Large-scale chaos in the neutral-line Hamiltonian

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The Hamiltonian $H=\{p_x^2+p_y^2+[(y^2-\delta x^2)/2]^2\}/2$ is shown to exhibit large-scale chaos over a wide range of the parameter δ. Such Hamiltonians are physically interesting in plasma physics, deriving from the motion of a charged particle in a neutral-line magnetic field. We suggest that this physically realistic Hamiltonian may be a candidate for a fully chaotic system with no stable periodic orbits.

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Until Dahlqvist and Russberg [1] (henceforth DR) exhibited a finite region of regular motion, dynamics in the x^2y^2 potential and similar quartic potentials had been conjectured from numerical evidence (see [1]) to be globally chaotic. By globally chaotic we mean that no stable periodic orbits exist. In this paper we study the Hamiltonian for a charged particle in a neutral-line magnetic field, which yields the x^2y^2 potential as a limiting case, and which may yet be a candidate for global chaos. Using the coordinate system of DR, the potential we have studied is

$$V(x,y)=[2\delta^{1/2}xy+(1-\delta)y^2]^2/8\ , \tag{1}$$

which reduces to the x^2y^2 potential when the parameter δ becomes 1. In this paper we present numerical evidence that a slight decrease in δ from $\delta=1$ destroys the DR regular region. Regular motion also exists for small δ [2]. However, in a search of the allowed phase space over a wide range of δ $(0.257\,686\,2<\delta<0.999\,806\,682)$, we have found no stable regular motion.

We further note that the class of Hamiltonians (1) are physically interesting. The x^2y^2 Hamiltonian is known to plasma physicists as deriving from the motion of a charged particle in a hyperbolic magnetic field (an "X-type" magnetic neutral line) with asymptotes perpendicular. The Hamiltonian for a positively charged particle in a general linear neutral-line magnetic field can be expressed in dimensionless units by

$$H=[p_x^2+p_y^2+(P_z+A_z)^2]/2 \tag{2a}$$

with vector potential

$$A_z=(y^2-\delta x^2)/2\ , \tag{2b}$$

where P_z is the conserved z momentum (which we will take equal to zero) and A_z describes a magnetic field with hyperbolic field lines, vanishing along the z axis, i.e., the neutral line. The parameter δ is related to the angle 2θ between the hyperbola asymptotes by $\tan\theta=\delta^{1/2}$. Figure 1 shows the contours of A_z, which coincide with the magnetic-field lines (and contours of the effective potential) for this two-dimensional field. Particle dynamics in magnetic neutral-line fields has been studied in the space physics and fusion literature [3], where such field structures occur, for example, as the result of plasma instabilities leading to magnetic reconnection. Taking $P_z=0$ in (2), and rotating axes by θ, we return to the potential in DR coordinates (1).

We have studied the dynamics in the neutral-line Hamiltonian with a predictor-corrector algorithm precise to 11 digits per time step. Since the energy can be scaled away when $P_z=0$, we chose a fixed energy of $H_0=0.5$. The results as δ is decreased from 1 are shown in Fig. 2. The DR region undergoes a period-doubling bifurcation near $\delta=0.999\,87$ [Fig. 2(c)]. No further bifurcations were observed, but the two resultant regions simply shrink to unobservability by $\delta=\delta_1$, where $0.999\,821\,865<\delta_1<0.998\,218\,66$. Four other very small regular regions [roughly at $(3.146\,220,0.001\,334)$, $(3.146\,264,0.001\,227)$, $(3.147\,424,0.001\,783)$, and $(3.147\,566,0.001\,713)$] are visible at $\delta=0.9999$ [Fig. 2(b)]. These four regions coalesce into two regions which grow to a maximum size near $\delta=0.999\,83$ [Fig. 2(d)], then shrink rapidly to unobservability for $0.999\,806\,682<\delta_U<0.999\,806\,683$. No regular regions of this type were observed for $\delta<\delta_U$ to the precision of

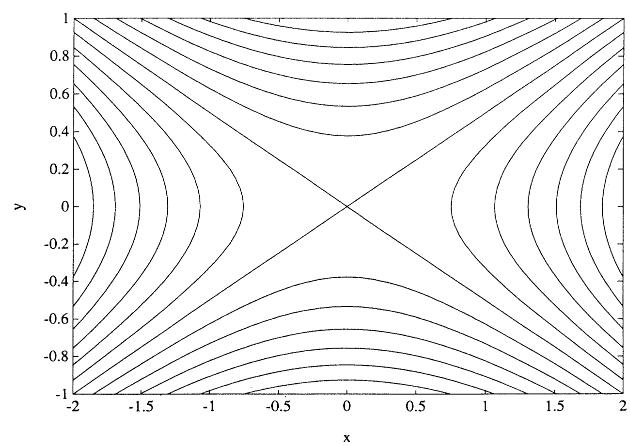

FIG. 1. Contours of the vector potential for the magnetic neutral-line field, representing the magnetic-field lines.

our calculation. The large blank regions in the figures represent chaotic orbits which rapidly leave the vicinity. These regions fill in very slowly as orbits are followed to long times.

The surface-of-section (SOS) method does not prove that these regular regions actually shrink to zero, but we can gain some confidence as follows. We plot in Fig. 3 the length L of the long axis of the DR region as a function of δ, fitting a curve of the form $L=m(\delta-\delta_1)^n$. The fit is best (correlation coefficient $R=0.994$) with $\delta_1=0.999\,821\,868$, $n=0.504$, and $m=0.069$. The other regions behave similarly, vanishing more rapidly. Thus we conjecture that δ_U is the smallest δ for which a finite regular region associated with the DR periodic orbit ex-

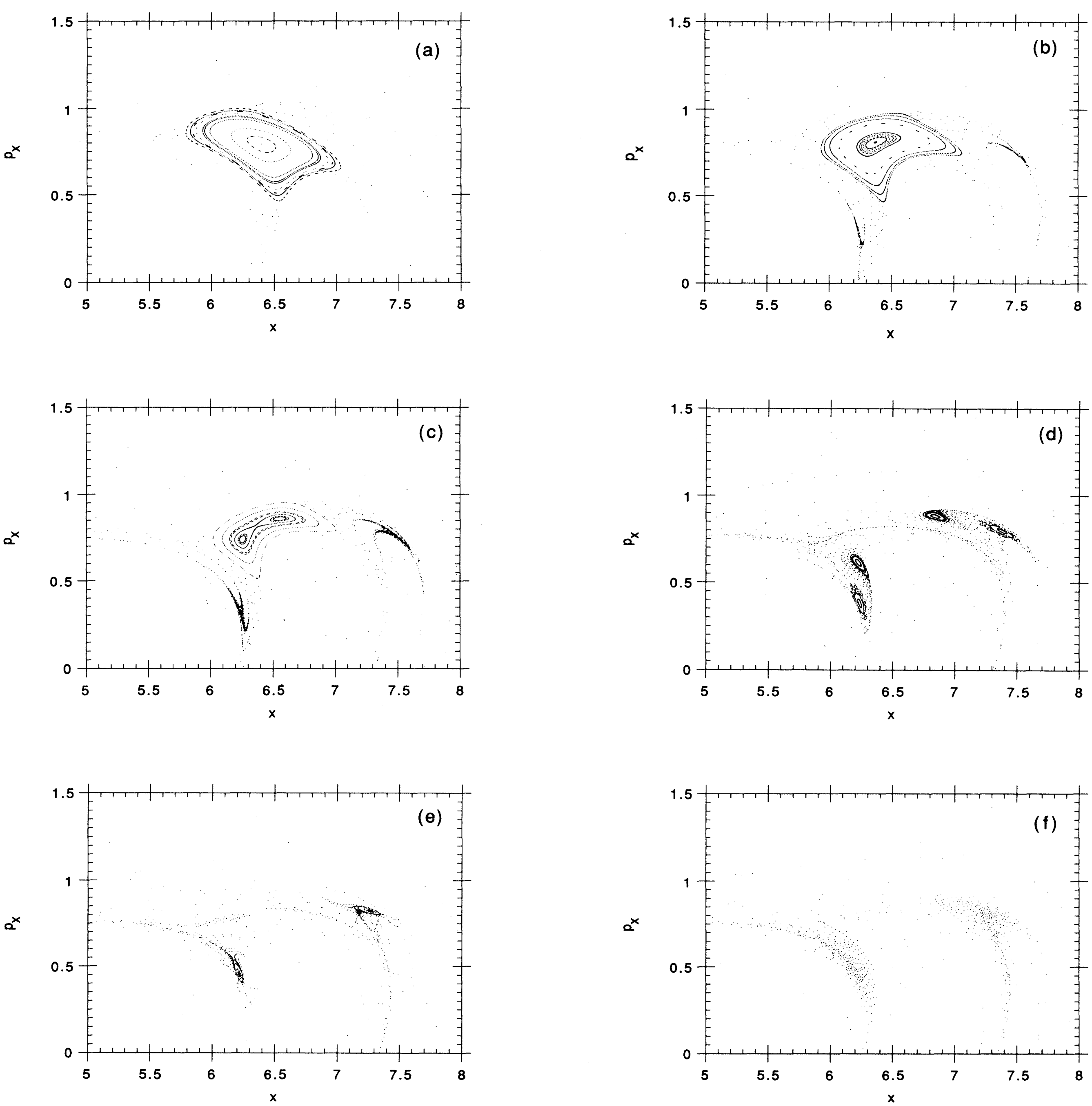

FIG. 2. Sequence of surface-of-section plots in the x-p_x plane at $y=0$, as the parameter δ is decreased from 1. (a) $\delta=1$, (b) $\delta=0.9999$, (c) $\delta=0.999\,87$, (d) $\delta=0.999\,83$, (e) $\delta=0.999\,81$, (f) $\delta=0.999\,80$. The axes have been scaled for readability: x shown is $(x-3.14)\times10^3$; p_x shown is $(p_x-0.001)\times10^3$.

ists.

Of course, the DR regular region is not the only possibility. Burkhart *et al.* [2] have shown that an unrelated regular region exists for small δ, which disappears by $\delta=0.3$. Our more detailed calculations show it to bifurcate and shrink to unobservability at $\delta_L=0.257\,686\,2$. Besides extensive searching near these two known regular regions, we have searched at values of δ between δ_L and δ_U in steps of 0.1 and have found no stable regular orbits. We therefore suggest that the neutral-line Hamiltonian (2) with $P_z=0$ is a candidate for global chaos in this range of $\delta_L<\delta<\delta_U$.

Our search method is a standard surface-of-section technique: We run many chaotic orbits out to long times until the allowed region in x-P_x space is visually dense with SOS points. Then we manually search for regular orbits with initial conditions in any "holes" or less dense regions. When regular orbits do exist we follow them as δ is varied until they are no longer observable to the precision of our calculation. SOS plots ranged from 2000 to 50 000 points, depending on whether regular regions were found. We note that in the coordinate system of (2) the accessible x-v_x phase space is bounded, so we can estimate our effective resolution simply: for an approximately uniform point distribution, the maximum fraction of phase space an unresolved stable island could occupy is on the order of N^{-2} for N SOS points. Thus, at our coarsest resolution, we would not observe a stable island occupying up to $(2.5\times10^{-5})\%$ of the accessible phase space, with considerably better resolution for larger N and in areas where stable regions existed at nearby values of δ.

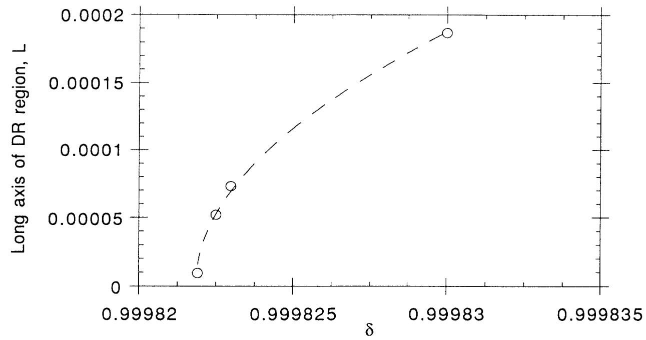

FIG. 3. The length L of the long axis of the DR regular region as a function of δ. The curve $L=0.069(\delta-\delta_U)^{0.504}$ is a best fit when $\delta_U=0.999\,821\,868$.

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[3] For example, see R. F. Martin, Jr. and T. W. Speiser, in *Physics of Space Plasmas (1991),* edited by T. Chang, G. B. Crew, and J. R. Jasperse, SPI Conference Proceedings and Reprint Series, No. 11 (Scientific Publishers, Cambridge, MA, 1992), p. 311; R. F. Martin, Jr., J. Geophys. Res. **91**, 11 985 (1986), and references therein.