{\text{eff}}/\delta\psi_0^* = 0$. A possible solution, which corresponds to the symmetric phase, is always $\psi_0 = 0$. However, at temperatures low enough a second solution is possible and obeys

$$\sum_{n=1}^{\infty} n \frac{\Gamma^{(2n)}}{(n!)^2} |\psi_0|^{2n-2} = 0. \quad (\text{A2})$$

Expanding the effective action around an arbitrary solution of Eq. (A2) by means of $\psi_0 = (n_0)^{1/2} + a_{0,0}/\sqrt{\hbar\beta V}$ gives for the quadratic part

$$S_{\text{eff}}^{(2)}[a^*, a] = S_{11} a_{0,0}^* a_{0,0} + S_{12} (a_{0,0} a_{0,0} + a_{0,0}^* a_{0,0}^*), \quad (\text{A3})$$

with

$$S_{11} = \sum_{n=1}^{\infty} \left[\frac{n!}{(n-1)!} \right]^2 (n_0)^{n-1} \frac{\Gamma^{(2n)}}{(n!)^2}, \quad (\text{A4})$$

$$S_{12} = \sum_{n=2}^{\infty} \frac{n!}{2!(n-2)!} (n_0)^{n-1} \frac{\Gamma^{(2n)}}{(n!)^2}.$$

From the definition of the normal and anomalous self-energies we know, on the other hand, that $S_{11} \equiv \hbar \Sigma_{11}(\mathbf{0}, 0) - \mu$ and $S_{12} \equiv \hbar \Sigma_{12}(\mathbf{0}, 0)/2$ [cf. Eq. (20)]. Therefore the Hugenholtz-Pines theorem is provided in all orders of perturbation theory if we show that $S_{11} - 2S_{12} = 0$. Substituting Eq. (A4) this means that

$$S_{11} - 2S_{12} = \Gamma^{(2)} + \sum_{n=2}^{\infty} \frac{n}{(n!)^2} (n_0)^{n-1} \Gamma^{(2n)}$$

should be equal to zero, which is clearly the case due to the stationarity condition (A2) for the solution $\psi_0 = (n_0)^{1/2}$.

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- [1] *The Quantum Hall Effect*, edited by R. E. Prange and S. M. Girvin (Springer, Berlin, 1990).
- [2] P. W. Anderson, *Science* **235**, 1196 (1987); C. M. Varma, P. Littlewood, S. Schmitt-Rink, E. Abrahams, and A. Ruckenstein, *Phys. Rev. Lett.* **63**, 1996 (1989).
- [3] N. D. Mermin, *Phys. Rev.* **176**, 250 (1968); N. D. Mermin and H. Wagner, *Phys. Rev. Lett.* **22**, 1133 (1966); P. C. Hohenberg, *Phys. Rev.* **158**, 383 (1967).
- [4] J. M. Kosterlitz and D. J. Thouless, *J. Phys. C* **6**, 1181 (1973).
- [5] J. S. Langer and M. E. Fisher, *Phys. Rev. Lett.* **19**, 560 (1967).
- [6] D. R. Nelson and J. M. Kosterlitz, *Phys. Rev. Lett.* **39**, 1201 (1977).
- [7] I. Rudnick, *Phys. Rev. Lett.* **40**, 1454 (1978).
- [8] D. J. Bishop and J. D. Reppy, *Phys. Rev. Lett.* **40**, 1727 (1978).
- [9] For a review we refer to the papers by T. J. Greytak and D. Kleppner, in *New Trends in Atomic Physics*, edited by C. Grynberg and R. Stora (North-Holland, Amsterdam, 1984), p. 1125; and I. F. Silvera and J. T. M. Walraven, in *Progress in Low Temperature Physics*, edited by D. F. Brewer (North-Holland, Amsterdam, 1986), Vol. 10, p. 139.
- [10] B. V. Svistunov, T. W. Hijmans, G. V. Shlyapnikov, and J. T. M. Walraven, *Phys. Rev. B* **43**, 13412 (1991).
- [11] H. T. C. Stoof, *Phys. Rev. Lett.* **66**, 3148 (1991); *Phys. Rev. A* **45**, 8398 (1992).
- [12] V. N. Popov, *Theor. Math. Phys.* **11**, 565 (1972); *Functional Integrals in Quantum Field Theory and Statistical Physics* (Reidel, Dordrecht, 1983), Chap. 6.
- [13] D. S. Fisher and P. C. Hohenberg, *Phys. Rev. B* **37**, 4936 (1988).
- [14] M. Schick, *Phys. Rev. A* **3**, 1067 (1971).
- [15] See, for example, J. W. Negele and H. Orland, *Quantum Many-Particle Systems* (Addison-Wesley, New York, 1988).
- [16] See, for example, D. J. Amit, *Field Theory, the Renormalization Group, and Critical Phenomena* (World Scientific, Singapore, 1984).
- [17] N. N. Bogoliubov, *J. Phys. (Moscow)* **11**, 23 (1947).
- [18] W. Glöckle, *The Quantum Mechanical Few-Body Problem* (Springer, Berlin, 1983).
- [19] H. T. C. Stoof, L. P. H. de Goey, W. M. H. M. Rovers, P. S. M. Kop Jansen, and B. J. Verhaar, *Phys. Rev. A* **38**, 1248 (1988).
- [20] N. M. Hugenholtz and D. Pines, *Phys. Rev.* **116**, 489 (1958).
- [21] J. P. H. W. van den Eijnde, C. J. Reuver, and B. J. Verhaar, *Phys. Rev. B* **28**, 6309 (1983).
- [22] H. T. C. Stoof, J. M. V. A. Koelman, and B. J. Verhaar, *Phys. Rev. B* **38**, 4688 (1988).