Contributions to the electromagnetic wave theory of bounded homogeneous anisotropic media

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An electromagnetic wave theory of bounded homogeneous anisotropic media is developed by using the method of angular spectrum expansion. The series and integral representations of the circular cylindrical wave functions and the spherical wave functions of the first, second, third, and fourth kind for homogeneous anisotropic media are obtained. Each coefficient of the Fourier series of a circular cylindrical wave function is a one-dimensional finite range of integration and every coefficient of the spherical-harmonic-function series is a two-dimensional finite range of integration. The addition theorem of wave functions for anisotropic media can be derived from that of wave functions for isotropic media. Weyl's method of deriving the scalar Green's function in isotropic media is generalized to the study of the dyadic Green's function in anisotropic media. The cold homogeneous magnetoplasma is considered as an illustrative example. For a cold homogeneous magnetoplasma, simplified series representations of wave functions and dyadic Green's functions are given. The distributional singular behavior of the dyadic Green's functions in the source region is investigated and taken into account by solving the static problem and the boundary integral equation is derived.

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I. INTRODUCTION

The spherical wave functions of homogeneous isotropic media were obtained by Mie as early as 1908 [1], and the dyadic Green's functions of homogeneous anisotropic media at large distances for the source were also worked out by Lighthill more than 30 years ago [2]. However, so far there are no corresponding eigenfunctions for anisotropic media [3–6] and no general methods for the common representations of the dyadic Green's functions except in the rectangular coordinate system and in the Fourier-transform domain [7].

Recently, there has been a growing interest in the theory of bounded homogeneous anisotropic media [8–11]. Although the electromagnetic wave theory of unbounded homogeneous anisotropic media is well known [12–14], it can only be numerically treated by the method of finite element, the method of moment, etc. [15–17] in the bounded cases.

The eigenfunctions are very important for the electromagnetic problems in both isotropic and anisotropic media, because the three-dimensional moment methods [17], the coupled-dipole method [18], and the integral-equation technique [3] are all difficult for the computation in the resonance region. Spherical wave functions of the first kind for anisotropic media are useful in designing ferrite circulators and resonators [19,6], studying modulation incident light beams [3], and analyzing characteristics of dense random media with moderate-size particles [20]. The circular cylindrical wave functions of the first kind for anisotropic media are also useful in studying the scattering and guiding of waves by anisotropic cylinders [15,16,21]. In order to apply the efficient recursive algorithm developed by Chew to the electromagnetic scattering by many scatterers and multilayered scatterers of anisotropic media and apply the multiple-scattering theory to random media, vector wave functions of all kinds and their addition theorems are required [19,22]. In fact, in a series of papers [23,24], Chew and Wang showed that the eigenfunctions are helpful for the problems involving both homogeneous and inhomogeneous media. Their excellent works inspire the author to develop the theory of wave functions for anisotropic media.

The point-source radiation in anisotropic media is a subject in plasma physics, ionosphere physics, and materials science [25,12]. Their far-region fields can be calculated by the saddle-point method, which has been used to analyze many nonlinear phenomena [12]. The dyadic Green's functions in the form of separation of variables are required to study Raman and fluorescent scattering by active molecules embedded in a particle [26,27] and to establish T-matrix formulation from Huygens's principle and extinction theorem for homogeneous anisotropic media [22,28]. The general representation of dyadic Green's functions is also required to set up integral equations [29] and study the scattering by weakly nonlinear target [30,31]. For the cold magnetoplasma discussed in this paper, the general representations of dyadic Green's functions in the rectangular coordinate system exist [25], which is very useful for the far fields and the problems in planar-layered media. But there are no counterparts in circular cylindrical and spherical coordinate systems, which are required for the problems in cylindrically and spherically layered media.

The immediate motivation of this work was the desire to obtain the T matrix for a ferrite sphere, which is fundamental in material science [19] and develop the multiple-scattering theory in anisotropic background media which is basic in microwave remote sensing [10]. It would take far too much space to describe here all of the major advances in the subject that we are treating and so we will cite here mainly those papers that have had a direct impact on this work and a few others that deal with closely related topics.

Of greatest importance to this work is the series of papers by Uzunoglu and co-workers [32,3] and the series of papers by Monzon and Damaskos [33,34]. In the early work by Uzunoglu, Cottis, and Fikioris [32], they treated the gyroelectric-cylinder problem analytically. The anisotropy axis of the gyrotropic medium was assumed to coincide with the cylinder axis. Soon thereafter, Monzon and Damaskos treated the scattering by an anisotropic circular cylinder numerically and stated that because of the complex form of the series solution, it results in no substantial improvement in the formulation. Since then, all the authors have treated this kind of problem numerically in conjunction with the method of angular spectrum expansion. The numerical results and comparisons presented in the papers by the above authors were also verified by the other authors [35–37]. In this paper, we shall use the same angular spectrum representations proved by the above papers as the starting point of our theory.

The following facts are also of great importance to this work. First, it is well known that in the static limit the scattering properties of an anisotropic sphere are similar to that of an isotropic ellipsoid [38,39]. Secondly, a harmonic function in the spheroidal coordinate system is a series of that in the spherical coordinate system [40]. Thirdly, a vector spheroidal wave function can be expanded in terms of a series of vector spherical wave functions [41] and a vector elliptic cylindrical wave function can be expanded in terms of a series of vector circular cylindrical wave functions [42]. Therefore, instead of searching the finite-term representation of the wave functions for an anisotropic medium as Monzon and Damaskos did, we study the possibility of series representations of the wave functions for an anisotropic medium. This approach leads to the more obvious answer to the problem, which has not been solved by many mathematicians, physicists, and engineers for a long time [4–6,20].

It is observed that the series representation of elliptical cylindrical wave functions and spheroidal wave functions of the second, third, and fourth kind for an isotropic medium can be obtained by replacing the Bessel functions of first kind in the series representation of elliptical cylindrical wave functions and spheroidal wave functions of first kind by the Bessel functions of second, third, and fourth kind [41,42]. This implies that if a series of Bessel functions of first kind is a solution of a differential equation, when the Bessel functions of first kind are replaced by the Bessel functions of second, third, and fourth kind, the series also is a solution to the problem. This is the key method of this paper, which solves the open problem about the fields in an annular region presented by Monzon and Damaskos [33,34].

The organization of this paper is as follows. In Sec. II we solve the vector wave equation in the circular cylindrical coordinate system and give the Fourier series representations, the integral representations, and the addition theorems of vector wave functions of first, second, third, and fourth kind for homogeneous anisotropic media. We follow a similar procedure in the spherical coordinate system in Sec. III. We turn in Sec. IV to the evaluation of the dyadic Green's functions due to the solenoidal part of the current **J** in both spherical and circular cylindrical coordinate systems by means of the method of angular spectrum expansion. In Sec. V we derive the dyadic Green's function due to the lamellar part of the current **J** through the solution of the corresponding static problem. We also discuss the singularity in the source region and derive the boundary integral equation for the homogeneous anisotropic media. Section VI concludes the work with a discussion of the related problem.

In the following analysis and $\exp(-i\omega t)$ time dependence is assumed and is suppressed throughout.

II. CYLINDRICAL WAVE FUNCTIONS IN ANISOTROPIC MEDIA

In this section, the vector wave functions in the circular cylindrical coordinate system for a homogeneous anisotropic medium are presented. The proposed method is based on the angular spectrum representations of the field in a simply connected domain of an anisotropic medium. The solution of wave functions is obtained by a Fourier transformation technique in conjunction with the concept of characteristic waves.

Assume a magnetoplasma characterized by a dielectric tensor in a rectangular coordinate system

$$\overleftrightarrow{\epsilon}=\epsilon_0\begin{bmatrix}\epsilon_1 & -i\epsilon_2 & 0\\ i\epsilon_2 & \epsilon_1 & 0\\ 2 & 0 & \epsilon_3\end{bmatrix}=\epsilon_0\overleftrightarrow{\epsilon}_r \tag{1}$$

and scalar magnetic permeability $\mu=\mu_0$, where ϵ_0 and μ_0 are the free-space permittivity and permeability, respectively, and $\epsilon_1,\epsilon_2,\epsilon_3$ are constants. The vector wave equation, in the case of $\overleftrightarrow{\epsilon}$ given from (1), is written in the form

$$\nabla\times\nabla\times\mathbf{E}(\mathbf{r})-k_0^2\overleftrightarrow{\epsilon}_r\mathbf{E}(\mathbf{r})=\mathbf{0}\ , \tag{2}$$

where $k_0=\omega(\epsilon_0\mu_0)^{1/2}$ is the free-space wave number. The solution of the wave equation (2) can be examined in the Fourier domain by the transformation

$$\mathbf{E}(\mathbf{r})=\int_{-\infty}^{+\infty}dk_z\int_0^{\infty}dk_\rho\int_0^{2\pi}k_\rho d\varphi_k e^{i\mathbf{k}\cdot\mathbf{r}}\mathbf{E}(\mathbf{k})\ . \tag{3}$$

Substituting (3) into (2) yields

$$\int_{-\infty}^{+\infty}dk_z\int_0^{\infty}dk_\rho\int_0^{2\pi}k_\rho d\varphi_k[k^2\overleftrightarrow{\mathbf{I}}-\mathbf{k}\mathbf{k}-k_0^2\overleftrightarrow{\epsilon}_r]\times\mathbf{E}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}=0\ , \tag{4}$$

where $\overleftrightarrow{\mathbf{I}}=\hat{\mathbf{x}}\hat{\mathbf{x}}+\hat{\mathbf{y}}\hat{\mathbf{y}}+\hat{\mathbf{z}}\hat{\mathbf{z}}$ is the unit dyadic, $k^2=\mathbf{k}\cdot\mathbf{k}$, and $\mathbf{k}=k_z\hat{\mathbf{z}}+\mathbf{k}_\rho$, $\mathbf{k}_\rho=k_x\hat{\mathbf{x}}+k_y\hat{\mathbf{y}}$, $\boldsymbol{\rho}=x\hat{\mathbf{x}}+y\hat{\mathbf{y}}$.

For nontrivial solution of $\mathbf{E}(\mathbf{k})$, the determinant of the matrix in the square brackets of Eq. (4) operating on $\mathbf{E}(\mathbf{k})$ must be equal to zero. Hence [32]

$$\epsilon_1k_\rho^4+[(k_z^2-k_0^2\epsilon_1)(\epsilon_1+\epsilon_3)+k_0^2\epsilon_2]k_\rho^2+[(k_z^2-k_0^2\epsilon_1)+k_0^4\epsilon_2^2]\epsilon_3=0\ . \tag{5}$$

The roots of (5) are designated in the following as $k_\rho=k_{\rho i}$ $(i=1,2,3,4)$, and the corresponding eigenvectors

$\mathbf{M}_i$ are obtained by letting $k_\rho = k_{\rho i}$ in the following equations.

$$\mathbf{E}(k_z,k_\rho,\varphi_k) = E_z(k_z,k_\rho,\varphi_k)\{[A(k_z,k_\rho)\cos\varphi_k - B(k_z,k_\rho)\sin\varphi_k]\hat{\mathbf{x}} + [A(k_z,k_\rho)\sin\varphi_k + B(k_z,k_\rho)\cos\varphi_k]\hat{\mathbf{y}} + \hat{\mathbf{z}}\}/D$$
$$= E_z(k_z,k_\rho,\varphi_k)\mathbf{M}(k_z,k_\rho,\varphi_k) , \tag{6}$$

$$A(k_z,k_\rho) = k_z^2 k_\rho (k_z^2 + k_\rho^2 - k_0^2\epsilon_1) , \quad B(k_z,k_\rho) = i k_z k_\rho k_0^2 \epsilon_2 , \tag{7}$$

$$D = (k_z^2 - k_0^2\epsilon_1)(k_\rho^2 + k_z^2 - k_0^2\epsilon_1) - k_0^4\epsilon_2^2 . \tag{8}$$

Returning to Eq. (3), we get

$$\mathbf{E}(\mathbf{r}) = \int_{-\infty}^{+\infty} e^{ik_z} \int_0^{2\pi} d\varphi_k \sum_{i=1}^{4} k_{\rho i}\mathbf{M}_i(k_z,k_{\rho i},\varphi_k) \times \exp[i\mathbf{k}_{\rho i}\cdot\mathbf{r}]E_z(k_{\rho i},\varphi_k) \times A(k_z)dk_z . \tag{9}$$

Note that the integrand function in Eq. (9) is periodic with respect to φ_k and that

$$k_{\rho 3} = -k_{\rho 1}, \quad k_{\rho 4} = -k_{\rho 2} . \tag{10}$$

It is easy to end up with the expression [32]

$$\mathbf{E}(\mathbf{r}) = \int_{-\infty}^{+\infty} A(k_z)e^{ik_z z}dk_z \int_0^{2\pi} d\varphi_k \sum_{q=1}^{2} \mathbf{M}_q(k_z,k_{\rho q},\varphi_k) \times \exp[i\mathbf{k}_{\rho q}\cdot\boldsymbol{\rho}] \times C_{qz}(\varphi_k) , \tag{11}$$

where

$$E_z = A(k_z)C_{qz}(\varphi_k) . \tag{12}$$

The special form of (11) suggests the use of the well-known expansions [33,42]

$$e^{i\mathbf{k}_{\rho q}\cdot\boldsymbol{\rho}} = \sum_{m=-\infty}^{+\infty} i^m J_m(k_{\rho q}\rho)e^{-im\varphi_k}e^{im\varphi} , \tag{13}$$

$$C_{qz} = \sum_{n=-\infty}^{+\infty} a_{nq}e^{in\varphi_k} . \tag{14}$$

We finish with the following expression:

$$\mathbf{E}(\mathbf{r}) = \sum_{q=1}^{2} \int_{-\infty}^{+\infty} A(k_z)e^{ik_z z}dk_z \sum_{n=-\infty}^{+\infty} a_{nq} \sum_{m=-\infty}^{+\infty} e^{im\varphi}\left[\int_0^{2\pi} i^m J_m(k_{\rho q}\rho)e^{i(n-m)\varphi_k}\mathbf{M}_q(k_z,k_{\rho q},\varphi_k)d\varphi_k\right]$$
$$= \sum_{q=1}^{2}\sum_{n=-\infty}^{+\infty} a_{nq}\int_{-\infty}^{+\infty} A(k_z)dk_z \left\{ e^{ik_z z}\sum_{m=-\infty}^{+\infty} e^{im\varphi}\left[\int_0^{2\pi} i^m J_m(k_{\rho q}\rho)e^{i(n-m)\varphi_k}\mathbf{M}_q(k_z,k_{\rho q},\varphi_k)d\varphi_k\right]\right\}$$
$$= \sum_{q=1}^{2}\sum_{n=-\infty}^{+\infty} a_{nq}\int_{-\infty}^{+\infty} A(k_z)dk_z\mathbf{E}_{nq}(k_z,\rho,\varphi,z) , \tag{15}$$

$$\mathbf{E}_{nq}(k_z,\rho,\varphi,z) = \sum_{m=-\infty}^{+\infty} e^{im\varphi}\mathbf{E}_{nmq}(k_z,\rho)e^{ik_z z} , \tag{16}$$

$$\mathbf{E}_{nmq}(k_z,\rho) = \int_0^{2\pi} i^m J_m(k_{\rho q}\rho)e^{i(n-m)\varphi_k}\mathbf{M}_q(k_z,k_{\rho q},\varphi_k)d\varphi_k . \tag{17}$$

It is noted that $\mathbf{E}_{nq}(k_z,\rho,\varphi,z)$ is one of the eigenfunctions of vector wave equations (2). It is also noted that after the replacement of Bessel functions of the first kind in Eq. (16) by Bessel functions of the other kind, $E_{nq}(k_z,\rho,\varphi,z)$ is one of the eigenfunctions of Eq. (2) too. Therefore we may use Eqs. (15)–(17) as the general definition of cylindrical wave functions of various kinds for an anisotropic medium if $J_m(k_{\rho q}\rho)$ is replaced by the Bessel functions of the corresponding kind.

Equation (17) can also be expanded into a form resembling the vector wave solution. For this purpose, introducing the vector wave functions $\mathbf{L},\mathbf{M},\mathbf{N}$ discussed in Tai's textbook [43] [see also Eq. (22) and Appendix A], after some straightforward algebra we obtain

$$\mathbf{E}_{nq}^{(i)}(k_z,\rho,\varphi,z) = \int_0^{2\pi}\left[\sum_{m=-\infty}^{+\infty} a_{nmq}(\varphi_k)\mathbf{M}_m^{(i)}(k_{\rho q},\mathbf{r}) + \beta_{nmq}(\varphi_k)\mathbf{N}_m^{(i)}(k_{\rho q},\mathbf{r}) + \gamma_{nmq}(\varphi_k)\mathbf{L}_m^{(i)}(k_{\rho q},\mathbf{r})\right]d\varphi_k$$
$$(i=1,2,3,4) , \tag{18}$$

where

$$A_m = i^m e^{i(n-m)\varphi_k} , \tag{19a}$$

$$\alpha_{nmq} = \frac{k_z}{2k_{\rho q}}[(A_{m-1} - A_{m+1})M_{xq} + (A_{m-1} + A_{m+1})M_{yq}] + M_{zq} A_m , \tag{19b}$$

$$\beta_{nmq} = -\frac{k_z}{2k_{\rho q}}[(A_{m+1} + A_{m-1})M_{xq} + (A_{m-1} - A_{m+1})M_{yq}] , \tag{19c}$$

$$\gamma_{nmq} = [ik_{\rho q}(M_{xq}\cos\varphi_k + M_{yq}\sin\varphi_k) + M_{zq} ik_z] A_m . \tag{19d}$$

The integral representations of circular cylindrical wave functions of various kinds for anisotropic media are as follows:

$$\mathbf{E}_{nq}^{(i)}(k_z,\rho,\varphi,z) = \int_{C_i} \mathbf{M}_q(k_z,k_{\rho q},\varphi_k) e^{in\varphi_k} e^{i\mathbf{k}_{\rho q}\cdot\boldsymbol{\rho}} d\varphi_k e^{ik_z z} \quad (i=1,2,3,4) , \tag{20}$$

where c_i is the complex integration path of the circular cylindrical wave functions of ith kind for isotropic media [42]. From the formalism of this section, it is easy to see that Eq. (22) is valid for $i=1$. Substituting the integral representations of the circular cylindrical wave function [42] into Eq. (18), exchanging the order of integration, comparing the series of wave functions of second, third, and fourth kind with that of the first kind, and recalling the integral representation of the wave function of the first kind, we end the proof.

The addition theorems of circular cylindrical wave functions for anisotropic media can be directly obtained by using the counterpart of isotropic media [22,42] in Eq. (18) [41].

The above method is easily adapted to solve the wave equation for the more general anisotropic media. By inspecting Sec. 7.2 of Stratton's treatise [42], it is easy to see that our circular cylindrical wave-function theory for anisotropic media is an extension of that for isotropic media.

For the magnetoplasma, after some straightforward algebra, by grouping properly the terms involved in the integration and by introducing the **L**, **M**, **N** cylindrical vector wave functions [32], we end up with the following expressions:

$$\mathbf{E}_{nq}^{(i)}(k_z,\rho,\varphi,z) = \pi i^n [A_M(k_z,k_{\rho q})\mathbf{M}_n^{(i)}(\mathbf{r},k_z,k_{\rho q}) + A_N(k_z,k_{\rho q})\mathbf{N}_n^{(i)}(\mathbf{r},k_z,k_{\rho q}) + A_L(k_z,k_{\rho q})\mathbf{L}_n^{(i)}(\mathbf{r},k_z,k_{\rho q})] , \tag{21}$$

where

$$\mathbf{M}_n^{(i)}(\mathbf{r},k_z,k_{\rho q}) = \left[\frac{in}{\rho} Z_n^{(i)}(k_{\rho q}\rho)\hat{\boldsymbol{\rho}} - \frac{\partial Z_n^{(i)}(k_{\rho q}\rho)}{\partial\rho}\hat{\boldsymbol{\varphi}}\right] \times \exp[i(n\varphi + k_z z)] , \tag{22a}$$

$$\mathbf{N}_n^{(i)}(\mathbf{r},k_z,k_{\rho q}) = \frac{1}{k_q}\left[ik_z \frac{\partial Z_n^{(i)}(k_{\rho q}\rho)}{\partial\rho}\hat{\boldsymbol{\rho}} - \frac{nk_z}{\rho} Z_n^{(i)}(k_{\rho q}\rho)\hat{\boldsymbol{\varphi}} + k_{\rho q}^2 Z_n^{(i)}(k_{\rho q}\rho)\hat{\mathbf{z}}\right] \times \exp[j(n\varphi + k_z z)] , \tag{22b}$$

$$\mathbf{L}_n^{(i)}(\mathbf{r},k_z,k_{\rho q}) = -\left[\frac{\partial Z_n^{(i)}(k_{\rho q}\rho)\hat{\boldsymbol{\rho}}}{\partial\rho} + \frac{n}{\rho} Z_n^{(i)}(k_{\rho q}\rho)\hat{\boldsymbol{\varphi}} + ik_z Z_n^{(i)}(k_{\rho q}\rho)\hat{\mathbf{z}}\right] \exp[i(n\varphi + k_z z)] , \tag{22c}$$

$$A_M = -2k_z k_0^2 \epsilon_2 / D , \tag{22d}$$

$$A_N = -2k_q(k_q^2 - k_0^2\epsilon_1)/D + 2[1 + k_{\rho q}^2(k_q^2 - k_0^2\epsilon_1)/D]/k_q , \tag{22e}$$

$$A_L = -2ik_z[1 + k_{\rho q}^2(k_q^2 - k_0^2\epsilon_1)/D]/k_q^2 , \tag{22f}$$

$$k_q = (k_{\rho q}^2 + k_z^2)^{1/2} , \tag{22g}$$

and

$$Z_n^{(i)}(k_{\rho q}\rho) = \begin{cases} J_n(k_{\rho q}\rho) , & i=1 \\ Y_n(k_{\rho q}\rho) , & i=2 \\ J_n(k_{\rho q}\rho) + iY_n(k_{\rho q}\rho) = H_n^{(1)}(k_{\rho q}\rho) , & i=3 \\ J_n(k_{\rho q}\rho) - iY_n(k_{\rho q}\rho) = H_n^{(2)}(k_{\rho q}\rho) , & i=4 \end{cases} \tag{22h}$$

is the nth-order Bessel function of the ith kind. D is given in Eq. (8).

It is well known that in a gyroelectric medium, the electromagnetic fields can be derived by two scalar wave functions [13]. In this way we get the same results. This is a theoretical verification of the method of plane-wave angular spectrum.

III. SPHERICAL WAVE FUNCTIONS IN ANISOTROPIC MEDIA

Following a procedure similar to that of Sec. II, we find the eigenwave angular spectrum expansions of electromagnetic fields inside a spherical region of homogeneous anisotropic media as follows [3,12–14]:

$$\mathbf{E}(\mathbf{r}) = \sum_{n=1}^{2} \int_0^{2\pi} d\varphi_k \int_0^{\pi} d\theta_k k_n \sin\theta_k C_n(\mathbf{k}) \times \left[\sum_{j=1}^{3} E_{jn}(\theta_k,\varphi_k)\hat{\mathbf{e}}_j\right] e^{i\mathbf{k}_n\cdot\mathbf{r}} , \tag{23}$$

$$\mathbf{H}(\mathbf{r})=\sum_{n=1}^{2}\int_0^{2\pi}d\varphi_k\int_0^{\pi}d\theta_k k_n\sin\theta_k C_n(\mathbf{k})\times\left[\sum_{j=1}^{3}H_{jn}(\theta_k,\varphi_k)\hat{\mathbf{e}}_j\right]e^{i\mathbf{k}_n\cdot\mathbf{r}}\ , \tag{24}$$

where $\hat{\mathbf{e}}_j$ is the unit vector in the jth direction; $C_n(\mathbf{k})$ is the amplitude function to be determined; k_n the wave number satisfying the dispersion equation, and the quantities in the square brackets are the electric field and magnetic field of the nth eigenwave. For the homogeneous magnetoplasma discussed in this paper, k_n and $\mathbf{E}_n$ are determined by Eqs. (5)–(8) with $k_\rho=k\cos\theta_k$, $k_z=k\sin\theta_k$. The theoretical analysis and the numerical results in Refs. [3,32–37] give the proofs of Eqs. (23) and (24).

A standard way of expanding the unknown angular spectrum amplitude is to use the series of orthogonal complete harmonic functions on a spherical surface [42]:

$$C_n(\mathbf{k})=\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}b_{nlm}P_l^m(\cos\theta_k)e^{im\varphi_k}\ , \tag{25}$$

where $P_l^m(\cos\theta)$ is the Legendre function. The particular form of (23) and (24) suggests the use of identity

$$e^{i\mathbf{k}_n\cdot\mathbf{r}}=\sum_{l'=0}^{\infty}\sum_{m'=-l'}^{+l'}A_{l'm'}(\theta_k,\varphi_k)j_{l'}(k_nr)P_{l'}^{m'}(\cos\theta)e^{im'\varphi}\ , \tag{26}$$

$$A_{l'm'}(\theta_k,\varphi_k)=i^{l'}(2l'+1)\frac{(l'-m')!}{(l'+m')!}P_{l'}^{m'}(\cos\theta_k)e^{-im'\varphi_k}\ .$$

As shown in Appendix A we have

$$\sum_{j=1}^{3}E_{jn}(\theta_k,\varphi_k)\hat{\mathbf{e}}_j e^{i\mathbf{k}_n\cdot\mathbf{r}}=\sum_{l'=0}^{\infty}\sum_{m'=-l'}^{+l'}\alpha_{nl'm'}\mathbf{M}_{l'm'}^{(1)}(k_nr)+\beta_{nl'm'}\mathbf{N}_{l'm'}^{(1)}(k_nr)+\gamma_{nl'm'}\mathbf{L}_{nl'm'}^{(1)}(k_nr)\ , \tag{27}$$

where $\mathbf{L}^{(1)},\mathbf{M}^{(1)},\mathbf{N}^{(1)}$ are vector spherical wave functions of the first kind. $\mathbf{L},\mathbf{M},\mathbf{N}$ and α,β,γ are given in Appendix A. Substituting Eqs. (25) and (27) into Eq. (23), we find that

$$\mathbf{E}(\mathbf{r})=\sum_{n=1}^{2}\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}b_{nlm}\mathbf{E}_{nlm}(\mathbf{r})\ , \tag{28}$$

$$\mathbf{E}_{nlm}(\mathbf{r})=\int_0^{\pi}\int_0^{2\pi}k_n\sin\theta_k d\theta_k d\varphi_k P_l^m(\cos\theta_k)e^{im\varphi_k}\times\sum_{l'=0}^{\infty}\sum_{m'=-l'}^{+l'}[\alpha_{nl'm'}\mathbf{M}_{l'm'}^{(1)}(k_n\mathbf{r})+\beta_{nl'm'}\mathbf{N}_{l'm'}^{(1)}(k_n\mathbf{r})+\gamma_{nl'm'}\mathbf{L}_{l'm'}^{(1)}(k_n\mathbf{r})]\ . \tag{29}$$

Equation (29) is the definition of spherical wave functions of the first kind and the solution to the Maxwell equations. Because spherical Bessel functions and spherical Hankel functions satisfy the same equation and the same recursive relations, when the spherical Bessel functions are replaced by spherical Bessel functions of the other kinds (second, third, and fourth), Eq. (29) also satisfies the Maxwell equations. Thus we have the general definitions of spherical wave functions for anisotropic media as follows:

$$\mathbf{E}_{nlm}^{(i)}(\mathbf{r})=\int_0^{\pi}\int_0^{2\pi}k_n\sin\theta_k d\theta_k d\varphi_k P_l^m(\cos\theta_k)e^{im\varphi_k}\times\sum_{l'm'}\alpha_{nl'm'}\mathbf{M}_{l'm'}^{(i)}(k_n\mathbf{r})+\beta_{nl'm'}\mathbf{N}_{l'm'}^{(i)}(k_n\mathbf{r})+\gamma_{nl'm'}\mathbf{L}_{l'm'}^{(i)}(k_n\mathbf{r})\quad(i=1,2,3,4)\ , \tag{30}$$

where $\mathbf{L}^{(i)},\mathbf{M}^{(i)},\mathbf{N}^{(i)}$ are vector spherical wave functions of ith kind; $\mathbf{L}^{(i)},\mathbf{M}^{(i)}$, $\mathbf{N}^{(i)}$ and α,β,γ are given in Appendix A.

From (30), it is evident that spherical eigenwave functions of all kinds for anisotropic media are accordingly determined when the media parameters are given.

We can separate the function of (θ,φ) and the functions of r in Eq. (30) using three vector spherical harmonic functions [20]:

$$\mathbf{E}_{nl'm'}^{(i)}(\mathbf{r})=\sum_{l,m}\sum_{j=1}^{3}\mathbf{V}_{jlm}(\theta,\varphi)Z_{njlm}^{(i)}(k_nr)\ , \tag{31}$$

where $Z_{njlm}^{(i)}(k_nr)$ is a double-integral representation in a finite domain $[0,\pi]\times[0,2\pi]$. $Z_{njlm}^{(i)}(k_nr)$ and $\mathbf{V}_{jlm}(\theta,\varphi)$ are given in Appendix B.

From the orthogonality of $\mathbf{V}_{jlm}$ on a spherical surface, we treat the scattering of an anisotropic sphere by plane waves strictly by solving the linear equations whose coefficient of each is a double integral only. But in the literature [3], the linear equations are derived from Galerkin's method whose coefficient of each is a double integral involved in an infinite double series. The algorithmic complexity of Ref. [3] is reduced considerably. For a homogeneous magnetoplasma, our method is even more efficient because the integral about φ_k can be carried out analytically as shown in Appendix B.

We can directly obtain the addition theorem of spherical wave functions for homogeneous anisotropic media by using that for homogeneous isotropic media in (30) [41].

From the derivation of this section, it is obvious that the integral representation of spherical wave functions of first kind for homogeneous anisotropic media is

$$\mathbf{E}^{(1)}_{nlm}(\mathbf{r})=\int_0^{\pi}\int_0^{2\pi}P_l^m(\cos\theta_k)e^{im\varphi_k}\times\sum_{j=1}^{3}E_{jn}(\theta_k,\varphi_k)\hat{\mathbf{e}}_j e^{i\mathbf{k}_n\cdot\mathbf{r}}k_n\times\sin\theta_k d\theta_k d\varphi_k \ . \quad (32)$$

Substituting the integral representations of spherical wave functions [44],

$$h_{l'}^{(1)}(k_n r)P_{l'}^{m'}(\cos\theta)e^{im'\varphi}=\frac{1}{4\pi i^{l'}}\int_0^{2\pi}\int_B e^{i\mathbf{k}_{n'}\cdot\mathbf{r}}P_{l'}^{m'}(\cos u)e^{im'v}\sin u\, dv\, du \ , \quad (33a)$$

$$j_{l'}(k_n r)P_{l'}^{m'}(\cos\theta)e^{im'\varphi}=\frac{1}{4\pi i^{l'}}\int_0^{2\pi}\int_0^{\pi} e^{i\mathbf{k}_{n'}\cdot\mathbf{r}}P_{l'}^{m'}(\cos u)e^{im'v}\sin u\, dv\, du \ , \quad (33b)$$

$$\mathbf{k}_n\cdot\mathbf{r}=k_n r[\cos\theta\cos u+\sin\theta\sin u\cos(\varphi-v)] \ , \quad (33c)$$

into Eq. (30), exchanging the order of integration, comparing the series obtained, and recalling Eq. (31), we obtain the integral representations of the spherical wave functions of third kind for homogeneous anisotropic media

$$\mathbf{E}^{(3)}_{nlm}(\mathbf{r})=\int_B\int_0^{2\pi}P_l^m(\cos\theta_k)e^{im\varphi_k}\times\sum_{j=1}^{3}E_{jn}(\theta_k,\varphi_k)\hat{\mathbf{e}}_j e^{i\mathbf{k}_n\cdot\mathbf{r}}k_n\times\sin\theta_k d\theta_k d\varphi_k \ , \quad (34a)$$

where B is the specific complex integration path [44]. So we give the general integral representations of spherical wave functions of all kinds for homogeneous anisotropic media

$$\mathbf{E}^{(i)}_{nlm}(\mathbf{r})=\int_{C_i}\int_0^{2\pi}P_l^m(\cos\theta_k)e^{im\varphi_k}\times\sum_{j=1}^{3}E_{jn}(\theta_k,\varphi_k)\hat{\mathbf{e}}_j e^{i\mathbf{k}_n\cdot\mathbf{r}}k_n\times\sin\theta_k d\theta_k d\varphi_k \ , \quad (34b)$$

where C_i is the corresponding integration path for isotropic media [44].

It is shown that in Eq. (34a), the integration paths of anisotropic cases are the same as that of isotropic cases for outgoing waves, and Eq. (34b) is returned to the isotropic cases when k_n, E_{jn} are independent of θ_k and φ_k by inspecting Sec. 7.12 of Stratton's treatise [42]. It is evident that our spherical wave-function theory for anisotropic media unifies that for the isotropic media [42].

IV. DYADIC GREEN'S FUNCTIONS IN ANISOTROPIC MEDIA

It is well known that the electric field due to the lamellar part of the current $\mathbf{J}$ is given by the solution of the static problem and the electric field due to the solenoidal part of the current $\mathbf{J}$ is given by the solution of the dyadic wave equation [45,46].

The dyadic Green's functions due to the solenoidal part of the current $\mathbf{J}$, $\overleftrightarrow{\mathbf{G}}(\mathbf{r},\mathbf{r}')$, satisfy the vector wave equation,

$$\nabla\times\nabla\times\overleftrightarrow{\mathbf{G}}(\mathbf{r},\mathbf{r}')-\omega^2\mu_0\epsilon_0\overleftrightarrow{\epsilon}_r\cdot\overleftrightarrow{\mathbf{G}}(\mathbf{r},\mathbf{r}')=-\overleftrightarrow{\mathbf{I}}\delta(\mathbf{r}-\mathbf{r}') \ , \quad (35)$$

where ω is the angular frequency and μ_0 is the isotropic permeability of the medium. Using the Fourier transform and the identity

$$\delta(\mathbf{r}-\mathbf{r})=\frac{1}{8\pi^3}\int_{-\infty}^{+\infty}d^3k\, e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \ , \quad (36)$$

we obtain [12–14]

$$\overleftrightarrow{\mathbf{G}}(\mathbf{k},\mathbf{r}')=[\mathbf{k}\mathbf{k}-|\mathbf{k}|^2\overleftrightarrow{\mathbf{I}}+\omega^2\mu_0\epsilon_0\overleftrightarrow{\epsilon}_r]^{-1}e^{-i\mathbf{k}\cdot\mathbf{r}'} \ . \quad (37)$$

The dispersion relation is considered as a general eigenvalue problem of Hermite matrices for a lossless homogeneous anisotropic media; the three-dimensional Fourier transform of the dyadic Green's function $\overleftrightarrow{\mathbf{G}}(\mathbf{r},\mathbf{r}')$ can be represented as [12–14]

$$\overleftrightarrow{\mathbf{G}}(\mathbf{k},\mathbf{r}')=\int_{-\infty}^{+\infty}d^3\mathbf{r}\overleftrightarrow{\mathbf{G}}(\mathbf{r},\mathbf{r}')e^{-i\mathbf{k}\cdot\mathbf{r}}=\sum_{n=1}^{2}\frac{\mathbf{E}_n\mathbf{E}_n^{*}e^{-i\mathbf{k}\cdot\mathbf{r}'}}{(k^2-k_n^2)N_n^2} \ , \quad (38)$$

where $\mathbf{E}_n$ and $N_n=\mathbf{E}_n^{*}\cdot\overleftrightarrow{\epsilon}_r\cdot\mathbf{E}_n$ are the eigenwave vector and normalized value of the nth eigenwave vector.

The inverse Fourier transform of Eq. (38) is

$$\overleftrightarrow{\mathbf{G}}(\mathbf{r},\mathbf{r}')=\int_{-\infty}^{+\infty}d^3\mathbf{k}\sum_{n=1}^{2}\frac{\mathbf{E}_n^{*}\mathbf{E}_n e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}}{(k^2-k_n^2)N_n^2} \ . \quad (39)$$

The general representations in the spherical coordinate system have not been obtained so far, although many researchers have treated the problems in cylindrical coordinate systems [25].

From

$$\mathbf{E}(\mathbf{r})=\int\overleftrightarrow{\mathbf{G}}(\mathbf{r},\mathbf{r}')\cdot\mathbf{J}(\mathbf{r}')d\mathbf{r}' \quad (40)$$

we can see that it represents outgoing waves which can be expanded by the spherical wave functions of third kind for anisotropic media, in which the integral for θ_k is along path B as shown in Sec. III. This is supported by Weyl in calculating the point-source radiation in isotropic media [42]. As $k_\rho=k_n\sin\theta_k$ and k_n is a bounded function of θ_k,φ_k, if and only if the integration path B for θ_k is chosen, the correct complex integration path for k_ρ may be obtained as shown by Weyl [42]. So we have

$$\overleftrightarrow{\mathbf{G}}(\mathbf{r},\mathbf{r}')=\frac{1}{8\pi^3}\int_0^{2\pi}d\varphi_k\int_B d\theta_k\int_{-\infty}^{+\infty}\sum_{n=1}^{2}\frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}}{(k^2-k_n^2)N_n^2}\mathbf{E}_n^*\mathbf{E}_n dk\sin\theta_k k^2$$
$$=\frac{i}{8\pi^2}\int_0^{2\pi}\int_B d\varphi_k d\theta_k\sum_{n=1}^{2}\frac{e^{i\mathbf{k}_n\cdot(\mathbf{r}-\mathbf{r}')}}{N_n^2}\mathbf{E}_n^*\mathbf{E}_n\sin\theta_k k_n\ . \qquad (41)$$

In terms of the relation between integral representation and the series representation of spherical wave functions of third kind for anisotropic media as well as the relation between the spherical wave functions for anisotropic media of third kind and that of first kind, we have

$$\overleftrightarrow{\mathbf{G}}(\mathbf{r},\mathbf{r}')=\frac{i}{8\pi^2}\sum_{n=1}^{2}\int_0^{2\pi}d\varphi_k\int_0^{\pi}d\theta_k\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}A_{lm}(\theta_k,\varphi_k)\frac{\mathbf{E}_n\mathbf{E}_n^*}{N_n^2}h_l^{(1)}(k_n|\mathbf{r}-\mathbf{r}'|)P_l^m(\cos\theta')e^{im\varphi'}\sin\theta_k k_n\ , \qquad (42)$$

where (θ',φ') means $\theta_{rr'},\varphi_{rr'}$ [19] and $A_{lm}(\theta_k,\varphi_k)$ is defined by Eq. (A4).

For the cold homogeneous magnetoplasma, $N_n=\mathbf{E}_n^*\cdot\epsilon_r\cdot\mathbf{E}_n$ is independent of φ_k. Using the formula (B8) and (B9) given in Appendix B, we can manipulate analytically the integral about φ_k.

Equation (42) can be written in the form of separation of variables when $r<r'$ and $r>r'$.

We have derived the dyadic Green's function for isotropic media and obtained the same result using the method described above. So the present theory is a natural extension of Weyl's method.

The above method is easily adapted to the circular cylindrical coordinate system. The result is

$$\overleftrightarrow{\mathbf{G}}(\mathbf{r},\mathbf{r}')=\frac{i}{8\pi}\int_{-\infty}^{+\infty}e^{ik_z(z-z')}\sum_{q=1}^{2}\int_0^{2\pi}d\varphi_k\mathbf{M}_q^*\mathbf{M}_q 1/(\mathbf{M}_q^*\cdot\overleftrightarrow{\epsilon}\cdot\mathbf{M}_q)\sum_{m=-\infty}^{+\infty}i^m e^{im\varphi_k}H_m^{(1)}(k_{\rho q}|\rho-\rho'|)e^{im\varphi'}\ , \qquad (43)$$

where φ' means $\varphi_{rr'}$. For the magnetoplasma, it is easy to see that the integral about φ_k can be evaluated analytically in terms of Eqs. (6)–(8) and then the series is converted into a few terms.

V. SCALAR DYADIC GREEN'S FUNCTION AND BOUNDARY INTEGRAL EQUATION

The dyadic Green's function due to the lamellar part of the current $\mathbf{J}$, $\overleftrightarrow{\mathbf{G}}^l$, can be deduced in a few simple steps beginning with the solution of the static problem [38].

$$\nabla\cdot\overleftrightarrow{\epsilon}\cdot\nabla G=\epsilon_1\frac{\partial^2 G}{\partial x^2}+\epsilon_1\frac{\partial^2 G}{\partial y^2}+\epsilon_3\frac{\partial^2 G}{\partial z^2}=-\delta(\bar{r}-\bar{r}')\ . \qquad (44)$$

The solution to Eq. (44) is

$$G=\frac{1}{4\pi\epsilon_1(\epsilon_3)^{1/2}}\frac{1}{(x^2/\epsilon_1+y^2/\epsilon_1+z^2/\epsilon_3)^{1/2}}\ , \qquad (45)$$

$$\overleftrightarrow{\mathbf{G}}^l=\frac{1}{k_0^2}\nabla\nabla G\ . \qquad (46)$$

In the Huygens principle for the anisotropic media [22], if the field point $\mathbf{r}$ is located in the boundary of the media, the singularity of the dyadic Green's function has to be properly accounted for. Usually, we can evaluate the divergent integral by using an exclusion volume surrounding $\mathbf{r}$. If we choose the exclusion volume as a sphere, we have [47]

$$\mathbf{E}=\overleftrightarrow{\mathbf{S}}\cdot\mathbf{J}\ , \qquad (47)$$

$$\overleftrightarrow{\mathbf{S}}=\frac{1}{4\pi k_0^2(\epsilon_1^2\epsilon_3)^{1/2}}\times\int_{\partial v}\nabla\frac{1}{(x^2/\epsilon_1+y^2/\epsilon_1+z^2/\epsilon_3)^{1/2}}dS\hat{\mathbf{n}}\ . \qquad (48)$$

The tensor $\overleftrightarrow{\mathbf{S}}$ can be evaluated by elementary functions [39]. With the above results, we derived the boundary integral equations for a homogeneous anisotropic medium easily [22].

$$-\mathbf{E}_{\rm inc}(\mathbf{r})=\tfrac{1}{2}\overleftrightarrow{\mathbf{S}}\cdot\mathbf{E}(\mathbf{r})+⨍_s dS'\{-j\omega\hat{\mathbf{n}}\times\mathbf{H}(\mathbf{r}')\cdot\overleftrightarrow{\mathbf{G}}_1(\mathbf{r}',\mathbf{r})+\hat{\mathbf{n}}\times\mathbf{E}(\mathbf{r}')\cdot\nabla'\times\overleftrightarrow{\mathbf{G}}_1(\mathbf{r}',\mathbf{r})\}\ , \qquad (49)$$

where $\overleftrightarrow{\mathbf{G}}_1(\mathbf{r},\mathbf{r}')=\overleftrightarrow{\mathbf{G}}(\mathbf{r},\mathbf{r}')+\overleftrightarrow{\mathbf{G}}^l(\mathbf{r},\mathbf{r}')$, with $\overleftrightarrow{\mathbf{G}}$ and $\overleftrightarrow{\mathbf{G}}^l$ given by (42) and (46) respectively, and $\mathbf{E}_{\rm inc}(\mathbf{r})$ is the incident field.

VI. CONCLUSION AND DISCUSSION

We have developed a method to transform the eigenwave theory of unbounded homogeneous anisotropic media to the eigenfunction theory of bounded homogeneous anisotropic media. Our wave-function theory for anisotropic media unifies the classical wave-function theory for isotropic media [42]. We have also found a unified method to study the dyadic Green's functions in both isotropic and anisotropic media. The method is expounded by taking the cold homogeneous magnetoplasma as an example. Although only a magnetoplasma is considered in this paper, because of the symmetry of

Maxwell's equations the present analysis is directly applicable to ferrite materials as well [19,38]. This study shows that for an arbitrary homogeneous anisotropic region, the electromagnetic fields can be expanded by a series of the eigenwave functions; each of the series is also a series just like the isotropic case in the spheroidal coordinate system [48]. So we can use the simple method of mode matching to analyze the guiding, resonance, radiation, and scattering in anisotropic media. The computational-complexity analysis shows that the algorithmic complexity of an anisotropic spherical scatterer is less than that of an isotropic nonspherical one and that the complexity of a spherically layered anisotropic scatterer is less than that of an isotropic nonspherically layered one. The theory developed in this paper is applicable to any waves and fields, such as the elastic waves in elastic anisotropic media, whose eigenwave numbers and eigenwave vectors can be determined and to any bounded coordinate systems, such as elliptic cylindrical and spheroidal coordinate systems. The canonical solutions of wave functions and dyadic Green's functions for anisotropic media given in this paper are useful in the further studying of problems in anisotropic cylindrically layered and spherically layered structures [22,43].

The present work, in comparison with previous works, has the following differences.

(i) Although the method of angular spectrum expansion has been successfully used to solve the problem in a simple connected domain by many authors, so far the extension to the annular domain is unavailable. We solve this open problem.

(ii) Except for the theoretical work of the early paper for the special medium and in a special coordinate system [32], all the authors used the representations of the plane-wave angular spectrum to simplify the numerical computation; we utilized the verified representations for developing the electromagnetic wave theory of bounded homogeneous anisotropic media. This theory includes that of isotropic media. So there are counterparts between the present theory and the classical one [42]. This is helpful in the formulation of boundary-value problems. Therefore the focal point of this paper is different from that of previous works.

(iii) As discussed in Sec. III of this paper, the theory developed in this paper simplified the numerical computation considerably. For the problems needing the internal fields [26,27], the advantage of the method presented in this paper is more obvious. Invoking the asymptotic expression of Bessel functions for large order [49], we can easily treat the truncation of series appearing in this paper.

(iv) The present theory facilitates the utilization of the character of media. As described in this paper, all the integrals about φ_k can be computed analytically for a magnetoplasma. It is appropriate to point out at this time that this paper's simplified theory for a cold homogeneous magnetoplasma is easily generalized to the case of a moving uniaxial medium, a moving gyrotropic medium [13], and the bianisotropic medium discussed in Sec. 3.3d of Ref. [13].

(v) Although our theory overcomes the difficulty of separation of variables [4] from the view of mathematical physics, the essence of the present theory is the physical insight into the problem.

There are many topics in the electromagnetic wave theory of bounded homogeneous anisotropic media to be studied which are left for future papers on applications.

APPENDIX A: EXPANSION OF A VECTOR PLANE WAVE

Let

$$\hat{\mathbf{x}}e^{i\mathbf{k}_n\cdot\mathbf{r}}=\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\alpha^x_{nlm}\mathbf{M}^{(1)}_{lm}+\beta^x_{nlm}\mathbf{V}^{(1)}_{lm}+\gamma^x_{nlm}\mathbf{L}^{(1)}_{lm}\ , \tag{A1a}$$

$$\hat{\mathbf{y}}e^{i\mathbf{k}_n\cdot\mathbf{r}}=\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\alpha^y_{nlm}\mathbf{M}^{(1)}_{lm}+\beta^y_{nlm}\mathbf{N}^{(1)}_{lm}+\gamma^y_{nlm}\mathbf{L}^{(1)}_{lm}\ , \tag{A1b}$$

$$\hat{\mathbf{z}}e^{i\mathbf{k}_n\cdot\mathbf{r}}=\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\alpha^z_{nlm}\mathbf{M}^{(1)}_{lm}+\beta^z_{nlm}\mathbf{N}^{(1)}_{lm}+\gamma^z_{nlm}\mathbf{L}^{(1)}_{lm}\ , \tag{A1c}$$

where [42]

$$\mathbf{L}^{(i)}_{lm}(k_n r,\theta,\varphi)=\nabla\Psi^{(i)}_{lm}\ , \tag{A2a}$$

$$\mathbf{M}^{(i)}_{lm}(k_n r,\theta,\varphi)=\nabla\times(\mathbf{r}\Psi^{(i)}_{lm})\ , \tag{A2b}$$

$$\mathbf{N}^{(i)}_{lm}(k_n r,\theta,\varphi)=\frac{1}{k_n}\nabla\times\mathbf{M}^{(i)}_{lm}\ , \tag{A2c}$$

and

$$\Psi^{(i)}_{lm}=z^{(i)}_l(k_n r)P^m_l(\cos\theta)e^{im\varphi}\ , \tag{A3a}$$

$$z^{(i)}_l(k_n r)=\begin{cases} j_l(k_n r)\ , & i=1\\ y_l(k_n r)\ , & i=2\\ j_l(k_n r)+iy_l(k_n r)=h^{(1)}_l(k_n r)\ , & i=3\\ j_l(k_n r)-iy_l(k_n r)=h^{(2)}_l(k_n r)\ , & i=4\ .\end{cases} \tag{A3b}$$

Taking the divergence and curl of (A1a), substituting the identity (26), we get

$$\gamma^x_{nlm}=-i\sin\theta_k\cos\varphi_k\frac{A_{lm}}{k_n}\ ,\quad A_{lm}=i^l(2l+1)\frac{(l-m)!}{(l+m)!}P^m_l(\cos\theta_k)e^{-im\varphi_k}\ , \tag{A4}$$

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}A_{lm}\mathbf{M}^{x(1)}_{lm}=\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\alpha^x_{nlm}k_n\mathbf{N}^{(1)}_{lm}+\beta^x_{nlm}k_n\mathbf{M}^{(1)}_{lm}\ . \tag{A5}$$

In terms of the formula given in Ref. [50], we obtain

$$\alpha^x_{nlm}=\frac{k_n}{2l(l+1)}\left\{\frac{m}{2l+3}[(l+m+1)(l+m+2)A_{l+1,m+1}-A_{l-1,m+1}]\right.$$

$$\left.+\frac{m+1}{2l-1}[-(l-m-1)(l-m)A_{l+1,m-1}+A_{l-1,m-1}]\right\}, \tag{A6}$$

$$\alpha^x_{n10}=\tfrac{3}{10}k_n A_{12}\ , \tag{A7}$$

$$\beta^x_{nlm}=\frac{k_n}{2l(l+1)}[(l-m)(l+m+1)A_{l+1,m}+A_{l-1,m}]\ , \tag{A8}$$

$$\beta^x_{n10}=\tfrac{1}{2}k_n A_{11}\ . \tag{A9}$$

Similarly, we get the $\alpha^y_{nlm}, \beta^y_{nlm}, \gamma^y_{nlm}, \alpha^z_{nlm}, \beta^z_{nlm}$, and γ^z_{nlm}. So we have

$$\alpha_{nlm}=E_{nx}\alpha^x_{nlm}+E_{ny}\alpha^y_{nlm}+E_{nz}\alpha^z_{nlm}\ , \tag{A10}$$

$$\beta_{nlm}=E_{nx}\beta^x_{nlm}+E_{ny}\beta^y_{nlm}+E_{nz}\beta^z_{nlm}\ , \tag{A11}$$

$$\gamma_{nlm}=E_{nx}\gamma^x_{nlm}+E_{ny}\gamma^y_{nlm}+E_{nz}\gamma^z_{nlm}\ . \tag{A12}$$

APPENDIX B: REPRESENTATIONS OF $Z^{(i)}_{njlm}(k_n r)$ AND $\mathbf{V}_{jlm}(\theta,\varphi)$

Let [20,44]

$$\mathbf{V}_{1lm}(\theta,\varphi)=\mathbf{P}_{lm}(\theta,\varphi)=\hat{\mathbf{r}}P_l^m(\cos\theta_k)e^{im\varphi_k}\ , \tag{B1}$$

$$\mathbf{V}_{2lm}(\theta,\varphi)=\mathbf{B}_{lm}(\theta,\varphi)=r\nabla P_l^m(\cos\theta)e^{im\varphi}=\mathbf{r}\times\mathbf{C}_{lm}(\theta,\varphi)$$

$$=\hat{\boldsymbol{\theta}}\frac{dP_l^m(\cos\theta)}{d\theta}e^{im\varphi}+\hat{\boldsymbol{\varphi}}\frac{im}{\sin\theta}P_l^m(\cos\theta)e^{im\varphi}\ , \tag{B2}$$

$$\mathbf{V}_{3lm}(\theta,\varphi)=\mathbf{C}_{lm}(\theta,\varphi)=\hat{\boldsymbol{\theta}}\frac{im}{\sin\theta}P_l^m(\cos\theta)e^{im\varphi}-\hat{\boldsymbol{\varphi}}\frac{dP_l^m(\cos\theta)}{d\theta}e^{im\varphi}\ . \tag{B3}$$

After some straightforward algebra, we get

$$Z^{(i)}_{njlm}(k_n r)=\int_0^\pi\int_0^{2\pi}k_n\sin\theta_k d\theta_k d\varphi_k P_{l'}^{m'}(\cos\theta_k)e^{im\varphi_k}f^{(i)}_{njlm}(k_n r)\ , \tag{B4}$$

$$f^{(i)}_{n1lm}(k_n r)=\beta_{nlm}\frac{l(l+1)z_l^{(i)}(k_n r)}{k_n r}+\gamma_{nlm}z_l'^{(i)}(k_n r)\ , \tag{B5}$$

$$f^{(i)}_{n2lm}(k_n r)=\beta_{nlm}\frac{[k_n r z_l^{(i)}(k_n r)]'}{k_n r}+\gamma_{nlm}\frac{z_l^{(i)}(k_n r)}{k_n r}\ , \tag{B6}$$

$$f^{(i)}_{n3lm}(k_n r)=\alpha_{nlm}z_l^{(i)}(k_n r)\ , \tag{B7}$$

where $z_l^{(i)}(k_n r)$ are given in Eq. (A3).

In terms of Eqs. (6)–(8), using the following formula [51]:

$$(2l+1)\sin\theta e^{i\varphi}P_l^m(\cos\theta)e^{im\varphi}=P_{l-1}^{m+1}(\cos\theta)e^{i(m+1)\varphi}-P_{l+1}^{m+1}(\cos\theta)e^{i(m+1)\varphi}\ , \tag{B8}$$

$$(2l+1)\sin\theta e^{-i\varphi}P_l^m(\cos\theta)e^{im\varphi}=(l-m+2)(l-m+1)P_{l+1}^{m-1}(\cos\theta)e^{i(m-1)\varphi}$$

$$-(l+m)(l+m-1)P_{l-1}^{m-1}(\cos\theta)e^{i(m-1)\varphi}\ , \tag{B9}$$

we can evaluate the integral about φ_k analytically.

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