Explosive instabilities of reaction-diffusion equations including pinch effects

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Particular solutions of reaction-diffusion equations for temperature are obtained for explosively unstable situations. As a result of the interplay between inertial, diffusion, pinch, and source processes, certain "bell-shaped" distributions may grow explosively in time while preserving the shape of the spatial distribution. The effect of the pinch, which requires a density inhomogeneity, is found to diminish the effect of diffusion, or inversely to support the inertial and source processes in creating the explosion. The results may be described in terms of elliptic integrals or, more simply, by means of expansions in the spatial coordinate. An application is the temperature evolution of a burning fusion plasma.

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Evolution equations of reaction-diffusion type play an important role in several branches of modern science. Applications are found in different fields of physics, such as plasma physics [1], laser and semiconductor physics, as well as in chemistry and biology. The reaction-diffusion equations also form an area of active research in mathematics [2–4]. From a fundamental point of view interesting phenomena such as localized nonlinear structures [5], and self-formation of such structures [1,2,6–8] have attracted considerable attention, as has the violation of Painlevé criteria [2].

For physical applications, considerations of the effects of boundaries and simultaneous effects of diffusion, sources, and losses are important [1,9–11]. In plasma physics it has been known for a long time that transport in plasmas does not always obey a simple diffusion equation, where the flux is proportional to the gradient of the quantity studied [12,13]. The total transport problem can in general be formulated in terms of a diffusion matrix where the diagonal terms are the usual diffusion coefficients and the off-diagonal elements describe the pinch effects [13].

Pinch effects in connection with drift-wave transport seem to have been discussed for the first time by Coppi and Spight [14] and by Antonsen, Coppi, and Englade [15]. More general expressions for the pinch effects caused by reactive drift modes are given in Nordman, Weiland, and Jarmén [16]. The tendency of equilibration of the density and temperature scale lengths, indicated by coefficients of order 1 for the pinch terms, remains for the generalized model. The equilibrium effects on the scale lengths give the system a stiffness that contributes to profile consistency [17].

The purpose of the present investigation is to study the reaction-diffusion equation for the evolution of temperature for explosively unstable situations, taking into account the pinch effects. For the occurrence of temperature pinch an inhomogeneity of the density is necessary. In the present work the density profile is assumed constant in time and, for simplicity, chosen identical in form to that of the temperature profile. Even if for practical cases [13] a time variation of the density occurs, it is of principal interest to consider analytically the role of a temperature pinch caused by a time-independent "bell-shaped" density inhomogeneity. The present study is limited to one-dimensional situations, and to free cases (no other boundary than that introduced by the fixed density profile).

The temperature evolution is assumed to be described by the following equation, namely,

$$\frac{\partial T}{\partial t}=a\frac{\partial}{\partial x}\left[T^{\delta}\frac{\partial T}{\partial x}\right]-k\frac{\partial}{\partial x}\left[T^{\delta+1}\frac{1}{n}\frac{\partial n}{\partial x}\right]+cT^{p}n^{\beta}\ , \quad (1)$$

where the terms including spatial derivatives refer to ordinary diffusion and pinch, respectively, and where the last term on the right-hand side of Eq. (1) accounts for the heating, T and n denoting the temperature and density of the plasma. For a tokamak fusion plasma diffusion caused by drift-wave turbulence for temperature gradient driven modes [16] the temperature exponent δ is given by $\delta=1.5$. For a burning fusion plasma the exponent p may be chosen $p=3-2$ and even smaller for extremely hot plasmas, i.e., $T=25$ keV. The coefficients of diffusion a, pinch k, and heat source c are all considered constant and may all be of order 1. The exponent β is chosen equal to 1. Losses, e.g., radiation losses by bremsstrahlung, may be represented by a term $-eT^{q}n$ but are here neglected for the high temperatures considered [1].

For explosive-type solutions the following similarity form of the solutions is assumed, namely,

$$T(x,t)=(t_0-t)^{\mu}\phi(\xi)\ , \quad (2)$$

$$\xi=x/(t_0-t)^{\nu}\ , \quad (3)$$

and assuming, furthermore,

$$n=n_0\frac{\phi}{\phi_0}=h\phi,\quad h=\frac{n_0}{\phi_0}\ , \tag{4}$$

where n_0 and ϕ_0 refer to the central (maximum) values of the corresponding variables.

It is convenient to introduce new variables of space and time; accordingly,

$$(ch/a_T)^{1/2}x \rightarrow x\ , \tag{5}$$

$$cht \rightarrow t\ . \tag{6}$$

Introducing the expressions (2), (3), and (4) in Eq. (1) and matching powers of (t_0-t) leads to the results

$$\mu=-1/(p-1),\quad \nu=0\ . \tag{7}$$

When the pinch effect is considered, $k\neq 0$, it is necessary that $p=\delta+1$ in order to have a solution of the form

$$T=(t_0-t)^{-1/(p-1)}\phi(x)\ , \tag{8}$$

whereas for $k=0$ the expression (2) is a solution with $\nu=\frac{1}{2}(1+\mu\delta)$ if $p\neq\delta+1$ $(p\neq 1)$.

For $p>\delta+1$ the solution in the absence of the pinch effect $(k=0)$ is dominated by the creative nonlinear heating term and corresponds to "collapse," whereas for $p<\delta+1$ the diffusion process dominates and causes an evolution towards an infinite width in a finite time, "anticollapse" [7,8]. For $k\neq 0$ $(k>0)$, and with the further assumption of equal shapes of the profiles for T and n (4), the form of the pinch term in Eq. (1) assures similar behavior. If the exponent of the T in the pinch term was increased the pinch effect became of more importance, counteracting the diffusion (aiding the heating) and thus leading to collapse even for $p=\delta+1$. An anticollapse situation would, correspondingly, occur if the exponent of T in the pinch term was decreased. When the relation (4) is fulfilled the central expansion method [1] can be used to study the corresponding coupled equations for the evolution in time of the amplitude and width of the temperature profile in these more complicated cases.

In expression (8) the time of explosion $t_\infty=t_0$ can be expressed by

$$t_\infty=T_0^{-(p-1)}[(2p+1)/(p-1)]^{(p-1)/p} \qquad (p=\delta+1,\ p\neq 1)\ , \tag{9}$$

where $T_0=T(0,0)$.

Introducing for the ratio of the pinch and diffusion coefficients,

$$K=k/a\ , \tag{10}$$

the remaining equation becomes

$$(1-K)\frac{d}{dx}\left[\phi^{\delta}\frac{d\phi}{dx}\right]=\frac{1}{p-1}\phi-\phi^{p+1}\ . \tag{11}$$

Multipling both sides of Eq. (11) by $\phi^{\delta}d\phi/dx$ and integrating, one obtains

$$x-x_0=\pm\frac{\sqrt{2}}{2}(1-K)^{1/2}(p-1)^{1/2}(\delta+2)^{1/2}\int_{\phi_0}^{\phi}\frac{d\phi}{[\phi^{2-\delta}-\phi_c^{2-\delta}-(\phi^{p+2-\delta}-\phi_c^{p+2-\delta})/\phi_0^{p}]^{1/2}}\ , \tag{12}$$

where

$$\phi_0=[(p+2+\delta)/(p-1)(\delta+2)]^{1/p}\ .$$

In relation (12) ϕ_c denotes a constant of integration. One notices that $\phi_c=0$ and $\phi_c=\phi_0$ give identical results. From Eq. (12) one has, for $x_0=0, \varphi_c=0$,

$$x=\pm\frac{\sqrt{2}}{2}(1-K)^{1/2}(p-1)^{1/2}(\delta+2)^{1/2}\times\int_{\phi_0}^{\phi}\frac{d\phi}{\{\phi^{2-\delta}[1-(\phi/\phi_0)^p]\}^{1/2}}\quad(\phi<\phi_0)\ . \tag{13}$$

The integral in expression (13) may be evaluated in terms of elliptic integrals for certain values of p and δ. In the case where $p=3$, $\delta=2$ one has, for example,

$$I=\int_{\phi_0}^{\phi}\frac{d\phi}{\{\phi^{2-\delta}[1-(\phi/\phi_0)^p]\}^{1/2}}=-\phi_0\int_u^1\frac{dx}{\sqrt{1-x^3}}=-\frac{\phi_0}{4\sqrt{3}}F(\beta,\sin 75^\circ)\ ,$$

where

$$\beta=\arccos\frac{\sqrt{3}-1+u}{\sqrt{3}+1-u}$$

$$F(\varphi,k)=\int_0^{\varphi}\frac{d\alpha}{(1-k^2\sin^2\alpha)^{1/2}}=\int_0^{\sin\varphi}\frac{dx}{\sqrt{[(1-x^2)(1-k^2x^2)]^{1/2}}}\quad(u=\phi/\phi_0)\ . \tag{14}$$

However, the integral I in (14) may be approximated in the central domain of the plasma $(\phi\leq\phi_0)$ by the expression

$$I=-\frac{2\phi_0^{\delta/2}}{p}[1-(\phi/\phi_0)^p]^{1/2}\ . \tag{15}$$

It follows from relations (13) and (15) that

$$x^2/l^2\cong 1-(\phi/\phi_0)^p\ , \tag{16}$$

where

$$l^2=\frac{2}{p^2}(1-K)\left[\frac{p+2+\delta}{(p-1)(\delta+2)}\right]^{\delta/p}(p-1)(\delta+2) \qquad (p\neq 1,\ p\neq 0)\ , \tag{17}$$

or

$$\phi/\phi_0 \cong 1-\left[\frac{x}{L}\right]^2 \quad (x \ll L)\ , \tag{18}$$

where

$$L^2 = pl^2\ , \tag{19}$$

with l^2 given by the relation (17), and where L plays the role of a width of the profile.

The evolution of the temperature can therefore be written

$$T = T_0(1-t/t_\infty)^{-1(p-1)}[1-(x/L)^2+\cdots]\ , \tag{20}$$

where t_∞, the time of explosion, is given by the expression (9) and the relations (17) and (19), where in (18) x is the normalized space coordinate, according to (5).

From the form of Eq. (11) one notices that the effect of the temperature pinch is to diminish the influence of diffusion by the factor $(1-K)$, or inversely to support the inertial and source processes, represented by the first and second terms on the right-hand side of the relation (11), respectively. From the expressions (17) and (19) the pinch will accordingly diminish the width of the localized solution, which will evolve explosively with unchanged shape.

As the temperature of a fusion plasma increases in time the exponent p of the temperature T in the source term will experience a degradation, which will limit the tendency of explosion and eventually lead to saturation [11], an effect which will also be supported by radiation losses and the influence of a finite boundary [1,9,11].

An extension to coupled equations for the evolution of the temperature and density profiles is a challenging problem. Numerical simulation studies of related questions have recently been performed [13,18]. These numerical simulations show that the dynamic coupling between temperature and density may cause saturation of the explosive growth in temperature, followed by a rapid decay in temperature and a simultaneous saturated explosion in density. These extreme cases mark the transition to situations where a coupling between temperature and density may lead to oscillations, approaching equilibria for cases where radiation losses and boundaries are considered [13,18].

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