Stress distribution of a hexagonally packed granular pile

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We present exact results for the stress distribution for a pyramid-shaped hexagonally packed granular pile in two dimensions. In particular, we show that the load acting on each grain at the bottom layer is identical. The real granular pile, however, has voids inside and thus may show some deviation from the present results.

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The stress distribution inside the granular pile remains an unsolved problem [1,2]. Unlike the case for fluids, the stress acting on a grain is not proportional to the total weight above it. This is evident when we watch sand flowing through an hourglass. The number of grains flowing out per unit time is constant and independent of the number remaining inside, which was the prime reason why it could be used as an accurate clock in ancient times. In an open system, when we drop grains of sand from above, they form a pyramid-shaped pile with a well-defined angle of repose. An interesting scientific question is what is the stress distribution of the pile, notably, the stress distribution at the bottom layer which is in contact with the floor? One might guess that the stress is proportional to the total weight and thus the force profile at the bottom layer might have the same pyramid shape. Only the direct experimental measurement will be able to tell us the correct answer. The purpose of this Brief Report is to present exact results for the stress distribution of the somewhat ideal granular pile shown in Fig. 1, where grains are packed in a hexagonal lattice. The true granular pile is randomly packed and has voids inside and thus this configuration might be too idealistic. One advantage of studying this ideal granular pile, however, is that one can determine exactly the stress distribution, which might have some relevance to the real system. The result of this investigation is quite surprising and against our intuition: the stress acting on each grain at the bottom layer is *identical.*

Consider the granular pile shown in Fig. 1. The floor that supports the pile is assumed to be *rough* [Fig. 2(c)] so that it provides necessary friction to the grains at the bottom layer. Otherwise, the pile becomes unstable. For simplicity, we use matrices a_{ij} to identify each grain, where i is the label for the layer and j is the order in the layer (Fig. 1). Note that there are total $n(n+1)/2$ grains. We then identify all the forces acting on each grain and draw the force diagram. The magnitude and

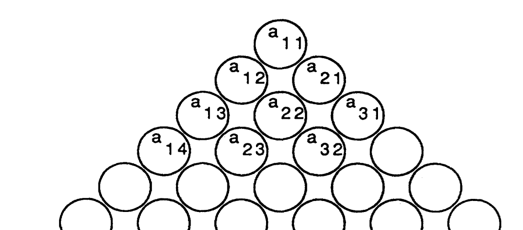

FIG. 1. A hexagonally packed granular pile. Each grain is identified with the matrix a_{ij}.

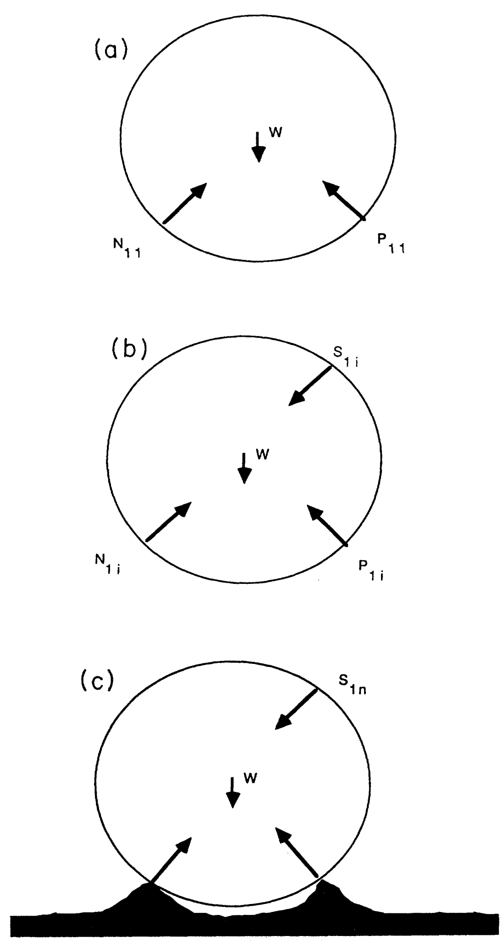

FIG. 2. Force diagrams for (a) a_{11}, (b) a_{1j} with $2 \le j \le n-1$, and (c) a_{1n}. The rough surface that supports the pile is somewhat exaggerated. For simplicity, a case is shown where the grain makes two contacts with the floor. Note, however, that the total sum of the vertical component of all the normal forces must balance out that coming from S_{1n} and the weight W.

direction of each force $\mathbf{F}_i$ is determined by the equilibrium conditions

$$\sum_x F_{ix}=0, \quad \sum_y F_{iy}=0 , \tag{1a}$$

$$\sum_i \tau_i=0 , \tag{1b}$$

where τ_i is a torque due to $\mathbf{F}_i$. Equation (1b) is automatically satisfied if we assume that there exists no friction among grains so that the normal force points toward the center. Further, we notice that all grains at same height are slightly apart from each other due to gravity. Thus the *horizontal* contact forces vanish. We now determine forces acting on each grain.

(a) Grain at the top a_{11}. The force diagram is shown in Fig. 2(a), where $\mathbf{N}_{11}$ and $\mathbf{P}_{11}$ are the normal forces due to contact with other grains and W is the weight of the grain, mg, where m is the mass and g is the acceleration due to gravity. By symmetry, $|\mathbf{N}_{11}|=|\mathbf{P}_{11}|$. Also, from (1a), we find $2|\mathbf{N}_1|\sin\theta=mg$ with $\theta=\pi/3$. Thus we find $|\mathbf{N}_{11}|=W/\sqrt{3}$.

(b) Grains in the first layer, a_{1j}. For each grain in the first layer, the force diagrams are identical for $2\le j\le n-1$ [Fig. 2(b)]. By invoking Newton's third law, we find $S_{1j}=N_{1,j-1}$. Now (1a) gives

$$N_{1j}-P_{1j}=S_{1j} , \tag{2a}$$

$$N_{1j}+P_{1j}=S_{1j}+2W/\sqrt{3} , \tag{2b}$$

whose solutions are easy to find:

$$P_{1j}=W/\sqrt{3} , \tag{3a}$$

$$N_{1j}=N_{1,j-1}+W/\sqrt{3}=jW/\sqrt{3} . \tag{3b}$$

(c) Normal force acting on a_{1n}. The force diagram is schematically shown in Fig. 2(c), where we assume that there are only two contact points between the floor and the grain. This is necessary to satisfy Eq. (1b). We, however, stress that no matter how many contact points there are, the sum of the vertical component of all the normal force $N_{1n}^{(1)}$ must balance out the load provided by the weight of the grain W and the contact force $S_{1,n}$ which is equal to $N_{1,n-1}$. Hence, we find

$$N_{1n}^{(n)}=S_{1,n}\sin\theta+W=N_{1,n-1}\sin\theta+W=(n+1)W/2 , \tag{4}$$

where we used $N_{1,n-1}=(n-1)W/\sqrt{3}$. Since the total weight of the pile $W_T=\sum_{k=1}^{n}kW=n(n+1)W$ and there are n grains at the bottom layer, we find $N_{1n}^{(n)}=W_T/n$.

(d) Grains in the second layer, a_{2j}. Since the pile is symmetrical, the force diagram for a_{21} is the mirror image of a_{12}. Note that there are $n-1$ grains in the second layer. For grains a_{2j} with $2\le j\le n-2$, the force diagram is shown in Fig. 3. Applying Eq. (1a), we find the recursion relations

$$N_{2j}=S_{2j}+W/\sqrt{3} , \tag{5a}$$

$$P_{2j}=T_{2j}+W/\sqrt{3} , \tag{5b}$$

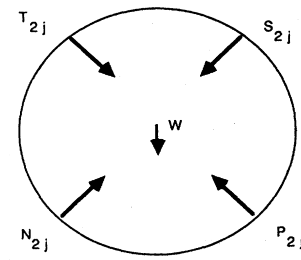

FIG. 3. Force diagrams for a_{2j} with $2\le j\le n-2$.

with the identities $S_{2j}=N_{2,j-1}$ and $T_{2j}=P_{1,j-1}=W/\sqrt{3}$ for $2\le j\le n-2$.

Recalling $T_{22}=P_{11}=W/\sqrt{3}$ and $N_{22}=2W/\sqrt{3}$, we find from (5a)

$$\begin{aligned}N_{2j}&=N_{2,j-1}+W/\sqrt{3}\\&=N_{22}+(j-2)W/\sqrt{3}=jW/\sqrt{3} \quad (2\le j\le n-2) ,\end{aligned} \tag{5c}$$

$$P_{2j}=2W/\sqrt{3} \quad (2\le j\le n-2) . \tag{5d}$$

For the grain at the bottom layer, $a_{2,n-1}$, the sum of the vertical components of all the normal forces $N_{2,n-1}^{(n)}$ can be found easily:

$$N_{2,n-1}^{(n)}=(S_{2,n-1}+T_{2,n-1})\sin\theta+W=(n+1)W/\sqrt{2} . \tag{6}$$

Note that the normal force $N_{2,n-1}^{(n)}=N_{1n}^{(n)}$.

(e) Grains in the ith layer, a_{ij}, for $3\le i\le n-1$. The force diagram for a_{ij} is the same as shown in Fig. 3 except for the labels. Therefore the equations for N_{ij} and P_{ij} must be the same as in (5a) and (5b). We find

$$N_{ij}=S_{ij}+W/\sqrt{3} , \tag{6a}$$

$$P_{ij}=T_{ij}+W/\sqrt{3} , \tag{6b}$$

with identities $S_{ij}=N_{i,j-1}$ and $T_{ij}=P_{i-1,j}$ for $2\le i\le n-i$. The solutions of (6a) and (6b) are

$$N_{ij}=N_{i1}+(j-1)W/\sqrt{3}=jW/\sqrt{3} , \tag{7a}$$

$$P_{ij}=P_{1j}+(i-1)W/\sqrt{3}=iW/\sqrt{3} , \tag{7b}$$

where we have used the fact that

$$P_{i1}=N_{1i}=iW/\sqrt{3}N_{i1}=P_{1i}=W/\sqrt{3} .$$

Hence, the normal force $N_{i,n-i+1}^{(n)}$ is given by

$$\begin{aligned}N_{i,n-i+1}^{(n)}&=(S_{i,n-i+1}+T_{i,n-i+1})\sin\theta+W\\&=(N_{i,n-i}+P_{i-1,n-i+1})\sin\theta+W\\&=[(n-i)+(i-1)+2]W/2\\&=(n+1)W/2 .\end{aligned} \tag{8}$$

Thus the normal force acting on each grain at the bottom layer is identical.

In conclusion, we have shown that for a hexagonally packed pyramid-shaped sandpile, the load acting on each grain at the bottom layer is identical. It would be quite interesting to actually measure the force profile and examine how the real sandpile differs from the ideal one considered in this paper. In passing, the following two points are in order. First, the real sandpile is not packed in a regular manner and there exist voids inside. It would be necessary to use the molecular-dynamics simulation methods to find out the stress distribution of the randomly packed granular piles [3]. Second, by a straightforward calculation, one can show that if grains identified with a_{1n} and a_{nn} are supported by the wall, then the wall will take the load and consequently the load acting on them is slightly reduced.

Note added in proof. After the submission of this paper, the author received a report by K. Liffman, D. Chan, and D. Hughes (unpublished). They obtained the same result presented in this paper. I wish to thank J. Lee for sending me this report.

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[1] See, for example, W. Reisner and E. Rothe, in *Bins and Bunkers for Handling Bulk Materials,* Series on Rock and Soil Mechanics (Trans Tech, Clausthal-Zellerfeld, West Germany, 1971).

[2] For a recent review on the general subject of granular materials, see H. Jaeger and S. Nagle, Science **255**, 1523 (1992), and references therein.

[3] J. Lee and H. Hermann (unpublished).