Negative-energy waves in an inhomogeneous force-free Vlasov plasma with sheared magnetic field

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The conditions for the existence of negative-energy electrostatic waves (which could be nonlinearly unstable and cause anomalous transport) are investigated for the case of an inhomogeneous force-free Vlasov-Maxwell equilibrium with sheared magnetic field. The method of investigation consists in evaluating the general expression for the second-order wave energy derived by Morrison and Pfirsch [Phys. Rev. A **40**, 3898 (1989); Phys. Fluids B **2**, 1105 (1990)] in the form given by Correa-Restrepo and Pfirsch [Phys. Rev. A **45**, 2512 (1992)]. In Cartesian coordinates, the equilibrium magnetic field is given by $\mathbf{B}^{(0)}=B^{(0)}(\sin ay\,\mathbf{e}_x+\cos ay\,\mathbf{e}_z)$. In this case, there is an electric current parallel to the magnetic field, and the charged particles of any species belong (according to the values of their constants of the motion) to either one of two essentially different groups, either to the group of gyrating particles (the overwhelming majority in all cases of interest), which move around the field lines, their motion being confined to a certain y region around $y=\mathcal{Y}_v$, or to the group of swinging particles, which move freely in the y direction. The two groups of particles must be investigated separately. Owing to the presence of the electric current associated with nonvanishing a, the equilibrium distribution function $f_\nu^{(0)}=f_\nu^{(0)}(\mathcal{H},\mathcal{U})$ [with $\mathcal{H}(\mathbf{v})$ the energy and $\mathcal{U}(\mathbf{v},y)$ a certain velocity variable] of at least one particle species ν *must* be anisotropic, unlike in the homogeneous case. If any $f_\nu^{(0)}$ has the property $v_y\partial f_\nu^{(0)}/\partial v_y>0$ for some $\mathcal{H}$ and $\mathcal{U}$, negative-energy waves exist for any wave number $\mathbf{k}$, irrespective of its magnitude and orientation. If $v_y(\partial f_\nu^{(0)}/\partial v_y)<0$ holds, only the waves with a component $\mathbf{k}_{\parallel 0}$ of $\mathbf{k}$ in the direction of $\mathbf{B}^{(0)}(y_0)$ can possess negative energy. If $v_y(\partial f_\nu^{(0)}/\partial v_y)<0$, but $A\equiv k_{\parallel 0}^2\langle w_{\parallel 0}\rangle\langle \mathbf{e}_B(y_0)\cdot(\partial f_\nu^{(0)}/\partial\mathbf{v})\rangle>0$ ($w_{\parallel 0}$ is a parallel velocity, the $\langle\ \rangle$ represent a certain averaging process), there are negative-energy waves with no restriction imposed on either $k_{\parallel 0}$ (other than $k_{\parallel 0}\neq 0$) or the spatial variation of the perturbation perpendicular to $\mathbf{B}^{(0)}$. This result agrees with that obtained for a homogeneous plasma by Morrison and Pfirsch [Phys. Fluids B **3** (2), 271 (1991)] in the context of drift-kinetic theory. If both $v_y(\partial f_\nu^{(0)}/\partial v_y)<0$ and $A<0$, negative-energy modes also exist. In this case, the characteristic length for the variation of the perturbation perpendicular to $\mathbf{B}^{(0)}$ is $\lesssim a^{-1}$, which, since a^{-1} is the shear length and is usually very large, is *not* an important restriction, and the possible *parallel* wave numbers are generally limited to a certain interval related to the magnitude of the gyroradius of the gyrating particles, this also being so in the homogeneous case. The results of that case are of course regained by taking the limit of vanishing shear, $a\rightarrow 0$. The present results show that large perpendicular wave numbers $k_\perp$ are *not* necessary for the existence of negative-energy waves in the system under consideration, a feature that enhances the relevance of these modes.

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I. INTRODUCTION

Considering arbitrary perturbations of general Vlasov-Maxwell equilibria, Morrison and Pfirsch [1,2] derived expressions for the second variation of the free energy and concluded that negative-energy perturbations (which are potentially dangerous because they may become nonlinearly unstable and cause anomalous transport [3,4]) exist in any Vlasov-Maxwell equilibrium whenever the unperturbed distribution function $f_\nu^{(0)}$ of any particle species ν deviates from monotonicity and/or isotropy in the vicinity of a single point, i.e., whenever the condition

$$(\mathbf{v}\cdot\mathbf{k})\left[\mathbf{k}\cdot\frac{\partial f_\nu^{(0)}}{\partial\mathbf{v}}\right]>0 \tag{1}$$

holds for any particle species ν for some position vector $\mathbf{x}$ and velocity $\mathbf{v}$ and for some vector $\mathbf{k}$. The proof of this result obtained by Morrison and Pfirsch was based on infinitely strongly localized perturbations. This raises the question of the degree of localization actually required for negative-energy modes to exist in a certain equilibrium. Studying a homogeneous Vlasov-Maxwell plasma with constant magnetic field, Correa-Restrepo and Pfirsch [5] showed that negative-energy waves exist for any deviation of the equilibrium distribution function of any of the species from monotonicity and/or isotropy, *without having to impose any restricting conditions on the perpendicular wave number* $k_\perp$, i.e., without requiring large $k_\perp$. These investigations are extended in the present paper to the more interesting case of an inhomogeneous, y-dependent, force-free equilibrium with a sheared magnetic field. Although the calculations are considerably more involved than in the case of the homogeneous magnetic field, substantial simplification of the problem is achieved by the introduction of appropriate coordinates in $\mathbf{v}$-y space and by a convenient representation of the perturbations. It is concluded that negative-energy modes exist in this particular inhomogeneous plasma as well whenever any of the equilibrium distribution functions deviates from monotonicity and/or isotropy (in fact, owing to the inhomogeneity of the configuration, the equilibrium distribution function of at least one particle species must be anisotropic), and that large perpendicular wave numbers are not required in this case either. If there is only anisotropy, the presence of shear merely requires that the perturbations have a characteristic variation length perpendicular to the equilibrium magnetic field $\mathbf{B}^{(0)}$ of the order $\lesssim a^{-1}$, which is *not* an important restriction; i.e., negative-energy modes persist without any major modification in the presence of shear, a feature which enhances their importance.

The equilibrium electromagnetic field is introduced in Sec. II, and the constants of the motion of the particles, from which the equilibrium distribution functions can be constructed, are derived. In Sec. III, the expression for the second-order wave energy from Refs. [1,2,5] is put in a simpler and more concise form by introducing a representation of the perturbations which is particularly appropriate to the equilibrium under consideration. The minimizing perturbations are obtained in Sec. IV, where the expression for the minimized energy is also obtained. In deriving this expression, the difference between gyrating and swinging particles plays a major role. Section V is devoted to an extensive discussion of the energy expression. This discussion leads to the main results, which are then summarized in Sec. VI.

A considerable part of the calculations is carried out in the appendices. Particularly convenient $\mathbf{v}$ space coordinates are introduced in Appendix A. The motion of the charged particles is exhaustively treated in Appendix B, and the two essentially different groups of particles, namely the gyrating particles (GP's) and the swinging particles (SP's) are introduced. In Appendix C, a first-order partial differential equation which appears in the minimization problem is solved by the method of characteristics. Appendix D introduces two different coordinates systems in $\mathbf{v}$-y space which are particularly appropriate to the treatment of the two different groups of particles. In Appendix E, several quantities which appear in the expression for the minimized wave energy are calculated, in particular several mean values along the particle orbits. Finally, in Appendix F, an expression is derived for the perturbed electric charge density, and it is shown that this can be made to vanish by an appropriate nontrivial choice of the perturbations.

II. EQUILIBRIUM ELECTROMAGNETIC FIELD AND DISTRIBUTION FUNCTIONS

The magnetic field of the equilibrium under consideration has constant magnitude and straight field lines which have a constant twist as one proceeds in a given direction. Associated with this shear of the magnetic field, there is an equilibrium electric current.

In Cartesian coordinates x, y, and z with unit basis $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$, the equilibrium vector potential $\mathbf{A}^{(0)}$ and the corresponding magnetic field $\mathbf{B}^{(0)}$ are given by

$$\mathbf{A}^{(0)}=\frac{B^{(0)}}{a}[-\sin ay\,\mathbf{e}_x+(1-\cos ay)\mathbf{e}_z]\ , \tag{2}$$

$$\mathbf{B}^{(0)}=B^{(0)}(\sin ay\,\mathbf{e}_x+\cos ay\,\mathbf{e}_z)\ . \tag{3}$$

The y-independent part of the vector potential is such that $\mathbf{A}^{(0)}$ remains well defined in the limit $a\rightarrow 0$.

The electric current density associated with this magnetic field is

$$\mathbf{j}^{(0)}=-\frac{c}{4\pi}a\mathbf{B}^{(0)}\ , \tag{4}$$

where c is the velocity of light. $\mathbf{j}^{(0)}$, of course, vanishes as $a\rightarrow 0$ and the equilibrium magnetic field becomes homogeneous.

It is assumed here that there is no equilibrium electric field $\mathbf{E}^{(0)}$. The Lagrangian of a particle of species ν with electric charge e_ν and mass m_ν is then given by

$$\begin{aligned}L_\nu&=\frac{m_\nu}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2)+\frac{e_\nu}{c}\mathbf{v}\cdot\mathbf{A}^{(0)}\\&=\frac{m_\nu}{2}(\dot{x}^2+\dot{y}^2+\dot{z}^2)\\&\quad+\frac{m_\nu\omega_\nu}{a}[-\dot{x}\sin ay+\dot{z}(1-\cos ay)]\ ,\end{aligned} \tag{5}$$

where we have set $\omega_\nu\equiv e_\nu B^{(0)}/m_\nu c$. The canonical momenta derived from Eq. (5) are

$$p_{x\nu}=m_\nu\dot{x}-\frac{m_\nu\omega_\nu}{a}\sin ay\ , \tag{6}$$

$$p_{y\nu}=m_\nu\dot{y}\ , \tag{7}$$

$$p_{z\nu}=m_\nu\dot{z}+\frac{m_\nu\omega_\nu}{a}(1-\cos ay)\ , \tag{8}$$

and the Hamiltonian is therefore given by

$$H_\nu=\frac{1}{2m_\nu}\left[\left[p_{x\nu}+\frac{m_\nu\omega_\nu}{a}\sin ay\right]^2+p_{y\nu}^2+\left[p_{z\nu}-\frac{m_\nu\omega_\nu}{a}(1-\cos ay)\right]^2\right]\ . \tag{9}$$

Since the Hamiltonian does not depend on either x, z, or t, the canonical momenta $p_{x\nu}$ and $p_{z\nu}$ and the energy $\mathcal{H}_\nu=(m_\nu/2)(\dot{x}^2+\dot{y}^2+\dot{z}^2)$ are constants of the motion.

The equilibrium distribution function $f_\nu^{(0)}$ for particles of species ν can be constructed from the constants of the motion $\mathcal{H}_\nu$, $p_{x\nu}$, and $p_{z\nu}$. The presence of an electric current, Eq. (4), requires $f_\nu^{(0)}$ to be anisotropic for at least one particle species ν.

Generally, the current density, as derived from the particle motion, is

$$\mathbf{j}^{(0)}=\sum_\nu e_\nu\int d^3v\,\mathbf{v}f_\nu^{(0)}\ , \tag{10}$$

which, taking into account Eqs. (3) and (4), yields

$$-\frac{c}{4\pi}aB^{(0)}\sin ay=\sum_\nu e_\nu\int d^3v\,\dot{x}f_\nu^{(0)}=\sum_\nu e_\nu N_\nu\langle\dot{x}\rangle_\nu\ , \tag{11}$$

$$0=\sum_\nu e_\nu\int d^3v\,\dot{y}f_\nu^{(0)}=\sum_\nu e_\nu N_\nu\langle\dot{y}\rangle_\nu\ , \tag{12}$$

and

$$-\frac{c}{4\pi}aB^{(0)}\cos ay=\sum_\nu e_\nu\int d^3v\,\dot z f_\nu^{(0)}=\sum_\nu e_\nu N_\nu\langle \dot z\rangle_\nu\,, \tag{13}$$

with N_ν the density of particles of species ν and $\langle \dot x\rangle_\nu$, $\langle \dot y\rangle_\nu$, and $\langle \dot z\rangle_\nu$ the mean values of the components of the velocity. Equation (12) is automatically satisfied since $f_\nu^{(0)}(\mathcal{H}_\nu,p_{x\nu},p_{z\nu})$ is symmetric in $\dot y$. Equations (11) and (13), however, impose constraints on $f_\nu^{(0)}(\mathcal{H}_\nu,p_{x\nu},p_{z\nu})$. In particular, these equations imply invariance under the transformation $\sin ay \Longrightarrow \cos ay$, $\dot x \Longrightarrow \dot z$. A combination of variables which is invariant under this transformation, and which suggests itself because it is the parallel particle velocity along $\mathbf{B}^{(0)}$, is $\dot x \sin ay+\dot z\cos ay$. This, however, is not a constant of the motion. On the other hand, as derived in Appendix C, an appropriate expression is given by

$$\mathcal{U}_\nu=\frac{a}{2\omega_\nu}\dot y^{\,2}+\dot x\sin ay+\dot z\cos ay =\frac{a}{2\omega_\nu}v_\perp^2\sin^2\phi+v_\parallel, \tag{14}$$

$v_\perp$, ϕ, and $v_\parallel$ being the local cylindrical velocity coordinates introduced in Appendix A. That $\mathcal{U}_\nu$ is indeed a constant of the motion becomes evident when it is expressed as

$$\mathcal{U}_\nu=\frac{a}{m_\nu\omega_\nu}\mathcal{H}_\nu-\frac{a}{2\omega_\nu m_\nu^2}(p_{x\nu}^2+p_{z\nu}^2)+\frac{p_{z\nu}}{m_\nu}\,. \tag{15}$$

The distribution functions we consider are therefore of the form $f_\nu^{(0)}=f_\nu^{(0)}(\mathcal{H}_\nu,\mathcal{U}_\nu)$. [Admissible distribution functions are, for example $f_\nu^{(0)}\sim\mathcal{U}_\nu^2\exp(-\text{const}\times\mathcal{H}_\nu)$, $f_\nu^{(0)}\sim\exp(-\text{const}_1\times\mathcal{H}_\nu-\text{const}_2\times\mathcal{U}_\nu^2)$, etc.] More explicitly, one has

$$f_\nu^{(0)}=\begin{cases} f_\nu^{(0)}\left[\mathcal{H}_\nu=\dfrac{m_\nu}{2}(\dot x^{\,2}+\dot y^{\,2}+\dot z^{\,2}),\ \ \mathcal{U}_\nu=\dfrac{a}{2\omega_\nu}\dot y^{\,2}+\dot x\sin ay+\dot z\cos ay\right], \\ f_\nu^{(0)}\left[\mathcal{H}_\nu=\dfrac{m_\nu}{2}(v_\perp^2+v_\parallel^2),\ \ \mathcal{U}_\nu=\dfrac{a}{2\omega_\nu}v_\perp^2\sin^2\phi+v_\parallel\right]. \end{cases} \tag{16}$$

Note that in the coordinates $(v_\perp,\phi,v_\parallel,y)$, *$f_\nu^{(0)}$ does not depend explicitly on y*, and this is the great advantage of using these coordinates for the problem under consideration. A functional dependence of the form given by Eqs. (16), together with the expressions for $\dot x(v_\perp,\phi,v_\parallel,y)$ and $\dot z(v_\perp,\phi,v_\parallel,y)$, Eqs. (A8) and (A10), derived in Appendix A yield, when inserted in Eqs. (11) and (13),

$$-\frac{c}{4\pi}aB^{(0)}\times\begin{Bmatrix}\sin ay\\ \cos ay\end{Bmatrix}=\begin{Bmatrix}\sin ay\\ \cos ay\end{Bmatrix}\times\sum_\nu e_\nu N_\nu\langle v_\parallel\rangle_\nu\,. \tag{17}$$

Evidently, the density of particles of species ν,

$$N_\nu=\int f_\nu^{(0)}(\mathcal{H}_\nu,\mathcal{U}_\nu)d^3v =\int f_\nu^{(0)}(\mathcal{H}_\nu,\mathcal{U}_\nu)v_\perp dv_\perp d\phi\, dv_\parallel\,, \tag{18}$$

and the mean parallel velocity of species ν,

$$\langle v_\parallel\rangle_\nu=\frac{1}{N_\nu}\int v_\parallel f_\nu^{(0)}(\mathcal{H}_\nu,\mathcal{U}_\nu)v_\perp dv_\perp d\phi\, dv_\parallel\,, \tag{19}$$

do not depend on y.

The following useful relations can be derived from Eqs. (16):

$$\left.\frac{\partial f_\nu^{(0)}}{\partial \mathbf{x}}\right|_{\mathbf{v}}=a(\dot x\cos ay-\dot z\sin ay)\left.\frac{\partial f_\nu^{(0)}}{\partial \mathcal{U}_\nu}\right|_{\mathcal{H}_\nu}\mathbf{e}_y =av_1\left.\frac{\partial f_\nu^{(0)}}{\partial \mathcal{U}_\nu}\right|_{\mathcal{H}_\nu}\mathbf{e}_y\,, \tag{20}$$

(v_1 is the component of the velocity in the direction of the vector $\mathbf{e}_1$ introduced in Appendix A),

$$\begin{aligned}\left.\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\right|_{\mathbf{x}}&=m_\nu\left.\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu}\right|_{\mathcal{U}_\nu}\mathbf{v}+\left.\frac{\partial f_\nu^{(0)}}{\partial \mathcal{U}_\nu}\right|\left[\frac{a}{\omega_\nu}v_\perp\sin\phi\,\mathbf{e}_y+\mathbf{e}_B\right]\\ &=\left[m_\nu\left.\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu}\right|_{\mathcal{U}_\nu}+\frac{a}{\omega_\nu}\left.\frac{\partial f_\nu^{(0)}}{\partial \mathcal{U}_\nu}\right|_{\mathcal{H}_\nu}\right]\mathbf{v} -\left[\frac{a}{\omega_\nu}\mathbf{w}-\mathbf{e}_B\right]\left.\frac{\partial f_\nu^{(0)}}{\partial \mathcal{U}_\nu}\right|_{\mathcal{H}_\nu}.\end{aligned} \tag{21}$$

Here, the projection $\mathbf{w}$ of the velocity in the planes perpendicular to $\mathbf{e}_y$ has been introduced, i.e.,

$$\mathbf{w}=\mathbf{v}-\dot y\mathbf{e}_y=\mathbf{v}-v_\perp\sin\phi\,\mathbf{e}_y=v_1\mathbf{e}_1+v_\parallel\mathbf{e}_B =v_\perp\cos\phi\,\mathbf{e}_1+v_\parallel\mathbf{e}_B\,. \tag{22}$$

A further useful quantity is

$$\mathcal{D}f_\nu^{(0)}(\mathcal{H}_\nu,\mathcal{U}_\nu)\equiv m_\nu\left.\frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu}\right|_{\mathcal{U}_\nu}+\frac{a}{\omega_\nu}\left.\frac{\partial f_\nu^{(0)}}{\partial \mathcal{U}_\nu}\right|_{\mathcal{H}_\nu} =2\left.\frac{\partial f_\nu^{(0)}}{\partial \dot y^{\,2}}\right|_{\mathbf{x},\dot x,\dot z}. \tag{23}$$

III. SECOND-ORDER WAVE ENERGY

In the context of Maxwell-Vlasov theory, Morrison and Pfirsch [1,2] derived expressions for the free energy δ^2H available upon arbitrary perturbations of an arbitrary equilibrium. In the absence of an equilibrium electric field, δ^2H can be expressed as [5]

$$\delta^2 H=\sum_\nu \int \frac{d^3x\, d^3v}{2m_\nu}\left\{\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\cdot\left[-\left[\mathbf{v}\cdot\frac{\partial G_\nu}{\partial \mathbf{x}}\right]\frac{\partial G_\nu}{\partial \mathbf{x}}-\left[\mathbf{a}_\nu^{(0)}\cdot\frac{\partial G_\nu}{\partial \mathbf{v}}\right]\left[\frac{e_\nu}{m_\nu c}\mathbf{B}^{(0)}\times\frac{\partial G_\nu}{\partial \mathbf{v}}+2\frac{\partial G_\nu}{\partial x}\right]\right.\right.$$
$$\left.+\frac{e_\nu}{m_\nu c}G_\nu \mathbf{v}\times\frac{\partial}{\partial \mathbf{v}}\left[\mathbf{B}^{(0)}\cdot\frac{\partial G_\nu}{\partial x}\right]\right]+\frac{\partial f_\nu^{(0)}}{\partial \mathbf{x}}\cdot\left[-\left[\frac{\partial G_\nu}{\partial \mathbf{x}}\cdot\frac{\partial G_\nu}{\partial \mathbf{v}}\right]\mathbf{v}+(d_\nu G_\nu)\frac{\partial G_\nu}{\partial \mathbf{v}}\right]$$
$$\left.+f_\nu^{(0)}\left[\frac{e_\nu}{c}\delta\mathbf{A}\right]^2-2\frac{e_\nu}{c}\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\cdot\left[d_\nu(G_\nu\delta\mathbf{A})-G_\nu\frac{\partial}{\partial \mathbf{x}}(\mathbf{v}\cdot\delta\mathbf{A})\right]\right\}+\frac{1}{8\pi}\int d^3x(\delta E^2+\delta B^2)\ , \tag{24}$$

where $G_\nu(\mathbf{x},\mathbf{v})$ is a generating function for the perturbation of the particle position and velocity, $\delta\mathbf{A}$ is the perturbation of the vector potential, and $\delta E^2/8\pi$ and $\delta B^2/8\pi$ are the perturbations in the electric- and magnetic-field energy densities. The operator d_ν is the equilibrium Vlasov operator, i.e.,

$$d_\nu=\mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}}+\mathbf{a}_\nu^{(0)}\cdot\frac{\partial}{\partial \mathbf{v}},\quad \mathbf{a}_\nu^{(0)}=\frac{e_\nu}{m_\nu c}\mathbf{v}\times\mathbf{B}^{(0)} \tag{25}$$

(in the absence of an equilibrium electric field). Using velocity coordinates $v_\perp$, ϕ, and $v_\parallel$, one has

$$\mathbf{a}_\nu^{(0)}\cdot\frac{\partial}{\partial \mathbf{v}}=-\omega_\nu\frac{\partial}{\partial\phi}\bigg|_{\mathbf{x},v_\perp,v_\parallel} \tag{26}$$

and

$$\begin{aligned}d_\nu&=\mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}}\bigg|_{\mathbf{v}}-\omega_\nu\frac{\partial}{\partial\phi}\bigg|_{\mathbf{x},v_\perp,v_\parallel}\\&=\mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}}\bigg|_{v_\perp,\phi,v_\parallel}-av_\perp v_\parallel\sin\phi\cos\phi\frac{\partial}{\partial v_\perp}\bigg|_{\mathbf{x},\phi,v_\parallel}+av_\parallel\sin^2\phi\frac{\partial}{\partial\phi}\bigg|_{\mathbf{x},v_\perp,v_\parallel}+av_\perp^2\sin\phi\cos\phi\frac{\partial}{\partial v_\parallel}\bigg|_{\mathbf{x},v_\perp,\phi}-\omega_\nu\frac{\partial}{\partial\phi}\bigg|_{\mathbf{x},v_\perp,v_\parallel}\ .\end{aligned} \tag{27}$$

By taking into account the identity

$$\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\cdot\frac{e_\nu}{m_\nu c}G_\nu\mathbf{v}\times\frac{\partial}{\partial \mathbf{v}}\left[\mathbf{B}^{(0)}\cdot\frac{\partial G_\nu}{\partial \mathbf{x}}\right]=\omega_\nu\frac{\partial}{\partial \mathbf{v}}\cdot\left[G_\nu\left[\mathbf{e_B}\cdot\frac{\partial G_\nu}{\partial \mathbf{x}}\right]\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\times\mathbf{v}\right]-\omega_\nu\left[\mathbf{e}_B\cdot\frac{\partial G_\nu}{\partial \mathbf{x}}\right]\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\cdot\mathbf{v}\times\frac{\partial G_\nu}{\partial \mathbf{v}} \tag{28}$$

and Eqs. (20)–(23), the expression for $\delta^2 H$ can be put in a more convenient form, namely,

$$\delta^2 H=\sum_\nu\int\frac{d^3x\,d^3v}{2m_\nu}\left\{-[\mathcal{D}f_\nu^{(0)}](d_\nu G_\nu)^2+\frac{\partial f_\nu^{(0)}}{\partial\mathcal{U}_\nu}\left[\left[\frac{a}{\omega_\nu}\mathbf{w}-\mathbf{e}_B\right]\cdot\frac{\partial G_\nu}{\partial \mathbf{x}}\right]d_\nu G_\nu+f_\nu^{(0)}\left[\frac{e_\nu}{c}\delta\mathbf{A}\right]^2\right.$$
$$\left.-2\frac{e_\nu}{c}\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\cdot\left[d_\nu(G_\nu\delta\mathbf{A})-G_\nu\frac{\partial}{\partial \mathbf{x}}(\mathbf{v}\cdot\delta\mathbf{A})\right]\right\}+\frac{1}{8\pi}\int d^3x(\delta E^2+\delta B^2)\ . \tag{29}$$

Since the equilibrium is independent of x and z, an appropriate ansatz for the generating function $G_\nu(\mathbf{x},\mathbf{v})$ is

$$G_\nu(\mathbf{x},\mathbf{v})=\tfrac{1}{2}[g_\nu(y,\mathbf{v})e^{i\mathbf{k}_{xz}\cdot\mathbf{x}}+g_\nu^*(y,\mathbf{v})e^{-i\mathbf{k}_{xz}\cdot\mathbf{x}}]\ . \tag{30}$$

The wave vector $\mathbf{k}_{xz}$ introduced here is defined by

$$\mathbf{k}_{xz}=k_x\mathbf{e}_x+k_z\mathbf{e}_z \tag{31}$$

and therefore lies on the planes of the $\mathbf{B}^{(0)}$ lines. G_ν is obviously a real function, since g_ν^* is the complex conjugate of g_ν.

The investigation is now limited to purely electrostatic perturbations, i.e., we choose

$$\delta\mathbf{A}=0\ . \tag{32}$$

Inserting Eq. (30) into Eq. (29) and then integrating over a periodicity surface s,

$$s=\int_{x_0}^{x_0+(2\pi/k_x)}\int_{z_0}^{z_0+(2\pi/k_z)}dx\,dz\ , \tag{33}$$

yields

$$\delta^2 H=\sum_\nu \frac{s}{4m_\nu}\int d^3v\,dy\left[-[\mathcal{D}f_\nu^{(0)}]|[\hat{d}_\nu g_\nu+i(\mathbf{v}\cdot\mathbf{k}_{xz})g_\nu]|^2 +\frac{i}{2}\frac{\partial f_\nu^{(0)}}{\partial \mathcal{U}_\nu}\left[\frac{a}{\omega_\nu}\mathbf{w}-\mathbf{e}_B\right]\cdot\mathbf{k}_{xz}[g_\nu\hat{d}_\nu g_\nu^*-g_\nu^*\hat{d}_\nu g_\nu-2i(\mathbf{v}\cdot\mathbf{k}_{xz})g_\nu g_\nu^*]\right]+\frac{1}{8\pi}\int d^3x\,\delta E^2\ , \tag{34}$$

where the operator $\hat{d}_\nu$ has been introduced. $\hat{d}_\nu$ is the equilibrium Vlasov operator for functions which depend on $\mathbf{v}$ and y, but not on x or z, i.e.,

$$\begin{aligned}\hat{d}_\nu&=\dot{y}\frac{\partial}{\partial y}\bigg|_{\mathbf{v}}-\omega_\nu\frac{\partial}{\partial\phi}\bigg|_{y,v_\perp,v_\parallel}\\&=v_\perp\sin\phi\frac{\partial}{\partial y}\bigg|_{v_\perp,\phi,v_\parallel}-av_\perp v_\parallel\sin\phi\cos\phi\frac{\partial}{\partial v_\perp}\bigg|_{y,\phi,v_\parallel}\\&\quad+(av_\parallel\sin^2\phi-\omega_\nu)\frac{\partial}{\partial\phi}\bigg|_{y,v_\perp,v_\parallel}\\&\quad+av_\perp^2\sin\phi\cos\phi\frac{\partial}{\partial v_\parallel}\bigg|_{y,v_\perp,\phi}\ .\end{aligned}\tag{35}$$

It is convenient to represent the complex function $g_\nu(y,\mathbf{v})$ as

$$g_\nu(y,v_\perp,\phi,v_\parallel)=\Psi_\nu(y,v_\perp,\phi,v_\parallel)e^{i\Gamma_\nu(y,v_\perp,\phi,v_\parallel)}\ , \tag{36}$$

where Ψ_ν and Γ_ν are *real* functions. Since g_ν is single valued, Ψ_ν and Γ_ν are subject to the periodicity conditions

$$\Psi_\nu(y,v_\perp,\phi+2\pi,v_\parallel)=\Psi_\nu(y,v_\perp,\phi,v_\parallel) \tag{37}$$

and

$$\Gamma_\nu(y,v_\perp,\phi+2\pi,v_\parallel)=\Gamma_\nu(y,v_\perp,\phi,v_\parallel)+2\pi n_\nu\ , \tag{38}$$

n_ν being any integer number, i.e., $n_\nu=0,\pm1,\ldots$.

Inserting Eq. (36) in Eq. (34) yields

$$\delta^2H=\sum_\nu\frac{s}{4m_\nu}\int d^3v\,dy\left[-[\mathcal{D}f_\nu^{(0)}]\{(\hat{d}_\nu\Psi_\nu)^2+\Psi_\nu^2[\hat{d}_\nu\Gamma_\nu+(\mathbf{v}\cdot\mathbf{k}_{xz})]^2\}+\Psi_\nu^2\frac{\partial f_\nu^{(0)}}{\partial\mathcal{U}_\nu}\left[\frac{a}{\omega_\nu}\mathbf{w}-\mathbf{e}_B\right]\cdot\mathbf{k}_{xz}[\hat{d}_\nu\Gamma_\nu+(\mathbf{v}\cdot\mathbf{k}_{xz})]\right] +\frac{1}{8\pi}\int d^3x\,\delta E^2\ , \tag{39}$$

which is the general expression for the second-order energy of *electrostatic* perturbations of the equilibrium considered. Note that δ^2H is a functional of Ψ_ν, which appears as Ψ_ν and $\hat{d}_\nu\Psi_\nu$, and of Γ_ν, *which appears only as* $\hat{d}_\nu\Gamma_\nu$.

IV. EXTREMIZATION OF THE SECOND-ORDER WAVE ENERGY

In order to minimize the wave energy with respect to Γ_ν, we now consider the variation of δ^2H brought about by a variation $\delta\Gamma_\nu$ of Γ_ν. This quantity can easily be calculated as

$$\begin{aligned}\delta_{\Gamma_\nu}(\delta^2H)&=\delta^2H(\Gamma_\nu+\delta\Gamma_\nu)-\delta^2H(\Gamma_\nu)\\&=\sum_\nu\frac{s}{4m_\nu}\int d^3v\,dy\left[\hat{d}_\nu\left\{\delta\Gamma_\nu\Psi_\nu^2\left[-2[\mathcal{D}f_\nu^{(0)}][\hat{d}_\nu\Gamma_\nu+(\mathbf{v}\cdot\mathbf{k}_{xz})]+\frac{\partial f_\nu^{(0)}}{\partial\mathcal{U}_\nu}\left[\frac{a}{\omega_\nu}\mathbf{w}-\mathbf{e}_B\right]\cdot\mathbf{k}_{xz}\right]\right\}\right.\\&\qquad\left.-\delta\Gamma_\nu\hat{d}_\nu\left\{\Psi_\nu^2\left[-2[\mathcal{D}f_\nu^{(0)}](\hat{d}_\nu\Gamma_\nu+\mathbf{v}\cdot\mathbf{k}_{xz})+\frac{\partial f_\nu^{(0)}}{\partial\mathcal{U}}\left[\frac{a}{\omega_\nu}\mathbf{w}-\mathbf{e}_B\right]\cdot\mathbf{k}_{xz}\right]\right\}\right]\ .\end{aligned}\tag{40}$$

It follows from Eq. (38) that the variation of Γ_ν, $\delta\Gamma_\nu$, must be periodic in ϕ, i.e.,

$$\delta\Gamma_\nu(v_\perp,\phi+2\pi,v_\parallel,y)=\delta\Gamma_\nu(v_\perp,\phi,v_\parallel,y)\ . \tag{41}$$

Since only derivatives of Γ_ν appear in Eq. (39), $\delta\Gamma_\nu$ can be taken to vanish at the boundaries. Therefore, Eq. (40) reduces to

$$\delta_{\Gamma_\nu}(\delta^2H)=-\sum_\nu\frac{s}{4m_\nu}\int d^3v\,dy\,\delta\Gamma_\nu\hat{d}_\nu\left\{\Psi_\nu^2\left[-2[\mathcal{D}f_\nu^{(0)}](\hat{d}_\nu\Gamma_\nu+\mathbf{v}\cdot\mathbf{k}_{xz})+\frac{\partial f_\nu^{(0)}}{\partial\mathcal{U}_\nu}\left[\frac{a}{\omega_\nu}\mathbf{w}-\mathbf{e}_B\right]\cdot\mathbf{k}_{xz}\right]\right\}, \tag{42}$$

and, since $\delta\Gamma_\nu$ is arbitrary, the condition for the vanishing of $\delta_{\Gamma_\nu}(\delta^2H)$ is

$$\hat{d}_\nu\left\{\Psi_\nu^2\left[-2[\mathcal{D}f_\nu^{(0)}](\hat{d}_\nu\Gamma_\nu+\mathbf{v}\cdot\mathbf{k}_{xz})+\frac{\partial f_\nu^{(0)}}{\partial\mathcal{U}_\nu}\left[\frac{a}{\omega_\nu}\mathbf{w}-\mathbf{e}_B\right]\cdot\mathbf{k}_{xz}\right]\right\}=0\ . \tag{43}$$

According to the results of Appendix C, the solution of this equation is

$$-2\Psi_\nu^2[\mathcal{D}f_\nu^{(0)}](\hat{d}_\nu\Gamma_\nu+\mathbf{v}\cdot\mathbf{k}_{xz})+\Psi_\nu^2\frac{\partial f_\nu^{(0)}}{\partial\mathcal{U}_\nu}\left[\frac{a}{\omega_\nu}\mathbf{w}-\mathbf{e}_B\right]\cdot\mathbf{k}_{xz}=C_\nu(\mathcal{H}_\nu,\mathcal{U}_\nu,\mathcal{Y}_\nu)\,, \tag{44}$$

where C_ν is a function of the constants of the motion $\mathcal{H}_\nu$, $\mathcal{U}_\nu$, nd $\mathcal{Y}_\nu$ which still has to be determined from the boundary conditions on Γ_ν. Solving Eq. (44) for $\hat{d}_\nu\Gamma_\nu+\mathbf{v}\cdot\mathbf{k}_{xz}$ and inserting the result in Eq. (39) yields

$$\delta^2H=\sum_\nu\frac{s}{4m_\nu}\int d^3v\,dy[\mathcal{D}f_\nu^{(0)}]\left\{-(\hat{d}_\nu\Psi_\nu)^2+\Psi_\nu^2\frac{(\partial f_\nu^{(0)}/\partial\mathcal{U}_\nu)^2}{4[\mathcal{D}f_\nu^{(0)}]^2}\left[\left[\frac{a}{\omega_\nu}\mathbf{w}-\mathbf{e}_B\right]\cdot\mathbf{k}_{xz}\right]^2-\frac{1}{4[\mathcal{D}f_\nu^{(0)}]^2}\frac{C_\nu^2}{\Psi_\nu^2}\right\}. \tag{45}$$

Here, the electrostatic energy term $(1/8\pi)\int d^3x\,\delta E^2$ has been dropped, since the perturbed charge density can be made zero by an appropriate choice of the *signs* of Ψ_ν, which do not influence Eq. (45). This is explicitly shown in Appendix F.

According to the results of Appendix B, the particles of each species ν are divided into two classes, namely the gyrating particles, which move around the field lines and, at the same time, oscillate about the planes $y=\mathcal{Y}_\nu$, and the swinging particles, for which ϕ takes values only between $\phi_{\min}$ and $\phi_{\max}$, and which never complete a turn around the field lines, moving freely in the y direction. In Appendix D, coordinates in $\mathbf{v}$-y space are introduced which are particularly convenient for both kinds of particles. With these results taken into account, the wave energy is now split into two parts:

$$\delta^2H=(\delta^2H)_{\mathrm{GP}}+(\delta^2H)_{\mathrm{SP}}\,, \tag{46}$$

where $(\delta^2H)_{\mathrm{GP}}$ is the contribution of the gyrating particles, and $(\delta^2H)_{\mathrm{SP}}$ that of the swinging particles. With the definitions and results of Appendices D and E, these contributions can be concisely expressed as

$$(\delta^2H)_{\mathrm{GP}}=\sum_\nu\frac{s|\omega_\nu|}{4m_\nu^2}\int_{(\mathrm{GP})}\frac{d\mathcal{H}_\nu d\mathcal{U}_\nu d\mathcal{Y}_\nu d\phi}{|\hat{d}_\nu\phi|}[\mathcal{D}f_\nu^{(0)}]\left[-[\hat{d}_\nu\phi]^2\left[\frac{\partial\Psi_\nu}{\partial\phi}\right]^2+\left[\langle\hat{d}_\nu\phi\rangle_{\tau_\phi}\frac{(a_{\nu1}-b_{\nu1})}{2}k_\parallel(\mathcal{Y}_\nu)\right]^2\langle\Psi_\nu^2\rangle_{\tau_\phi}-\frac{\langle\hat{d}_\nu\phi\rangle_{\tau_\phi}^2}{\left\langle\frac{1}{\Psi_\nu^2}\right\rangle_{\tau_\phi}}\left[n_\nu+\frac{(a_{\nu1}+b_{\nu1})}{2}k_\parallel(\mathcal{Y}_\nu)\right]^2\right] \tag{47}$$

and

$$(\delta^2H)_{\mathrm{SP}}=\sum_\nu\frac{s|\omega_\nu|}{4m_\nu^2}\int_{(\mathrm{SP})}\frac{d\mathcal{H}_\nu d\mathcal{U}_\nu d\mathcal{Y}_\nu dy}{|\hat{d}_\nu y|}[\mathcal{D}f_\nu^{(0)}]\left[-[\hat{d}_\nu y]^2\left[\frac{\partial\Psi_\nu}{\partial y}\right]^2+\left[\langle\hat{d}_\nu y\rangle_{\tau_y}\frac{(a_{\nu2}-b_{\nu2})}{2}k_\parallel(\mathcal{Y}_\nu)\right]^2\langle\Psi_\nu^2\rangle_{\tau_y}+\frac{\langle\hat{d}_\nu y\rangle_{\tau_y}^2}{\left\langle\frac{1}{\Psi_\nu^2}\right\rangle_{\tau_y}}\left[\Delta\Gamma_\nu\frac{a}{2\pi}+\frac{(a_{\nu2}+b_{\nu2})}{2}k_\parallel(\mathcal{Y}_\nu)\right]^2\right], \tag{48}$$

where

$$\mathcal{D}f_\nu^{(0)}=m_\nu\frac{\partial f_\nu^{(0)}}{\partial\mathcal{H}_\nu}+\frac{a}{\omega_\nu}\frac{\partial f_\nu^{(0)}}{\partial\mathcal{U}_\nu}=\frac{1}{\dot{y}}\frac{\partial f_\nu^{(0)}}{\partial\dot{y}}\,, \tag{49}$$

$$a_{\nu1}(\mathcal{H}_\nu,\mathcal{U}_\nu)=\frac{1}{\langle\hat{d}_\nu\phi\rangle_{\tau_\phi}[\mathcal{D}f_\nu^{(0)}]}\left\langle\mathbf{e}_B(\mathcal{Y}_\nu)\cdot\frac{\partial f_\nu^{(0)}}{\partial\mathbf{v}}\right\rangle_{\tau_\phi}=\frac{q_\parallel}{\langle\hat{d}_\nu\phi\rangle_{\tau_\phi}}+\frac{1}{\langle\hat{d}_\nu\phi\rangle_{\tau_\phi}[\mathcal{D}f_\nu^{(0)}]}\frac{\Omega_\nu}{|\omega_\nu|}\frac{\partial f_\nu^{(0)}}{\partial\mathcal{U}_\nu}\,, \tag{50}$$

$$a_{\nu2}(\mathcal{H}_\nu,\mathcal{U}_\nu)=\frac{1}{\langle\hat{d}_\nu y\rangle_{\tau_y}[\mathcal{D}f_\nu^{(0)}]}\left\langle\mathbf{e}_B(\mathcal{Y}_\nu)\cdot\frac{\partial f_\nu^{(0)}}{\partial\mathbf{v}}\right\rangle_{\tau_y}=\frac{r_\parallel}{\langle\hat{d}_\nu y\rangle_{\tau_y}}+\frac{1}{\langle\hat{d}_\nu y\rangle_{\tau_y}[\mathcal{D}f_\nu^{(0)}]}\frac{\Omega_\nu}{|\omega_\nu|}\frac{\partial f_\nu^{(0)}}{\partial\mathcal{U}_\nu}\,, \tag{51}$$

$$b_{\nu1}(\mathcal{H}_\nu,\mathcal{U}_\nu)=\frac{q_\parallel}{\langle\hat{d}_\nu\phi\rangle_{\tau_\phi}}\,, \tag{52}$$

$$b_{\nu 2}(\mathcal{H}_\nu,\mathcal{U}_\nu)=\frac{r_\parallel}{\langle \hat{d}_\nu y\rangle_{\tau_y}} \; . \tag{53}$$

In deriving Eqs. (47) and (48), use has been made of the fact that $\hat{d}_\nu\phi$ and $\hat{d}_\nu y$ do not change sign for gyrating particles and for swinging particles, respectively. This yields for the integrals along the particle orbits

$$\begin{aligned}\oint_0^{2\pi}\Psi_\nu^2\frac{d\phi}{|\hat{d}_\nu\phi|}&=[\mathrm{sgn}(\hat{d}_\nu\phi)]\oint_0^{2\pi}\Psi_\nu^2\frac{d\phi}{\hat{d}_\nu\phi}\\&=[\mathrm{sgn}(\hat{d}_\nu\phi)]\langle\Psi_\nu^2\rangle_{\tau_\phi}\oint_0^{2\pi}\frac{d\phi}{\hat{d}_\nu\phi}\\&=\langle\Psi_\nu^2\rangle_{\tau_\phi}\oint_0^{2\pi}\frac{d\phi}{|\hat{d}_\nu\phi|},\end{aligned} \tag{54}$$

and correspondingly for swinging particles. *Note that the only $\mathcal{Y}_\nu$ dependence in the integrands in Eqs. (47) and (48) is given by $k_\parallel(\mathcal{Y}_\nu)$ and by Ψ_ν (if the arbitrary Ψ_ν is chosen to be dependent on $\mathcal{Y}_\nu$).*

V. DISCUSSION OF THE EXPRESSION FOR THE SECOND-ORDER WAVE ENERGY

A. Homogeneous equilibrium

The expression for $\delta^2 H$ which is valid in the homogeneous case is easily obtained from Eqs. (46)–(48) when one observes that, in that case, there is no electric current, and $a=0$. Therefore, there are no swinging particles, according to Appendix B. Also, $\mathcal{U}_\nu=v_\parallel=v_z$, $\mathbf{e}_B(\mathcal{Y}_\nu)=\mathbf{e}_B(y)=\mathbf{e}_z$, $\hat{d}_\nu\phi=-\omega_\nu$, $\mathcal{D}f_\nu^{(0)}=2\partial f_\nu^{(0)}/\partial v_\perp^2$, $b_{\nu 1}=-v_\parallel/\omega_\nu$, $a_{\nu 1}=-(v_\parallel/\omega_\nu)\alpha_\nu$, where

$$\alpha_\nu=\left[\frac{\partial f_\nu^{(0)}}{\partial v_z^2}\right]\Big/\left[\frac{\partial f_\nu^{(0)}}{\partial v_\perp^2}\right] . \tag{55}$$

Therefore

$$\begin{aligned}(a_{\nu 1}-b_{\nu 1})&=\frac{v_\parallel}{\omega_\nu}(1-\alpha_\nu),\\(a_{\nu 1}+b_{\nu 1})&=-\frac{v_\parallel}{\omega_\nu}(1+\alpha_\nu) .\end{aligned} \tag{56}$$

Transforming the volume element according to Appendix C and performing a trivial integration in y, one then obtains Eq. (43) of Ref. [5].

B. Inhomogeneous equilibrium

In the case of an inhomogeneous equilibrium, one has to consider the contribution from both groups of particles.

1. *The wave energy $(\delta^2 H)_{\mathrm{GP}}$ for gyrating particles*

The difference between gyrating and swinging particles is extensively treated in Appendix B. For all cases of interest, the condition for a particle to be a gyrating particle, namely $av_\parallel/\omega_\nu\le 1$, is satisfied for the vast majority of particles. This is easily seen if one introduces the gyroradius $(R_g)_{\mathrm{th}}$ corresponding to a thermal velocity $(v_\perp)_{\mathrm{th}}$, which yields $av_\parallel/\omega_\nu=a(R_g)_{\mathrm{th}}v_\parallel/(v_\perp)_{\mathrm{th}}$, and observes that $a^{-1}\gg(R_g)_{\mathrm{th}}$ for all cases of interest.

Owing to the symmetry of the system, one can set $k_\parallel(\mathcal{Y}_\nu)=k_{\parallel 0}\cos a\mathcal{Y}_\nu$ without any restriction, and one has to distinguish the following two cases:

a. $k_{\parallel 0}=0$, *i.e.*, $k_\parallel(\mathcal{Y}_\nu)=0$ *for all* $\mathcal{Y}_\nu$ *(wave propagation perpendicular to* $B^{(0)}$*).* In this case, there is wave propagation only in the direction y of the inhomogeneity and $(\delta^2 H)_{\mathrm{GP}}$ is given by

$$(\delta^2 H)_{\mathrm{GP}}=\sum_\nu\frac{s|\omega_\nu|}{4m_\nu^2}\int_{(\mathrm{GP})}\frac{d\mathcal{H}_\nu d\mathcal{U}_\nu d\mathcal{Y}_\nu d\phi}{|\hat{d}_\nu\phi|}[\mathcal{D}f_\nu^{(0)}]\times\left[-[\hat{d}_\nu\phi]^2\left[\frac{\partial\Psi_\nu}{\partial\phi}\right]^2-\frac{\langle\hat{d}_\nu\phi\rangle_{\tau_\phi}^2}{\langle 1/\Psi_\nu^2\rangle_{\tau_\phi}}n_\nu^2\right] . \tag{57}$$

Then, $\delta^2 H<0$ if $\mathcal{D}f_\nu^{(0)}=m_\nu\partial f_\nu^{(0)}/\partial\mathcal{H}_\nu+(a/\omega_\nu)\partial f_\nu^{(0)}/\partial\mathcal{U}_\nu=2\partial f_\nu^{(0)}/\partial\dot{y}^2>0$ for some $\mathcal{H}_{\nu 0},\mathcal{U}_{\nu 0}$ corresponding to gyrating particles, and for any particle species ν. This means that the presence of a local minimum with respect to $\dot{y}^2$ in

$$f_\nu^{(0)}(\mathcal{H}_\nu(\dot{x}^2,\dot{y}^2,\dot{z}^2),\mathcal{U}_\nu(\dot{x},\dot{y}^2,\dot{z},y)) \tag{58}$$

guarantees $\delta^2 H<0$, *without any restrictions in the spatial variation of the perturbations perpendicular to* $B^{(0)}$: it suffices to localize Ψ_ν ($\partial\Psi_\nu/\partial\phi$) is then also localized) to the region in $\mathcal{H}_\nu,\mathcal{U}_\nu$ where $\partial f_\nu^{(0)}/\partial\dot{y}^2>0$. Outside this region Ψ_ν vanishes. All other Ψ_μ are set equal to zero. The Ψ_ν corresponding to the swinging particles are likewise all set equal to zero, so that $(\delta^2 H)_{\mathrm{SP}}=0$. The sign of $\delta^2 H=(\delta^2 H)_{\mathrm{GP}}$ is then determined only by the sign of the integrand in the region of localization.

b. $k_\parallel(\mathcal{Y}_\nu)=k_{\parallel 0}\cos a\mathcal{Y}_\nu$ *does not vanish for all* $\mathcal{Y}_\nu$ *[the wave vector has a component in the direction of* $\mathbf{B}^{(0)}$ *for all* $\mathcal{Y}_\nu$, *except* $a\mathcal{Y}_\nu=\pm(\pi/2)\pm m\pi$]. If $\mathcal{D}f_\nu^{(0)}=m_\nu\partial f_\nu^{(0)}/\partial\mathcal{H}_\nu+(a/\omega_\nu)\partial f_\nu^{(0)}/\partial\mathcal{U}_\nu=2\partial f_\nu^{(0)}/\partial\dot{y}^2>0$ for some $\mathcal{H}_{\nu 0},\mathcal{U}_{\nu 0}$ corresponding to gyrating particles and for any species ν, one again localizes the perturbations Ψ_ν around these values, as in the preceding case. All Ψ_ν corresponding to swinging particles are set equal to zero; therefore, $(\delta^2 H)_{\mathrm{SP}}=0$. If $a_{\nu 1}=b_{\nu 1}$ (local isotropy), all terms in Eq. (47) are negative. If $a_{\nu 1}\neq b_{\nu 1}$, one can use n_ν to make the integrand in Eq. (47) negative. This is most easily shown if Ψ_ν is chosen independent of ϕ. In this case, the integrand in Eq. (47) is given by

$$-[\mathcal{D}f_\nu^{(0)}]\langle\hat{d}_\nu\phi\rangle_{\tau_\phi}^2\langle\Psi_\nu^2\rangle_{\tau_\phi}[n_\nu+a_{\nu 1}k_\parallel(\mathcal{Y}_\nu)]\times[n_\nu+b_{\nu 1}k_\parallel(\mathcal{Y}_\nu)] . \tag{59}$$

If $a_{\nu 1}b_{\nu 1}>0$, it suffices to take $n_\nu=0$ to make the expression (59) (and thus $\delta^2 H$) negative. For any $a_{\nu 1}b_{\nu 1}$, it is negative if the factors in the square brackets are either both positive or both negative. Both factors are positive if

$$n_\nu>-a_{\nu 1}k_\parallel(\mathcal{Y}_\nu) \text{ and } n_\nu>-b_{\nu 1}k_\parallel(\mathcal{Y}_\nu) . \tag{60}$$

Let $\max_{\mathcal{Y}_\nu}[-a_{\nu 1}k_\parallel(\mathcal{Y}_\nu)]$ be the maximum of $-a_{\nu 1}k_\parallel(\mathcal{Y}_\nu)$ with respect to $\mathcal{Y}_\nu$, and correspondingly for $-b_{\nu 1}k_{\parallel 0}(\mathcal{Y}_\nu)$. Then, choosing n_ν larger than the largest of the two maxima satisfies inequalities (60). This can be concisely expressed by

$$n_\nu > \max(\mathcal{H}_{\nu 0},\mathcal{U}_{\nu 0}) \equiv \max_{\mathcal{Y}_\nu}\{-a_{\nu 1}k_\parallel(\mathcal{Y}_\nu), -b_{\nu 1}k_\parallel(\mathcal{Y}_\nu)\} \ . \tag{61}$$

The expression (59) also is negative if both factors in the square brackets are negative, i.e., if

$$n_\nu < -a_{\nu 1}k_\parallel(\mathcal{Y}_\nu) \ \text{ and } \ n_\nu < -b_{\nu 1}k_\parallel(\mathcal{Y}_\nu) \ . \tag{62}$$

This is made possible by choosing

$$n_\nu < \min(\mathcal{H}_{\nu 0},\mathcal{U}_{\nu 0}) \equiv \min_{\mathcal{Y}_\nu}\{-a_{\nu 1}k_\parallel(\mathcal{Y}_\nu), -b_{\nu 1}k_\parallel(\mathcal{Y}_\nu)\} \ . \tag{63}$$

These choices of n_ν guarantee that the integrand in Eq. (47) [and therefore $(\delta^2 H)_{\rm GP}$, since the Ψ_ν are localized in $\mathcal{H}_\nu,\mathcal{U}_\nu$] be negative for $\mathcal{H}_{\nu 0},\mathcal{U}_{\nu 0}$ and *all* $\mathcal{Y}_\nu$.

Note that when $\mathcal{D}f_\nu^{(0)}(\mathcal{H}_{\nu 0},\mathcal{U}_{\nu 0})>0$, $\delta^2 H<0$ is possible without imposing any conditions on either $k_{\parallel 0}$, or the spatial variation of the perturbations perpendicular to $\mathbf{B}^{(0)}$.

If $\mathcal{D}f_\nu^{(0)} = m_\nu \partial f_\nu^{(0)}/\partial\mathcal{H}_\nu + (a/\omega_\nu)\partial f_\nu^{(0)}/\partial\mathcal{U}_\nu = 2\partial f_\nu^{(0)}/\partial\dot{y}^2 < 0$ for some $\mathcal{H}_{\nu 0},\mathcal{U}_{\nu 0}$ corresponding to gyrating particles of any species ν, one again localizes Ψ_ν around $\mathcal{H}_{\nu 0},\mathcal{U}_{\nu 0}$. All other Ψ_μ, and all Ψ_ν for swinging particles are set equal to zero. The positive contribution of $(\partial\Psi_\nu/\partial\phi)^2$ to the integral in Eq. (47) can be eliminated by choosing $\Psi_\nu = \Psi_\nu(\mathcal{H}_\nu,\mathcal{U}_\nu,\mathcal{Y}_\nu)$, i.e., $\partial\Psi_\nu/\partial\phi = 0$. In this case, $\delta^2 H$ is given by

$$\delta^2 H = (\delta^2 H)_{\rm GP} = \sum_\nu \frac{s|\omega_\nu|}{4m_\nu^2}\int_{\rm (GP)} \frac{d\mathcal{H}_\nu d\mathcal{U}_\nu d\phi}{|\hat{d}_\nu\phi|}\left[|\mathcal{D}f_\nu^{(0)}|\langle \hat{d}_\nu\phi\rangle_{\tau_\phi}^2 \int d\mathcal{Y}_\nu \langle \Psi_\nu^2\rangle_{\tau_\phi}(\mathcal{H}_\nu,\mathcal{U}_\nu,\mathcal{Y}_\nu)(n_\nu + a_{\nu 1}k_{\parallel 0}\cos a\mathcal{Y}_\nu) \times (n_\nu + b_{\nu 1}k_{\parallel 0}\cos a\mathcal{Y}_\nu)\right] . \tag{64}$$

Since Ψ_ν is localized in $\mathcal{H}_\nu,\mathcal{U}_\nu$ around $\mathcal{H}_{\nu 0},\mathcal{U}_{\nu 0}$, the condition for $\delta^2 H<0$ is

$$\int d\mathcal{Y}_\nu \langle \Psi_\nu^2\rangle_{\tau_\phi}(\mathcal{H}_{\nu 0},\mathcal{U}_{\nu 0},\mathcal{Y}_\nu) b_{\nu 1}^2 k_{\parallel 0}^2 \times \left[\frac{n_\nu}{b_{\nu 1}k_{\parallel 0}} + \frac{a_{\nu 1}}{b_{\nu 1}}\cos a\mathcal{Y}_\nu\right] \times \left[\frac{n_\nu}{b_{\nu 1}k_{\parallel 0}} + \cos a\mathcal{Y}_\nu\right] < 0 \ . \tag{65}$$

If $a_{\nu 1}b_{\nu 1}<0$, it is clear that choosing $n_\nu=0$ satisfies inequality (65) *without any condition being imposed on* $k_{\parallel 0}$, *except* $k_{\parallel 0}\neq 0$. To understand what $k_{\parallel 0}^2 a_{\nu 1}b_{\nu 1}<0$ means, consider Eqs. (50) and (52), which yield

$$a_{\nu 1}b_{\nu 1} = \frac{q_\parallel^2}{\langle \hat{d}_\nu\phi\rangle_{\tau_\phi}^2}\left[1 + \frac{1}{[\mathcal{D}f_n^{(0)}]q_\parallel}\frac{\Omega_\nu}{|\omega_\nu|}\frac{\partial f_\nu^{(0)}}{\partial\mathcal{U}_\nu}\right] = \frac{1}{\langle \hat{d}_\nu\phi\rangle_{\tau_\phi}^2}\frac{1}{[\mathcal{D}f_\nu^{(0)}]}q_\parallel\left\langle \mathbf{e}_B(\mathcal{Y}_\nu)\cdot\frac{\partial f_\nu^{(0)}}{\partial\mathbf{v}}\right\rangle_{\tau_\phi} . \tag{66}$$

Since $\mathcal{D}f_\nu^{(0)}<0$ was assumed, $k_{\parallel 0}^2 a_{\nu 1}b_{\nu 1}<0$ means that

$$k_{\parallel 0}^2 q_\parallel\left\langle \mathbf{e}_B(\mathcal{Y}_\nu)\cdot\frac{\partial f_\nu^{(0)}}{\partial\mathbf{v}}\right\rangle_{\tau_\phi} > 0 \ . \tag{67}$$

Since the mean values $\langle\ \rangle_{\tau_\phi}$ are built along the particle orbit while the particle completes a ϕ turn around the field lines, this result closely resembles that obtained for a homogeneous plasma by Pfirsch and Morrison [6], Eq. (144.b), in the context of drift-kinetic theory.

If $a_{\nu 1}b_{\nu 1}>0$, it can be shown that the inequality

$$\left[\frac{n_\nu}{b_{\nu 1}k_{\parallel 0}} + \frac{a_{\nu 1}}{b_{\nu 1}}\cos a\mathcal{Y}_\nu\right]\left[\frac{n_\nu}{b_{\nu 1}k_{\parallel 0}} + \cos a\mathcal{Y}_\nu\right] < 0 \tag{68}$$

can be satisfied in a certain $\mathcal{Y}_\nu$ interval by appropriately choosing the arbitrary $n_\nu/k_{\parallel 0}$, and that inequality (65) can then be satisfied by making a mild assumption concerning the dependence of Ψ_ν on $\mathcal{Y}_\nu$.

Inequality (68) is satisfied if one factor is positive and the other is negative, i.e., if

$$\frac{n_\nu}{b_{\nu 1}k_{\parallel 0}} + \frac{a_{\nu 1}}{b_{\nu 1}}\cos a\mathcal{Y}_\nu > 0 \ \text{ and } \ \frac{n_\nu}{b_{\nu 1}k_{\parallel 0}} + \cos a\mathcal{Y}_\nu < 0 \ , \tag{69}$$

or if

$$\frac{n_\nu}{b_{\nu 1}k_{\parallel 0}} + \frac{a_{\nu 1}}{b_{\nu 1}}\cos a\mathcal{Y}_\nu < 0 \ \text{ and } \ \frac{n_\nu}{b_{\nu 1}k_{\parallel 0}} + \cos a\mathcal{Y}_\nu < 0 \ . \tag{70}$$

These inequalities are equivalent to

$$\frac{a_{\nu 1}}{b_{\nu 1}}\cos a\mathcal{Y}_\nu > -\frac{n_\nu}{b_{\nu 1}k_{\parallel 0}} > \cos a\mathcal{Y}_\nu \tag{71}$$

and

$$\frac{a_{\nu 1}}{b_{\nu 1}}\cos a\mathcal{Y}_\nu < -\frac{n_\nu}{b_{\nu 1}k_{\parallel 0}} < \cos a\mathcal{Y}_\nu \ . \tag{72}$$

$n_\nu/k_{\|0}$ can be chosen in such a way that the inequalities

$$\frac{a_{\nu 1}}{b_{\nu 1}} > -\frac{n_\nu}{b_{\nu 1}k_{\|0}} > 1 \quad \text{for} \quad \frac{a_{\nu 1}}{b_{\nu 1}} > 1 \tag{73}$$

or

$$\frac{a_{\nu 1}}{b_{\nu 1}} < -\frac{n_\nu}{b_{\nu 1}k_{\|0}} < 1 \quad \text{for} \quad \frac{a_{\nu 1}}{b_{\nu 1}} < 1 \tag{74}$$

are satisfied. This means that inequalities (71) and (72) are satisfied for $\cos a\mathcal{Y}_\nu = 1$, and also in an interval around this value, as shall presently be shown.

If $a_{\nu 1}/b_{\nu 1} > 1$, inequalities (73) imply that $1 > -n_\nu/a_{\nu 1}k_{\|0} > b_{\nu 1}/a_{\nu 1} > 0$. One can then define a $\mathcal{Y}_\nu^{(0)}$ by the equations

$$\cos a\mathcal{Y}_\nu^{(0)} = -\frac{n_\nu}{a_{\nu 1}k_{\|0}} > 0, \quad a\mathcal{Y}_\nu^{(0)} > 0 \ . \tag{75}$$

In this case

$$\left[\frac{n_\nu}{b_{\nu 1}k_{\|0}} + \frac{a_{\nu 1}}{b_{\nu 1}}\cos a\mathcal{Y}_\nu\right]\left[\frac{n_\nu}{b_{\nu 1}k_{\|0}} + \cos a\mathcal{Y}_\nu\right] = \frac{a_{\nu 1}}{b_{\nu 1}}(-\cos a\mathcal{Y}_\nu^{(0)} + \cos a\mathcal{Y}_\nu) \times \left[\frac{n_\nu}{b_{\nu 1}k_{\|0}} + 1 + \cos a\mathcal{Y}_\nu - 1\right] , \tag{76}$$

where $a_{\nu 1}/b_{\nu 1} > 0$, $-\cos a\mathcal{Y}_\nu^{(0)} < 0$, $n_\nu/b_{\nu 1}k_{\|0} + 1 < 0$ after Eq. (73), $\cos a\mathcal{Y}_\nu - 1 \le 0$ for all $\mathcal{Y}_\nu$. Inequality (68) can therefore be satisfied if $\cos a\mathcal{Y}_\nu - \cos a\mathcal{Y}_\nu^{(0)} > 0$, i.e., if

$$-a\mathcal{Y}_\nu^{(0)} \le a\mathcal{Y}_\nu \le a\mathcal{Y}_\nu^{(0)} \equiv \arccos\left[-\frac{n_\nu}{a_{\nu 1}k_{\|0}}\right] , \tag{77}$$

and inequality (65) can be satisfied if $\Psi_\nu(\mathcal{Y}_\nu)$ is chosen to vanish whenever $\cos a\mathcal{Y}_\nu - \cos a\mathcal{Y}_\nu^{(0)} \le 0$, i.e., when its integrand is positive. The characteristic length for the variation of $\Psi_\nu(\mathcal{Y}_\nu)$ perpendicular to $\mathbf{B}^{(0)}$ can therefore be as large as a^{-1}, which is usually very large.

If $a_{\nu 1}/b_{\nu 1} < 1$, the assumption $a_{\nu 1}b_{\nu 1} > 0$ and inequalities (74) imply that $0 < a_{\nu 1}/b_{\nu 1} < -n_\nu/b_{\nu_1}k_{\|0} < 1$. One can then define a $\mathcal{Y}_\nu^{(1)}$ by the equations

$$\cos a\mathcal{Y}_\nu^{(1)} = -\frac{n_\nu}{b_{\nu 1}k_{\|0}} > 0 \ , \quad a\mathcal{Y}_\nu^{(1)} > 0 \ . \tag{78}$$

In this case

$$\left[\frac{n_\nu}{b_{\nu 1}k_{\|0}} + \cos a\mathcal{Y}_\nu\right]\left[\frac{n_\nu}{b_{\nu 1}k_{\|0}} + \frac{a_{\nu 1}}{b_{\nu 1}}\cos a\mathcal{Y}_\nu\right] = \frac{a_{\nu 1}}{b_{\nu 1}}(-\cos a\mathcal{Y}_\nu^{(1)} + \cos a\mathcal{Y}_\nu) \times \left[\frac{n_\nu}{a_{\nu 1}k_{\|0}} + 1 + \cos a\mathcal{Y}_\nu - 1\right] , \tag{79}$$

where $a_{\nu 1}/b_{\nu 1} > 0$, $-\cos a\mathcal{Y}_\nu^{(1)} < 0$, $n_\nu/a_{\nu 1}k_{\|0} + 1 < 0$ after Eq. (74), $\cos a\mathcal{Y}_\nu - 1 \le 0$ for all $\mathcal{Y}_\nu$. As in the preceding case, inequality (68) can be satisfied if

$$-a\mathcal{Y}_\nu^{(1)} \le a\mathcal{Y}_\nu \le a\mathcal{Y}_\nu^{(1)} \equiv \arccos\left[-\frac{n_\nu}{b_{\nu 1}k_{\|0}}\right] , \tag{80}$$

and (65) can be satisfied if $\Psi_\nu(\mathcal{Y}_\nu)$ is chosen to vanish whenever $\cos a\mathcal{Y}_\nu - \cos a\mathcal{Y}_\nu^{(1)} \le 0$. The characteristic length for the variation of $\Psi_\nu(\mathcal{Y}_\nu)$ perpendicular to $\mathbf{B}^{(0)}$ is, of course, as in the preceding case, i.e., it can be as large as $\sim a^{-1}$.

Inequalities (73) and (74) extend to the inhomogeneous case the results obtained for a homogeneous plasma in Ref. [5], Eqs. (49) and (50). From Eqs. (50) and (52), one obtains

$$\frac{a_{\nu 1}}{b_{\nu 1}} = 1 + \frac{1}{[\mathcal{D}f_\nu^{(0)}]}\frac{\Omega_\nu}{|\omega_\nu|}\frac{1}{q_\|}\frac{\partial f_\nu^{(0)}}{\partial \mathcal{U}_\nu} , \tag{81}$$

a quantity which can be interpreted as the local anisotropy of the distribution function, and which coincides with the previous definition of the anisotropy in the homogeneous case.

It has just been shown that, when $\mathcal{D}f_\nu^{(0)} > 0$, it is always possible to have $\delta^2 H < 0$ without any restriction on $k_{\|0}$ or the spatial variation of the perturbation perpendicular to $\mathbf{B}^{(0)}$. When $\mathcal{D}f_\nu^{(0)} < 0$ and $a_{\nu 1}$ and $b_{\nu 1}$ have different signs, i.e., when $q_\| \langle \mathbf{e}_B(\mathcal{Y}_\nu)\cdot(\partial f_\nu^{(0)}/\partial \mathbf{v}\rangle_{\tau_\phi} > 0$, it is also possible to have $\delta^2 H < 0$, without any restriction on $k_{\|0}$, except $k_{\|0} \neq 0$, and without any restrictions on the spatial variation of the perturbation perpendicular to $\mathbf{B}^{(0)}$. In the case that $\mathcal{D}f_\nu^{(0)} < 0$ and $a_{\nu 1}b_{\nu 1} > 0$, however, $k_{\|0}$ is restricted by inequalities (73) or (74), and the characteristic length for the variation of the perturbation Ψ_ν perpendicular to $\mathbf{B}^{(0)}$ must be of the order $\lesssim a^{-1}$, which is *not* an important restriction.

If $\mathcal{D}f_\nu^{(0)} < 0$ and $a_{\nu 1} = b_{\nu 1}$ for $\mathcal{H}_\nu = \mathcal{H}_{\nu 0}$, $\mathcal{U}_\nu = \mathcal{U}_{\nu 0}$, the equilibrium distribution function is *locally* monotonically decreasing and isotropic, and inequality (65) cannot be satisfied for these $\mathcal{H}_{\nu 0}, \mathcal{U}_{\nu 0}$. If $\mathcal{D}f_\nu^{(0)} < 0$ and $a_{\nu 1} = b_{\nu 1}$ for all $\mathcal{H}_\nu, \mathcal{U}_\nu$, then $f_\nu^{(0)} = f_\nu^{(0)}(\mathcal{H}_\nu)$, the equilibrium is everywhere isotropic and homogeneous, there is no electric current and $a = 0$. The equilibrium distribution function is a monotonically decreasing function of the particle energy, and no negative-energy modes are possible, in accordance with the general results obtained in Ref. [7].

2. *The wave energy* $(\delta^2 H)_{\rm SP}$ *for swinging particles*

These particles, for which the condition $av_\|/\omega_\nu > 1$ must be satisfied, do not have the same importance as the gyrating particles. They must, however, be treated whenever the equilibrium distribution functions allow arbitrarily large velocities.

Again, two cases concerning $k_{\|0}$ are distinguished:

a. $k_{\|0} = 0$, *i.e.,* $k_\|(\mathcal{Y}_\nu) = 0$ *for all* $\mathcal{Y}_\nu$ *(wave propagation perpendicular to* $B^{(0)}$*).* In this case, Eq. (48) yields

$$(\delta^2 H)_{\rm SP}=\sum_\nu \frac{s|\omega_\nu|}{4m_\nu^2}\int_{\rm (SP)} \frac{d\mathcal{H}_\nu d\mathcal{U}_\nu d\mathcal{Y}_\nu dy}{|\hat{d}_\nu y|}[\mathcal{D}f_\nu^{(0)}] \times\left[-[\hat{d}_\nu y]^2\left[\frac{\partial\Psi_\nu}{\partial y}\right]^2 - \frac{\langle \hat{d}_\nu y\rangle_{\tau_y}^2}{\langle 1/\Psi_\nu^2\rangle_{\tau_y}}\left[\frac{a}{2\pi}\Delta\Gamma_\nu\right]^2\right], \quad (82)$$

and $\delta^2 H<0$ if $\mathcal{D}f_\nu^{(0)}=m_\nu \partial f_\nu^{(0)}/\partial\mathcal{H}_\nu+(a/\omega_\nu)\partial f_\nu^{(0)}/\partial\mathcal{U}_\nu=2\partial f_\nu^{(0)}/\partial\dot{y}^{\,2}>0$ for some $\mathcal{H}_{\nu 0},\mathcal{U}_{\nu 0}$ corresponding to swinging particles and for any particle species ν. It suffices to localize Ψ_ν ($\partial\Psi_\nu/\partial y$ is then also localized) to the region in $\mathcal{H}_\nu,\mathcal{U}_\nu$ where $\partial f_\nu^{(0)}/\partial\dot{y}^{\,2}>0$. Outside this region Ψ_ν vanishes. All other Ψ_μ are set equal to zero. The Ψ_ν corresponding to the gyrating particles are also all set to zero, so that $(\delta^2 H)_{\rm GP}=0$. The sign of $\delta^2 H=(\delta^2 H)_{\rm SP}$ is then determined only by the sign of the integrand in the region of localization.

b. $k_\parallel(\mathcal{Y}_\nu)=k_{\parallel 0}\cos a\mathcal{Y}_\nu$ does not vanish for all $\mathcal{Y}_\nu$ (the wave vector has a component in the direction of $B^{(0)}$ for all $\mathcal{Y}_\nu$, except $a\mathcal{Y}_\nu=\pm\pi/2\pm m\pi$). If $\mathcal{D}f_\nu^{(0)}=m_\nu\partial f_\nu^{(0)}/\partial\mathcal{H}_\nu+(a/\omega_\nu)\partial f_\nu^{(0)}/\partial\mathcal{U}_\nu=2\partial f_\nu^{(0)}/\partial\dot{y}^{\,2}>0$ for some $\mathcal{H}_{\nu 0},\mathcal{U}_{\nu 0}$ corresponding to swinging particles and for any species ν, one again localizes the perturbations Ψ_ν around these values, as in the preceding case. All Ψ_ν corresponding to gyrating particles are set equal to zero; therefore, $(\delta^2 H)_{\rm GP}=0$. Following the same line of argumentation as in the case of gyrating particles, it is easily shown that $\delta^2 H<0$ is possible without imposing any conditions on either $k_{\parallel 0}$ or the spatial variation of the perturbation perpendicular to $\mathbf{B}^{(0)}$.

If

$$\mathcal{D}f_\nu^{(0)}=m_\nu\partial f_\nu^{(0)}/\partial\mathcal{H}_\nu+(a/\omega_\nu)\partial f_\nu^{(0)}/\partial\mathcal{U}_\nu =2\partial f_\nu^{(0)}/\partial\dot{y}^{\,2}<0$$

for some $\mathcal{U}_{\nu 0},\mathcal{H}_{\nu 0}$ corresponding to swinging particles of any species ν, one again localizes Ψ_ν around $\mathcal{H}_{\nu 0},\mathcal{U}_{\nu 0}$. All other Ψ_μ, and all Ψ_ν for gyrating particles are set equal to zero. The positive contribution of $(\partial\Psi_\nu/\partial y)^2$ to the integral in Eq. (48) can be eliminated by choosing $\Psi_\nu=\Psi_\nu(\mathcal{H}_\nu,\mathcal{U}_\nu,\mathcal{Y}_\nu)$, i.e., $\partial\Psi_\nu/\partial y=0$. In this case, $\delta^2 H$ is given by

$$\delta^2 H=(\delta^2 H)_{\rm SP}=\sum_\nu \frac{s|\omega_\nu|}{4m_\nu^2}\int_{\rm (SP)}\frac{d\mathcal{H}_\nu d\mathcal{U}_\nu dy}{|\hat{d}_\nu y|}\left[|\mathcal{D}f_\nu^{(0)}|\langle \hat{d}_\nu y\rangle_{\tau_y}^2\int d\mathcal{Y}_\nu\times\langle\Psi_\nu^2\rangle_{\tau_y}(\mathcal{H}_\nu,\mathcal{U}_\nu,\mathcal{Y}_\nu)\left[\frac{a}{2\pi}\Delta\Gamma_\nu+a_{\nu 2}k_{\parallel 0}\cos a\mathcal{Y}_\nu\right] \times\left[\frac{a}{2\pi}\Delta\Gamma_\nu+b_{\nu 2}k_{\parallel 0}\cos a\mathcal{Y}_\nu\right]\right]. \quad (83)$$

Since Ψ_ν is localized in $\mathcal{H}_\nu,\mathcal{U}_\nu$ around $\mathcal{H}_{\nu 0}\mathcal{U}_{\nu 0}$, the condition for $\delta^2 H<0$ is

$$\int d\mathcal{Y}_\nu\langle\Psi_\nu^2\rangle_{\tau_y}(\mathcal{H}_{\nu 0},\mathcal{U}_{\nu 0},\mathcal{Y}_\nu)b_{\nu 2}^2 k_{\parallel 0}^2 \times\left[\frac{a\Delta\Gamma_\nu}{2\pi b_{\nu 2}k_{\parallel 0}}+\frac{a_{\nu 2}}{b_{\nu 2}}\cos a\mathcal{Y}_\nu\right] \times\left[\frac{a\Delta\Gamma_\nu}{2\pi b_{\nu 2}k_{\parallel 0}}+\cos a\mathcal{Y}_\nu\right]<0\,. \quad (84)$$

If $a_{\nu 2}b_{\nu 2}<0$, choosing $\Delta\Gamma_\nu=0$ satisfies inequality (84) *without any condition being imposed on $k_{\parallel 0}$, except $k_{\parallel 0}\neq 0$.* If $a_{\nu 2}b_{\nu 2}>0$, one defines $a\mathcal{Y}_\nu^{(0)}$ and $a\mathcal{Y}_\nu^{(1)}$ as in Eqs. (75) and (78), but with $a_{\nu 2}$ and $b_{\nu 2}$ instead of $a_{\nu 1}$ and $b_{\nu 1}$. It can then be shown as in the case of gyrating particles that the inequality

$$\left[\frac{a\Delta\Gamma_\nu}{2\pi b_{\nu 2}k_{\parallel 0}}+\frac{a_{\nu 2}}{b_{\nu 2}}\cos a\mathcal{Y}_\nu\right]\left[\frac{a\Delta\Gamma_\nu}{2\pi b_{\nu 2}k_{\parallel 0}}+\cos a\mathcal{Y}_\nu\right]<0 \quad (85)$$

can be satisfied in the interval

$$-a\mathcal{Y}_\nu^{(0)}\le a\mathcal{Y}_\nu\le a\mathcal{Y}_\nu^{(0)}\equiv\arccos\left[-\frac{a\Delta\Gamma_\nu}{2\pi a_{\nu 2}k_{\parallel 0}}\right] \quad \text{for } \frac{a_{\nu 2}}{b_{\nu_2}}>1 \quad (86)$$

or

$$-a\mathcal{Y}_\nu^{(1)}\le a\mathcal{Y}_\nu\le a\mathcal{Y}_\nu^{(1)}\equiv\arccos\left[-\frac{a\Delta\Gamma_\nu}{2\pi b_{\nu 2}k_{\parallel 0}}\right] \quad \text{for } \frac{a_{\nu 2}}{b_{\nu 2}}<1 \quad (87)$$

if the inequalities

$$\frac{a_{\nu 2}}{b_{\nu 2}}>-\frac{a\Delta\Gamma_\nu}{2\pi b_{\nu 2}k_{\parallel 0}}>1 \quad \text{for } \frac{a_{\nu 2}}{b_{\nu 2}}>1\,, \quad (88)$$

$$\frac{a_{\nu 2}}{b_{\nu 2}}<-\frac{a\Delta\Gamma_\nu}{2\pi b_{\nu 2}k_{\parallel 0}}<1 \quad \text{for } \frac{a_{\nu 2}}{b_{\nu 2}}<1\,, \quad (89)$$

are satisfied. Therefore, inequality (84) can be satisfied if $\Psi_\nu(\mathcal{Y}_\nu)$ is chosen to vanish whenever $\mathcal{Y}_\nu$ does not satisfy (86) (for $a_{\nu 2}/b_{\nu 2}>1$) or (87) (for $a_{\nu 2}/b_{\nu 2}<1$).

Contrary to the case of gyrating particles, inequalities (88) and (89) do not impose any condition on $k_{\parallel 0}$, except $k_{\parallel 0}\neq 0$. $k_{\parallel 0}$ can be chosen arbitrarily, and then the arbitrary $\Delta\Gamma_\nu$ can be chosen so as to satisfy inequality (88) or (89). In the case of gyrating particles, on the other hand, one does not have the arbitrary $\Delta\Gamma_\nu$, but n_ν, which is not completely arbitrary because it must be an *integer* number.

It has just been shown that when $\mathcal{D}f_\nu^{(0)}>0$ for some $\mathcal{H}_\nu,\mathcal{U}_\nu$ corresponding to swinging particles, $\delta^2 H<0$ is always possible, without any restriction on $k_{\parallel 0}$ or on the spatial variation of the perturbation perpendicular to $\mathbf{B}^{(0)}$. When $\mathcal{D}f_\nu^{(0)}<0$ and $a_{\nu 2}$ and $b_{\nu 2}$ have different signs, i.e., when $r_\parallel\langle \mathbf{e}_B(\mathcal{Y}_\nu)\cdot(\partial f_\nu^{(0)}/\partial \mathbf{v}\rangle_{\tau_\phi}>0$, $\delta^2 H<0$ is also possible without any restriction on $k_{\parallel 0}$, except $k_{\parallel 0}\neq 0$, and without any restrictions on the spatial variation of the perturbation perpendicular to $\mathbf{B}^{(0)}$. In the case that $a_{\nu 2}b_{\nu 2}>0$, there is also no restriction on $k_{\parallel 0}$. However, in this case, the characteristic length for the variation of the perturbation Ψ_ν perpendicular to $\mathbf{B}^{(0)}$ must be of the order $\lesssim a^{-1}$, which is *not* an important restriction.

VI. CONCLUSIONS

In the case of an inhomogeneous, force-free Vlasov-Maxwell plasma with sheared magnetic field, waves of negative energy ($\delta^2 H<0$) exist for any local deviation from monotonicity [i.e., if $\mathcal{D}f_\nu^{(0)}\equiv m_\nu \partial f_\nu^{(0)}/\partial \mathcal{H}_\nu + (a/\omega_\nu)\partial f_\nu^{(0)}/\partial \mathcal{U}_\nu = 2\partial f_\nu^{(0)}/\partial \dot{y}^2>0$ for some $\mathcal{H}_\nu, \mathcal{U}_\nu$] *for any wave number* $\mathbf{k}$, irrespective of its magnitude and orientation. If $\partial f_\nu^{(0)}/\partial \dot{y}^2<0$, only the waves with a component $\mathbf{k}_{\parallel 0}$ of $\mathbf{k}_\parallel$ in the direction $\mathbf{B}^{(0)}(y_0)$ can possess negative energy. If $\partial f_\nu^{(0)}/\partial \dot{y}^2<0$, but $k_{\parallel 0}^2\langle \mathbf{w}\rangle\langle \mathbf{e}_B(y_0)\cdot(\partial f_\nu^{(0)}/\partial \mathbf{v})\rangle>0$ ($\langle \mathbf{w}\rangle$ is an averaged parallel velocity, the angles represent averages along the particle orbits), negative-energy waves also exist, *with no restriction imposed on either* $k_{\parallel 0}$ *(others than* $k_{\parallel 0}\neq 0$*) or the spatial variation of the perturbation perpendicular to* $\mathbf{B}^{(0)}$. This result agrees with, and closely resembles, that obtained for a homogeneous plasma by Pfirsch and Morrison [6], Eq. (144.b), in the context of drift-kinetic theory.

If both $\partial f_\nu^{(0)}/\partial \dot{y}^2<0$ and $k_{\parallel 0}^2\langle \mathbf{w}\rangle\langle \mathbf{e}_B(y_0)\cdot(\partial f_\nu^{(0)}/\partial \mathbf{v})\rangle<0$, negative-energy modes also exist. In this case, the characteristic length for the variation of the perturbation Ψ_ν perpendicular to $\mathbf{B}^{(0)}$ is of the order of the shear length a^{-1} (or smaller), and there is generally a restriction on the possible *parallel* wave numbers [conditions (73) and (74), which are limited to a certain interval, this also being so in the homogeneous case]. The results of that case are of course regained by taking the limit of vanishing shear, $a\rightarrow 0$. The present results show that large perpendicular wave numbers are *not* necessary for the existence of negative-energy waves in the system under consideration, a feature which enhances the relevance of these modes.

APPENDIX A: COORDINATES IN v SPACE

The magnetic field of the equilibrium considered is

$$\mathbf{B}^{(0)}=B^{(0)}\mathbf{e}_B,\quad \mathbf{e}_B=\sin ay\,\mathbf{e}_x+\cos ay\,\mathbf{e}_z\ , \tag{A1}$$

where x, y, and z are Cartesian coordinates and $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ are the corresponding unit basis vectors. For this configuration, it is convenient to introduce, besides the Cartesian velocities $\dot{x}$, $\dot{y}$, and $\dot{z}$, a *local* cylindrical coordinates system $v_\perp$, ϕ, and $v_\parallel$ in $\mathbf{v}$ space, which is particularly appropriate to the problem and, in fact, makes it tractable. This decomposition is, of course, space dependent, i.e., for a given vector $\mathbf{v}$, the components $v_\perp$, ϕ, and $v_\parallel$ will differ depending on where in $\mathbf{x}$ space the decomposition is carried out. In this system, $v_\perp$ is the magnitude of $\mathbf{v}_\perp$, the projection of $\mathbf{v}$ onto the plane perpendicular to $\mathbf{B}^{(0)}$, and $v_\parallel$ is the projection of $\mathbf{v}$ in the direction of the magnetic field. The remaining velocity coordinate ϕ is the angle between $\mathbf{v}_\perp$ and the vector $\mathbf{e}_1$ defined by

$$\mathbf{e}_1=\mathbf{e}_y\times\mathbf{e}_B=\cos ay\,\mathbf{e}_x-\sin ay\,\mathbf{e}_z\ . \tag{A2}$$

If one introduces unit basis vectors $\mathbf{e}_{v_\perp}$ and $\mathbf{e}_\phi$ in the direction of $\mathbf{v}_\perp$ and $\mathbf{B}^{(0)}\times\mathbf{v}_\perp$, respectively, then the following relations obtain:

$$\mathbf{e}_{v_\perp}=\cos\phi\,\mathbf{e}_1+\sin\phi\,\mathbf{e}_y$$
$$=\cos\phi\cos ay\,\mathbf{e}_x+\sin\phi\,\mathbf{e}_y-\cos\phi\sin ay\,\mathbf{e}_z\ , \tag{A3}$$

$$\mathbf{e}_\phi=-\sin\phi\,\mathbf{e}_1+\cos\phi\,\mathbf{e}_y$$
$$=-\sin\phi\cos ay\,\mathbf{e}_x+\cos\phi\,\mathbf{e}_y+\sin\phi\sin ay\,\mathbf{e}_z\ , \tag{A4}$$

and the velocity $\mathbf{v}$ can be expressed as

$$\mathbf{v}=v_\perp(y,\mathbf{v})\mathbf{e}_{v_\perp}(y,\mathbf{v})+v_\parallel(y,\mathbf{v})\mathbf{e}_B(y)\ . \tag{A5}$$

The relations between $v_\perp$, ϕ, and $v_\parallel$ and the Cartesian velocity coordinates $\dot{x}$, $\dot{y}$, and $\dot{z}$ are therefore given by

$$v_\perp=\dot{x}\cos\phi\cos ay+\dot{y}\sin\phi-\dot{z}\cos\phi\sin ay\ , \tag{A6}$$

$$v_\parallel=\dot{x}\sin ay+\dot{z}\cos ay\ , \tag{A7}$$

$$\dot{x}=v_\perp\cos\phi\cos ay+v_\parallel\sin ay\ , \tag{A8}$$

$$\dot{y}=v_\perp\sin\phi\ , \tag{A9}$$

$$\dot{z}=-v_\perp\cos\phi\sin ay+v_\parallel\cos ay\ . \tag{A10}$$

From the foregoing expressions, the following useful relations can be derived:

$$\left.\frac{\partial v_\perp}{\partial \mathbf{x}}\right|_{\mathbf{v}}=-av_\parallel\cos\phi\,\mathbf{e}_y\ ,\quad \left.\frac{\partial \phi}{\partial \mathbf{x}}\right|_{\mathbf{v}}=a\frac{v_\parallel}{v_\perp}\sin\phi\,\mathbf{e}_y\ ,\quad \left.\frac{\partial v_\parallel}{\partial \mathbf{x}}\right|_{\mathbf{v}}=av_\perp\cos\phi\,\mathbf{e}_y\ , \tag{A11}$$

$$\left.\frac{\partial v_\perp}{\partial \mathbf{v}}\right|_{\mathbf{x}}=\mathbf{e}_{v\perp},\quad \left.\frac{\partial \phi}{\partial \mathbf{v}}\right|_{\mathbf{x}}=\frac{\mathbf{e}_\phi}{v_\perp},\quad \left.\frac{\partial v_\parallel}{\partial \mathbf{v}}\right|_{\mathbf{x}}=\mathbf{e}_B\ , \tag{A12}$$

$$\left.\frac{\partial \mathbf{v}}{\partial v_\perp}\right|_{\mathbf{x},\phi,v_\parallel}=\mathbf{e}_{v_\perp}\ ,\quad \left.\frac{\partial \mathbf{v}}{\partial \phi}\right|_{\mathbf{x},v_\perp,v_\parallel}=v_\perp\mathbf{e}_\phi\ ,\quad \left.\frac{\partial \mathbf{v}}{\partial v_\parallel}\right|_{\mathbf{x},v_\perp,\phi}=\mathbf{e}_B\ . \tag{A13}$$

The volume element in $\mathbf{v}$ space is therefore given by the obvious expression

$$d^3v=v_\perp dv_\perp d\phi\, dv_\parallel\ . \tag{A14}$$

Given a function $G_\nu(\mathbf{x},\mathbf{v})$, one then has the following relations:

$$\left.\frac{\partial G_\nu}{\partial \mathbf{x}}\right|_{\mathbf{v}} = \left.\frac{\partial G_\nu}{\partial \mathbf{x}}\right|_{v_\perp,\phi,v_\parallel} - av_\parallel\cos\phi \left.\frac{\partial G_\nu}{\partial v_\perp}\right|_{\mathbf{x},\phi,v_\parallel}\mathbf{e}_y + a\frac{v_\parallel}{v_\perp}\sin\phi\left.\frac{\partial G_\nu}{\partial \phi}\right|_{\mathbf{x},v_\perp,v_\parallel}\mathbf{e}_y + av_\perp\cos\phi\left.\frac{\partial G_\nu}{\partial v_\parallel}\right|_{\mathbf{x},\phi,v_\perp}\mathbf{e}_y\ , \tag{A15}$$

$$\left.\frac{\partial G_\nu}{\partial \mathbf{v}}\right|_{\mathbf{x}} = \left.\frac{\partial G_\nu}{\partial v_\perp}\right|_{\mathbf{x},\phi,v_\parallel}\mathbf{e}_{v_\perp} + \left.\frac{\partial G_\nu}{\partial \phi}\right|_{\mathbf{x},v_\perp,v_\parallel}\frac{\mathbf{e}_\phi}{v_\perp} + \left.\frac{\partial G_\nu}{\partial v_\parallel}\right|_{\mathbf{x},v_\perp,\phi}\mathbf{e}_B\ . \tag{A16}$$

APPENDIX B: PARTICLE ORBITS, GYRATING PARTICLES, AND SWINGING PARTICLES

Owing to the fact that the canonical momenta $p_{x\nu}$ and $p_{z\nu}$, Eqs. (6) and (8) are constants of the motion, the particles moving in the magnetic field $\mathbf{B}^{(0)}=B^{(0)}(\sin ay\,\mathbf{e}_x+\cos ay\,\mathbf{e}_z)$ can be considered as effectively being in a one-dimensional potential $V(y)$ and, with the notation of Sec. II, the Hamiltonian, Eq. (9), can be expressed as

$$H_\nu=\frac{p_{y\nu}^2}{2m_\nu}+V_\nu(y)\ , \tag{B1}$$

with

$$V_\nu=\frac{m_\nu}{2}(\dot x^2+\dot z^2)=\mathcal{H}_\nu-\frac{m_\nu\omega_\nu}{a}\mathcal{U}_\nu+\frac{m_\nu\omega_\nu}{a}v_\parallel\ . \tag{B2}$$

By taking into account that

$$av_\parallel-\omega_\nu=a(\dot x\sin ay+\dot z\cos ay)-\omega_\nu = \frac{a}{m_\nu}\left[p_{x\nu}\sin ay+\left[p_{z\nu}-\frac{m_\nu\omega_\nu}{a}\right]\cos ay\right] \tag{B3}$$

and defining a frequency Ω_ν by

$$\Omega_\nu=|\omega_\nu|\left[\frac{a^2p_{x\nu}^2}{\omega_\nu^2m_\nu^2}+\left[\frac{ap_{z\nu}}{\omega_\nu m_\nu}-1\right]^2\right]^{1/2} =\frac{1}{m_\nu}[a^2p_{x\nu}^2+(ap_{z\nu}-m_\nu\omega_\nu)^2]^{1/2} =[a^2v_\perp^2\cos^2\phi+(av_\parallel-\omega_\nu)^2]^{1/2} =\left[\frac{2}{m_\nu}a^2\mathcal{H}_\nu-2a\,\omega_\nu\mathcal{U}_\nu+\omega_\nu^2\right]^{1/2} \tag{B4}$$

and an angle $a\mathcal{Y}_\nu$ by

$$\sin a\mathcal{Y}_\nu=-\frac{|\omega_\nu|}{\omega_\nu}\frac{ap_{x\nu}}{m_\nu\Omega_\nu}\ , \tag{B5}$$

$$\cos a\mathcal{Y}_\nu=\frac{|\omega_\nu|}{\Omega_\nu}\left[1-\frac{ap_{z\nu}}{\omega_\nu m_\nu}\right]\ , \tag{B6}$$

$av_\parallel/\omega_\nu-1$ and the effective potential $V(y)$ can then be expressed as

$$\frac{av_\parallel}{\omega_\nu}-1=-\frac{\Omega_\nu}{|\omega_\nu|}\cos a(\mathcal{Y}_\nu-y) \tag{B7}$$

and

$$V(y)=\frac{m_\nu}{2a^2}(\Omega_\nu^2+\omega_\nu^2)-\frac{m_\nu|\omega_\nu|\Omega_\nu}{a^2}\cos a(\mathcal{Y}_\nu-y)\ , \tag{B8}$$

respectively.

The particles moving in this periodic potential can be divided into two classes: those whose energy $\mathcal{H}_\nu$ is so large that they can overcome the potential barriers determined by the maximum value of $V_\nu(y)$, $(V_\nu)_{\max}$ and move freely in the y direction, since for them $\dot y$ never vanishes, and those (in fact, the overwhelming majority in all situations of interest) with energy lower than $(V_\nu)_{\max}$, which are trapped, their motion being confined to a certain y region around $y=\mathcal{Y}_\nu$. The maximum value of $V_\nu(y)$ can be determined from Eq. (B8) and is

$$(V_\nu)_{\max}=\frac{m_\nu}{2a^2}(\Omega_\nu+|\omega_\nu|)^2\ . \tag{B9}$$

The condition for a particle to move freely in the y direction is then

$$\mathcal{H}_\nu-(V_\nu)_{\max}>0\ . \tag{B10}$$

Although the energies for which this condition is satisfied are usually very high [corresponding to $av_\parallel/\omega_\nu-1\ge 0$, i.e., $a(R_g)_{\rm th}v_\parallel/(v_\perp)_{\rm th}-1\ge 0$, see Sec. V B 1], the corresponding particles (which play a particular role in guiding center theories [8,9]) must be taken into account when the distribution functions allow arbitrarily large velocities, for instance when one considers Maxwell distributions.

The two groups of particles, those oscillating about the planes $y=\mathcal{Y}_\nu$ and those moving freely along y with $\dot y$ vanishing nowhere, can be characterized by the behavior of the quantity $\hat d_\nu\phi$, which is, by the definition of $\hat d_\nu$ in Eq. (27), the rate of change of ϕ experienced by the moving particle, i.e., $\hat d_\nu\phi=(d\phi/dt)_{\text{along orbits}}$. When $\hat d_\nu\phi$ has no zeros (and ϕ therefore changes monotonically with time along the particle orbit), the particles gyrate around the field lines while they oscillate about the plane $y=\mathcal{Y}_\nu$. This is the group of the *gyrating particles.* If $\hat d_\nu\phi$ vanishes for a certain ϕ_0, it also vanishes for $\pi-\phi_0$ (and for $-\phi_0$ and $-\pi+\phi_0$). Calculating the second derivative of ϕ for these values, i.e., $[\hat d_\nu(\hat d_\nu\phi)](\phi=\phi_0)$, $[\hat d_\nu(\hat d_\nu\phi)](\phi=\pi-\phi_0)$, it can be shown that ϕ_0 and $\pi-\phi_0$ (the same is valid for $-\phi_0, -\pi+\phi_0$) are, respectively, minimum and maximum values of ϕ. *The velocity* $\mathbf{v}_\perp$ *of the particle swings between these two angles, the particle never completing a* ϕ *turn. At the same time, it moves freely in the y direction,* $\dot y=v_\perp\sin\phi$ *never vanishing. These are the swinging particles.* To understand this relation between $\hat d_\nu\phi$ and the behavior of the particles, consider the quantity $\mathcal{H}_\nu-(V_\nu)_{\max}$, which, taking into account Eq. (B9), reads

$$\mathcal{H}_\nu-(V_\nu)_{\max}=m_\nu\frac{\omega_\nu^2}{a^2}\left[\frac{a}{\omega_\nu}\mathcal{U}_\nu-1-\left[\frac{2}{m_\nu}\frac{a^2}{\omega_\nu^2}\mathcal{H}_\nu-2\frac{a}{\omega_\nu}\mathcal{U}_\nu+1\right]^{1/2}\right]$$
$$=m_\nu\frac{\omega_\nu^2}{a^2}\left\{\left[\frac{a}{\omega_\nu}v_\parallel-1\right]+\frac{1}{2}\frac{a^2}{\omega_\nu^2}v_\perp^2\sin^2\phi-\left[\left[\frac{a}{\omega_\nu}v_\parallel-1\right]^2+\frac{a^2}{\omega_\nu^2}v_\perp^2\cos^2\phi\right]^{1/2}\right\} . \tag{B11}$$

If $\hat{d}_\nu\phi=av_\parallel\sin^2\phi-\omega_\nu\neq0$ for all $v_\parallel,\phi$, then this means that $(a/\omega_\nu)v_\parallel-1<0$, since otherwise $\hat{d}_\nu\phi=0$ would be possible. Then, taking $\phi=0$, $v_\parallel=v_{\parallel0}(\mathcal{H}_\nu,\mathcal{U}_\nu,\phi=0)$, $v_\perp=v_{\perp0}(\mathcal{H}_\nu,\mathcal{U}_\nu,\phi=0)$, which does not mean any restriction, since $\mathcal{H}_\nu-(V_\nu)_{\max}$ is a constant of the motion, Eq. (B11) yields

$$\mathcal{H}_\nu-(V_\nu)_{\max}=-m_\nu\frac{\omega_\nu^2}{a^2}\left\{\left|\frac{a}{\omega_\nu}v_{\parallel0}-1\right|+\left[\left[\frac{a}{\omega_\nu}v_{\parallel0}-1\right]^2+\frac{a^2}{\omega_\nu^2}v_{\perp0}^2\right]^{1/2}\right\}<0 \tag{B12}$$

and the particles are therefore trapped.

If $\hat{d}_\nu\phi=av_\parallel\sin^2\phi-\omega_\nu=0$ for a ϕ_0 and a $v_{\parallel0}$, then

$$\frac{a}{\omega_\nu}v_{\parallel0}=\frac{1}{\sin^2\phi_0} , \tag{B13}$$

and, setting

$$c_0^2=\frac{a^2}{\omega_\nu^2}v_{\perp0}^2\frac{\sin^4\phi_0}{\cos^2\phi_0} , \tag{B14}$$

one obtains from Eq. (B11)

$$\mathcal{H}_\nu-(V_\nu)_{\max}=\frac{m_\nu}{2}\frac{\omega_\nu^2}{a^2}\frac{\cos^2\phi_0}{\sin^2\phi_0}[2+c_0^2-2(1+c_0^2)^{1/2}]$$
$$=\frac{m_\nu}{2}\frac{\omega_\nu^2}{a^2}\frac{\cos^2\phi_0}{\sin^2\phi_0}[(1+c_0^2)^{1/2}-1]^2>0 \tag{B15}$$

and the particles move freely in the y direction.

A quantity which plays a crucial role in determining the sign of δ^2H is $\mathcal{D}f_\nu^{(0)}=m_\nu\partial f_\nu^{(0)}/\partial\mathcal{H}_\nu+(a/\omega_\nu)\partial f_\nu^{(0)}/\partial\mathcal{U}_\nu$. If this quantity has a certain sign for $\mathcal{H}_\nu=\mathcal{H}_{\nu0}$, $\mathcal{U}_\nu=\mathcal{U}_{\nu0}$, one can determine to what kind of orbit these values correspond in the following way: by taking into account Eqs. (C8) and (C10), the parallel velocity can be expressed as $v_\parallel=v_\parallel(\mathcal{H}_{\nu0},\mathcal{U}_{\nu0},\phi)$, and $\hat{d}_\nu\phi$ is then given by

$$\hat{d}_\nu\phi=av_\parallel\sin^2\phi-\omega_\nu$$
$$=\pm\omega_\nu\left[1+\frac{2a^2}{m_\nu\omega_\nu^2}\mathcal{H}_{\nu0}\sin^4\phi-\frac{2a}{\omega_\nu}\mathcal{U}_{\nu0}\sin^2\phi\right]^{1/2} . \tag{B16}$$

If $a\mathcal{U}_{\nu0}/\omega_\nu\le0$, then $\hat{d}_\nu\phi\neq0$ for all ϕ and the particles with $\mathcal{H}_{\nu0},\mathcal{U}_{\nu0}$ are gyrating particles. If $a\mathcal{U}_{\nu0}/\omega_\nu>0$ and the expression on the right-hand side (rhs) of Eq. (B16) has no zeros, the particles are likewise gyrating particles. If $a\mathcal{U}_{\nu0}/\omega_\nu>0$, and if $\hat{d}_\nu\phi$ vanishes for any $\phi=\phi_0$, then the particles with $\mathcal{H}_{\nu0},\mathcal{U}_{\nu0}$ are swinging particles. It should again be stressed that, in most cases of interest, $a(R_g)_{\rm th}\ll1$, and that $av_\parallel/\omega_\nu=a(R_g)_{\rm th}v_\parallel/v_{\perp\rm th}>1$, which is necessary for a particle to be a swinging particle, is only possible for very high values of $v_\parallel$. Therefore, *the vast majority of particles are gyrating particles.*

APPENDIX C: SOLUTION OF THE EQUATION $\hat{d}_\nu X=0$

The extremization of the wave energy, Eq. (39), with respect to Γ_ν leads to an equation of the form

$$\hat{d}_\nu X=\hat{d}_\nu y\frac{\partial X}{\partial y}+\hat{d}_\nu v_\perp\frac{\partial X}{\partial v_\perp}+\hat{d}_\nu\phi\frac{\partial X}{\partial\phi}+\hat{d}_\nu v_\parallel\frac{\partial X}{\partial v_\parallel}=0 , \tag{C1}$$

where $\hat{d}_\nu y$, $\hat{d}_\nu v_\perp$, etc. represent the change with time of the variables along particle orbits, i.e.,

$$\hat{d}_\nu y=\dot{y}=v_\perp\sin\phi , \tag{C2}$$
$$\hat{d}_\nu v_\perp=-av_\perp v_\parallel\sin\phi\cos\phi , \tag{C3}$$
$$\hat{d}_\nu\phi=av_\parallel\sin^2\phi-\omega_\nu , \tag{C4}$$
$$\hat{d}_\nu v_\parallel=av_\perp^2\sin\phi\cos\phi . \tag{C5}$$

The solution of Eq. (C1) can be found by the method of characteristics, i.e., by solving the system

$$\frac{dy}{\hat{d}_\nu y}=\frac{dv_\perp}{\hat{d}_\nu v_\perp}=\frac{d\phi}{\hat{d}_\nu\phi}=\frac{dv_\parallel}{\hat{d}_\nu v_\parallel} . \tag{C6}$$

The equation

$$\frac{dv_\perp}{\hat{d}_\nu v_\perp}=\frac{dv_\parallel}{\hat{d}_\nu v_\parallel} \tag{C7}$$

is easily seen to be equivalent to $v_\perp dv_\perp+v_\parallel dv_\parallel=0$ and leads to the constant of the motion $\mathcal{H}_\nu$:

$$\mathcal{H}_\nu=\frac{m_\nu}{2}(v_\perp^2+v_\parallel^2) . \tag{C8}$$

Using this result, the equation

$$\frac{d\phi}{\hat{d}_\nu\phi}=\frac{dv_\parallel}{\hat{d}_\nu v_\parallel} \tag{C9}$$

can be integrated (carrying out some minor manipulations) and yields the constant of the motion $\mathcal{U}_\nu$:

$$\mathcal{U}_\nu=\frac{a}{2\omega_\nu}v_\perp^2\sin^2\phi+v_\parallel \ . \tag{C10}$$

The equations

$$\frac{dy}{\hat{d}_\nu y}=\frac{d\phi}{\hat{d}_\nu\phi},\quad \frac{dy}{\hat{d}_\nu y}=\frac{dv_\perp}{\hat{d}_\nu v_\perp} \tag{C11}$$

can be written as

$$\begin{aligned}&(av_\parallel\sin^2\phi-\omega_\nu)dy=v_\perp\sin\phi d\phi,\\&-av_\parallel\cos^2\phi dy=\cos\phi dv_\perp \ ,\end{aligned} \tag{C12}$$

respectively. The difference of these two equations yields

$$(av_\parallel-\omega_\nu)dy=d(-v_\perp\cos\phi), \tag{C13}$$

which, together with Eq. (B7), leads to

$$d\left[\sin a(\mathcal{Y}_\nu-y)+\frac{|\omega_\nu|}{\omega_\nu}\frac{a}{\Omega_\nu}v_\perp\cos\phi\right]=0 \ . \tag{C14}$$

This expression yields a convenient relation between the third constant of the motion, $\mathcal{Y}_\nu$, and y, $v_\perp$, and ϕ.

For gyrating particles, $y-\mathcal{Y}_\nu$ is bounded and is given by

$$y-\mathcal{Y}_\nu=\frac{1}{a}\arcsin\left[\frac{|\omega_\nu|a}{\omega_\nu\Omega_\nu}v_\perp\cos\phi\right], \tag{C15}$$

with

$$-\pi\le\arcsin\left[\frac{|\omega_\nu|a}{\omega_\nu\Omega_\nu}v_\perp\cos\phi\right]\le\pi \ . \tag{C16}$$

The meaning of $\mathcal{Y}_\nu$ becomes clear if one assumes that $(a/\Omega_\nu)v_\perp$ is small, i.e., $(a/\Omega_\nu)v_\perp\sim(a/\omega_\nu)v_\perp=a(R_g)_{\text{th}}v_\perp/(v_\perp)_{\text{th}}\ll1$. In that case

$$\mathcal{Y}_\nu\left[\frac{a}{\Omega_\nu}v_\perp\to0\right]=y-\frac{v_\perp}{\omega_\nu}\cos\phi \ , \tag{C17}$$

and $y=\mathcal{Y}_\nu$ is obviously the plane on which the guiding center is located.

For swinging particles (which move freely, with nonvanishing $\dot{y}$), y is unbounded and

$$y-\mathcal{Y}_\nu=\frac{1}{a}\arcsin\left[\frac{|\omega_\nu|a}{\omega_\nu\Omega_\nu}v_\perp\cos\phi\right]+l_\nu\frac{2\pi}{a} \ ,\qquad l_\nu=0,\pm1,\text{etc.} \tag{C18}$$

In this case, $y=\mathcal{Y}_\nu+l_\nu(2\pi/a)$ are the plasma on which $|\dot{y}|$ assumes its maximum value. When the particle swings from ϕ_0 to $\pi-\phi_0$ and back again to ϕ_0, $v_\perp$ returns to the same value, as can easily be seen when $v_\perp$ is expressed as $v_\perp(\mathcal{H}_\nu,\mathcal{U}_\nu,\sin^2\phi)$. At the same time, the particle moves from y_0 to $y_0+2\pi/a$. With these three independent integrals, the general solution of the equation $\hat{d}_\nu X=0$ is given by

$$\begin{aligned}&X(v_\perp,\phi,v_\parallel,y)\\&=X(\mathcal{H}_\nu(v_\perp,v_\parallel),\mathcal{U}_\nu(v_\perp,\phi,v_\parallel),\mathcal{Y}_\nu(v_\perp,\phi,v_\parallel,y)).\end{aligned} \tag{C19}$$

APPENDIX D: THE COORDINATES SYSTEMS $\mathcal{H}_\nu,\phi,\mathcal{U}_\nu,\mathcal{Y}_\nu$ (FOR GYRATING PARTICLES) AND $\mathcal{H}_\nu,\mathcal{U}_\nu,\mathcal{Y}_\nu,y$ (FOR SWINGING PARTICLES)

It is most convenient to introduce coordinates in y-$\mathbf{v}$ space which are particularly adapted to the motion of the particles. For gyrating particles, the coordinates $\mathcal{H}_\nu,\phi,\mathcal{U}_\nu,\mathcal{Y}_\nu$ are introduced. The relevant relations are Eqs. (C8), (C10), and (C15) [together with Eq. (B4)], which enable one to calculate the Jacobian

$$\Delta=\frac{\partial(\mathcal{H}_\nu,\phi,\mathcal{U}_\nu,\mathcal{Y}_\nu)}{\partial(v_\perp,\phi,v_\parallel,y)}=\begin{vmatrix}\frac{\partial\mathcal{H}_\nu}{\partial v_\perp}&\frac{\partial\mathcal{H}_\nu}{\partial\phi}&\frac{\partial\mathcal{H}_\nu}{\partial v_\parallel}&\frac{\partial\mathcal{H}_\nu}{\partial y}\\\frac{\partial\phi}{\partial v_\perp}&\frac{\partial\phi}{\partial\phi}&\frac{\partial\phi}{\partial v_\parallel}&\frac{\partial\phi}{\partial y}\\\frac{\partial\mathcal{U}_\nu}{\partial v_\perp}&\frac{\partial\mathcal{U}_\nu}{\partial\phi}&\frac{\partial\mathcal{U}_\nu}{\partial v_\parallel}&\frac{\partial\mathcal{U}_\nu}{\partial y}\\\frac{\partial\mathcal{Y}_\nu}{\partial v_\perp}&\frac{\partial\mathcal{Y}_\nu}{\partial\phi}&\frac{\partial\mathcal{Y}_\nu}{\partial v_\parallel}&\frac{\partial\mathcal{Y}_\nu}{\partial y}\end{vmatrix} \ . \tag{D1}$$

The quantities which appear in the functional determinant can easily be calculated. They are

$$\frac{\partial\mathcal{H}_\nu}{\partial v_\perp}=m_\nu v_\perp,\quad\frac{\partial\mathcal{H}_\nu}{\partial\phi}=0,\quad\frac{\partial\mathcal{H}_\nu}{\partial v_\parallel}=m_\nu v_\parallel,\quad\frac{\partial\mathcal{H}_\nu}{\partial y}=0 \ , \tag{D2}$$

$$\frac{\partial\phi}{\partial v_\perp}=0,\quad\frac{\partial\phi}{\partial\phi}=1,\quad\frac{\partial\phi}{\partial v_\parallel}=0,\quad\frac{\partial\phi}{\partial y}=0 \ , \tag{D3}$$

$$\begin{aligned}&\frac{\partial\mathcal{U}_\nu}{\partial v_\perp}=\frac{a}{\omega_\nu}v_\perp\sin^2\phi \ ,\\&\frac{\partial\mathcal{U}_\nu}{\partial\phi}=\frac{a}{\omega_\nu}v_\perp^2\sin\phi\cos\phi \ ,\\&\frac{\partial\mathcal{U}_\nu}{\partial v_\parallel}=1 \ ,\\&\frac{\partial\mathcal{U}_\nu}{\partial y}=0 \ ,\end{aligned} \tag{D4}$$

$$\begin{aligned}&\frac{\partial\mathcal{Y}_\nu}{\partial v_\perp}=-\frac{|\omega_\nu|}{\omega_\nu}\frac{1}{\Omega_\nu}\cos\phi\cos a(\mathcal{Y}_\nu-y),\\&\frac{\partial\mathcal{Y}_\nu}{\partial\phi}=\frac{|\omega_\nu|}{\omega_\nu}\frac{v_\perp}{\Omega_\nu}\sin\phi\cos a(\mathcal{Y}_\nu-y) \ ,\\&\frac{\partial\mathcal{Y}_\nu}{\partial v_\parallel}=-\frac{av_\perp}{\Omega_\nu^2}\cos\phi \ ,\\&\frac{\partial\mathcal{Y}_\nu}{\partial y}=1 \ .\end{aligned} \tag{D5}$$

From the relation

$$d\mathcal{H}_\nu d\phi\, d\mathcal{U}_\nu d\mathcal{Y}_\nu=|\Delta|dv_\perp d\phi\, dv_\parallel dy=|\Delta|\frac{d^3v\, dy}{v_\perp} \tag{D6}$$

one then obtains the volume element in y-$\mathbf{v}$ space:

$$d^3v\,dy=\left|\frac{\omega_\nu}{m_\nu(av_\parallel\sin^2\phi-\omega_\nu)}\right|d\mathcal{H}_\nu d\phi\,d\mathcal{U}_\nu d\mathcal{Y}_\nu$$
$$=\left|\frac{\omega_\nu}{m_\nu\hat{d}_\nu\phi}\right|d\mathcal{H}_\nu d\phi\,d\mathcal{U}_\nu d\mathcal{Y}_\nu\ . \quad \text{(D7)}$$

Since, as was seen in Appendix B, $\hat{d}_\nu\phi\neq 0$ for gyrating particles, the coordinates are well defined.

For swinging particles, the coordinates $\mathcal{H}_\nu$, $\mathcal{U}_\nu$, $\mathcal{Y}_\nu$, and y are introduced. Proceeding in a similar manner as in the case of the gyrating particles, the volume element is easily derived. One obtains

$$d^3v\,dy=\left|\frac{\omega_\nu}{m_\nu v_\perp\sin\phi}\right|d\mathcal{H}_\nu d\mathcal{U}_\nu d\mathcal{Y}_\nu dy$$
$$=\left|\frac{\omega_\nu}{m_\nu\hat{d}_\nu y}\right|d\mathcal{H}_\nu d\mathcal{U}_\nu d\mathcal{Y}_\nu dy\ . \quad \text{(D8)}$$

The coordinate system is well defined, since $\hat{d}_\nu y=\dot{y}=v_\perp\sin\phi$ does not vanish for the swinging particles.

For gyrating particles, the expression $\hat{d}_\nu\Gamma_\nu$ appearing in Eq. (44) takes the form

$$(\hat{d}_\nu\Gamma_\nu)_{\rm GP}=\hat{d}_\nu\Gamma_\nu(\mathcal{H}_\nu,\mathcal{U}_\nu,\mathcal{Y}_\nu,\phi)=(\hat{d}_\nu\phi)\frac{\partial\Gamma_\nu}{\partial\phi}\ , \quad \text{(D9)}$$

while for swinging particles one obtains

$$(\hat{d}_\nu\Gamma_\nu)_{\rm SP}=\hat{d}_\nu\Gamma_\nu(\mathcal{H}_\nu,\mathcal{U}_\nu,\mathcal{Y}_\nu,y)=(\hat{d}_\nu y)\frac{\partial\Gamma_\nu}{\partial y}\ . \quad \text{(D10)}$$

APPENDIX E: SOME USEFUL RELATIONS FOR THE EVALUATION OF $\delta^2 H$

It is convenient to introduce two *reference unit vectors* $\mathbf{e}_1(\mathcal{Y}_\nu)$ and $\mathbf{e}_B(\mathcal{Y}_\nu)$ defined by

$$\mathbf{e}_1(\mathcal{Y}_\nu)=\mathbf{e}_1(y=\mathcal{Y}_\nu)=\cos a\mathcal{Y}_\nu\mathbf{e}_x-\sin a\mathcal{Y}_\nu\mathbf{e}_z\ , \quad \text{(E1)}$$

$$\mathbf{e}_B(\mathcal{Y}_\nu)=\mathbf{e}_B(y=\mathcal{Y}_\nu)=\sin a\mathcal{Y}_\nu\mathbf{e}_x-\cos a\mathcal{Y}_\nu\mathbf{e}_z\ . \quad \text{(E2)}$$

Then, taking into account Eqs. (B7) and (C15), one obtains

$$\mathbf{e}_1(y)=\cos ay\mathbf{e}_x-\sin ay\mathbf{e}_z$$
$$=\cos a(\mathcal{Y}_\nu-y)\mathbf{e}_1(\mathcal{Y}_\nu)+\sin a(\mathcal{Y}_\nu-y)\mathbf{e}_B(\mathcal{Y}_\nu)$$
$$=-\frac{|\omega_\nu|}{\omega_\nu\Omega_\nu}[(av_\parallel-\omega_\nu)\mathbf{e}_1(\mathcal{Y}_\nu)+av_\perp\cos\phi\mathbf{e}_B(\mathcal{Y}_\nu)] \quad \text{(E3)}$$

and

$$\mathbf{e}_B(y)=\sin ay\mathbf{e}_x+\cos ay\mathbf{e}_z$$
$$=-\sin a(\mathcal{Y}_\nu-y)\mathbf{e}_1(\mathcal{Y}_\nu)+\cos a(\mathcal{Y}_\nu-y)\mathbf{e}_B(\mathcal{Y}_\nu)$$
$$=\frac{|\omega_\nu|}{\omega_\nu\Omega_\nu}[av_\perp\cos\phi\mathbf{e}_1(\mathcal{Y}_\nu)-(av_\parallel-\omega_\nu)\mathbf{e}_B(\mathcal{Y}_\nu)]\ . \quad \text{(E4)}$$

Solving these equations for $\mathbf{e}_1(\mathcal{Y}_\nu)$ and $\mathbf{e}_B(\mathcal{Y}_\nu)$, one obtains

$$\mathbf{e}_1(\mathcal{Y}_\nu)=-\frac{|\omega_\nu|}{\omega_\nu\Omega_\nu}[(av_\parallel-\omega_\nu)\mathbf{e}_1(y)-av_\perp\cos\phi\mathbf{e}_B(y)] \quad \text{(E5)}$$

and

$$\mathbf{e}_B(\mathcal{Y}_\nu)=-\frac{|\omega_\nu|}{\omega_\nu\Omega_\nu}[av_\perp\cos\phi\mathbf{e}_1(y)+(av_\parallel-\omega_\nu)\mathbf{e}_B(v)]\ . \quad \text{(E6)}$$

With the help of Eqs. (A1), (A2), (22), (B7), and (C14), the vector $(a/\omega_\nu)\mathbf{w}-\mathbf{e}_B$ appearing in Eq. (44) can be shown to be a constant of the motion:

$$\frac{a}{\omega_\nu}\mathbf{w}-\mathbf{e}_B=\frac{a}{\omega_\nu}v_\perp\cos\phi\mathbf{e}_1+\left[\frac{a}{\omega_\nu}v_\parallel-1\right]\mathbf{e}_B$$
$$=-\frac{\Omega_\nu}{|\omega_\nu|}\mathbf{e}_B(\mathcal{Y}_\nu)\ . \quad \text{(E7)}$$

It is also convenient to relate the velocity $\mathbf{w}=\mathbf{v}-\dot{y}\mathbf{e}_y$ to the vectors $\mathbf{e}_1(\mathcal{Y}_\nu)$ and $\mathbf{e}_B(\mathcal{Y}_\nu)$:

$$\mathbf{w}=v_\parallel\mathbf{e}_B+v_\perp\cos\phi\mathbf{e}_1$$
$$=\frac{|\omega_\nu|}{\omega_\nu\Omega_\nu}[(-av_\perp^2\cos^2\phi-av_\parallel^2+\omega_\nu v_\parallel)\mathbf{e}_B(\mathcal{Y}_\nu)+\omega_\nu v_\perp\cos\phi\mathbf{e}_1(\mathcal{Y}_\nu)]$$
$$=\frac{|\omega_\nu|}{\omega_\nu\Omega_\nu}\left[\left[-\frac{2a}{m_\nu}\mathcal{H}_\nu+2\omega_\nu\mathcal{U}_\nu-\omega_\nu v_\parallel\right]\mathbf{e}_B(\mathcal{Y}_\nu)+\omega_\nu v_\perp\cos\phi\mathbf{e}_1(\mathcal{Y}_\nu)\right]$$
$$=Q_\parallel(\mathcal{H}_\nu,\mathcal{U}_\nu,v_\parallel)\mathbf{e}_B(\mathcal{Y}_\nu)+\frac{|\omega_\nu|}{\Omega_\nu}v_\perp\cos\phi\mathbf{e}_1(\mathcal{Y}_\nu)\ , \quad \text{(E8)}$$

where

$$Q_\parallel(\mathcal{H}_\nu,\mathcal{U}_\nu,v_\parallel)=\frac{|\omega_\nu|}{\Omega_\nu}\left[-\frac{2a}{\omega_\nu m_\nu}\mathcal{H}_\nu+2\mathcal{U}_\nu-v_\parallel\right] \quad \text{(E9)}$$

has been introduced. Mean values for gyrating and for swinging particles, respectively, are now defined by the expressions

$$\langle\cdots\rangle_{\tau_\phi}=\left[\oint_0^{2\pi}\cdots\frac{d\phi}{\hat{d}_\nu\phi}\right]\Big/\left[\oint_0^{2\pi}\frac{d\phi}{\hat{d}_\nu\phi}\right]\ , \quad \text{(E10)}$$

$$\langle \cdots \rangle_{\tau_y} = \left[\oint_{y_0}^{y_0+(2\pi/a)} \cdots \frac{dy}{\hat{d}_\nu y} \right] \Big/ \left[\oint_{y_0}^{y_0+(2\pi/a)} \frac{dy}{\hat{d}_\nu y} \right] , \tag{E11}$$

where $\oint$ means that the integrals are taken along the particle orbits, i.e., at constant $\mathcal{H}_\nu, \mathcal{U}_\nu, \mathcal{Y}_\nu$. Then, taking $v_\perp = v_\perp(\mathcal{H}_\nu, \mathcal{U}_\nu, \sin^2\phi)$ into account yields

$$\langle v_\perp \cos\phi \rangle_{\tau_\phi} = 0 \tag{E12}$$

since positive and negative contributions compensate each other. On the other hand, expressing $v_\perp \cos\phi$ as a function of $\mathcal{H}_\nu$, $\mathcal{U}_\nu$, and y through Eq. (C15) and $\hat{d}_\nu y = \dot{y} = p_{x\nu}/m_\nu$ as a function of the same variables with the help of Eqs. (B1) and (B8), and taking into account that $\dot{y}$ does not change sign for swinging particles, one derives

$$\langle v_\perp \cos\phi \rangle_{\tau_y} = 0. \tag{E13}$$

Introducing now the *$\mathcal{Y}_\nu$-independent, mean parallel velocities* $q_\parallel(\mathcal{H}_\nu, \mathcal{U}_\nu)$ and $r_\parallel(\mathcal{H}_\nu, \mathcal{U}_\nu)$ as

$$q_\parallel(\mathcal{H}_\nu, \mathcal{U}_\nu) = \langle Q_\parallel(\mathcal{H}_\nu, \mathcal{U}_\nu, v_\parallel(\mathcal{H}_\nu, \mathcal{U}_\nu, \sin^2\phi)) \rangle_{\tau_\phi} \tag{E14}$$

and

$$r_\parallel(\mathcal{H}_\nu, \mathcal{U}_\nu) = \langle Q_\parallel(\mathcal{H}_\nu, \mathcal{U}_\nu, v_\parallel(\mathcal{H}_\nu, \mathcal{U}_\nu, a(\mathcal{Y}_\nu - y))) \rangle_{\tau_y} , \tag{E15}$$

respectively, these results allow concise expressions for the mean value of $\mathbf{w}$ and $\mathbf{w}\cdot\mathbf{k}_{xz}$, namely

$$\begin{aligned} \langle \mathbf{w} \rangle_{\tau_\phi} &= q_\parallel(\mathcal{H}_\nu, \mathcal{U}_\nu) \mathbf{e}_B(\mathcal{Y}_\nu) , \\ \langle \mathbf{w}\cdot\mathbf{k}_{xz} \rangle_{\tau_\phi} &= k_\parallel(\mathcal{Y}_\nu) q_\parallel(\mathcal{H}_\nu, \mathcal{U}_\nu) \end{aligned} \tag{E16}$$

and

$$\begin{aligned} \langle \mathbf{w} \rangle_{\tau_y} &= r_\parallel(\mathcal{H}_\nu, \mathcal{U}_\nu) \mathbf{e}_B(\mathcal{Y}_\nu) , \\ \langle \mathbf{w}\cdot\mathbf{k}_{xz} \rangle_{\tau_y} &= k_\parallel(\mathcal{Y}_\nu) r_\parallel(\mathcal{H}_\nu, \mathcal{U}_\nu) , \end{aligned} \tag{E17}$$

where the *parallel component of the wave vector*

$$k_\parallel(\mathcal{Y}_\nu) = \mathbf{k}_{xz}\cdot\mathbf{e}_B(\mathcal{Y}_\nu) \tag{E18}$$

has been introduced. These results, together with Eqs. (21), (22), (23), and (E7), lead to the relations

$$\begin{aligned} &\left\langle \mathbf{e}_B(\mathcal{Y}_\nu)\cdot\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}} \right\rangle_{\tau_\phi} (\mathcal{H}_\nu, \mathcal{U}_\nu) \\ &= \left[m_\nu \frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu} + \frac{a}{\omega_\nu}\frac{\partial f_\nu^{(0)}}{\partial \mathcal{U}_\nu} \right] q_\parallel(\mathcal{H}_\nu, \mathcal{U}_\nu) + \frac{\Omega_\nu}{|\omega_\nu|}\frac{\partial f_\nu^{(0)}}{\partial \mathcal{U}_\nu} \\ &= \mathcal{D} f_\nu^{(0)}(\mathcal{H}_\nu, \mathcal{U}_\nu) q_\parallel(\mathcal{H}_\nu, \mathcal{U}_\nu) + \frac{\Omega_\nu}{|\omega_\nu|}\frac{\partial f_\nu^{(0)}}{\partial \mathcal{U}_\nu} \end{aligned} \tag{E19}$$

and

$$\begin{aligned} &\left\langle \mathbf{e}_B(\mathcal{Y}_\nu)\cdot\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}} \right\rangle_{\tau_y} (\mathcal{H}_\nu, \mathcal{U}_\nu) \\ &= \left[m_\nu \frac{\partial f_\nu^{(0)}}{\partial \mathcal{H}_\nu} + \frac{a}{\omega_\nu}\frac{\partial f_\nu^{(0)}}{\partial \mathcal{U}_\nu} \right] r_\parallel(\mathcal{H}_\nu, \mathcal{U}_\nu) + \frac{\Omega_\nu}{|\omega_\nu|}\frac{\partial f_\nu^{(0)}}{\partial \mathcal{U}_\nu} \\ &= \mathcal{D} f_\nu^{(0)}(\mathcal{H}_\nu, \mathcal{U}_\nu) r_\parallel(\mathcal{H}_\nu, \mathcal{U}_\nu) + \frac{\Omega_\nu}{|\omega_\nu|}\frac{\partial f_\nu^{(0)}}{\partial \mathcal{U}_\nu} . \end{aligned} \tag{E20}$$

It is now straightforward to calculate the constant of the motion C_ν, which appears in Eq. (44). Setting

$$k_1(\mathcal{Y}_\nu) = \mathbf{k}_{xz}\cdot\mathbf{e}_1(\mathcal{Y}_\nu) , \tag{E21}$$

Eq. (44) can be written as

$$\begin{aligned} \frac{C_\nu}{\Psi_\nu^2} = &- k_\parallel(\mathcal{Y}_\nu)\frac{\Omega_\nu}{|\omega_\nu|}\frac{\partial f_\nu^{(0)}}{\partial \mathcal{U}_\nu} \\ &- 2[\mathcal{D} f_\nu^{(0)}] \left[\frac{|\omega_\nu|}{\Omega_\nu} k_1(\mathcal{Y}_\nu) v_\perp \cos\phi \right. \\ &\left. + k_\parallel(\mathcal{Y}_\nu) Q_\parallel(\mathcal{H}_\nu, \mathcal{U}_\nu, \mathcal{Y}_\nu) + \hat{d}_\nu \Gamma_\nu \right] . \end{aligned} \tag{E22}$$

For both kinds of particles, the gyrating particles and the swinging particles, particularly appropriate coordinates were introduced in Appendix D. By means of these coordinates, the constant of the motion of C_ν can be determined from Eq. (E22) and the boundary conditions. For gyrating particles, the boundary conditions, Eq. (38), expressed in the coordinates $\mathcal{H}_\nu$, $\mathcal{U}_\nu$, ϕ, and $\mathcal{Y}_\nu$, are

$$\Gamma_\nu(\mathcal{H}_\nu, \mathcal{U}_\nu, \phi + 2\pi, \mathcal{Y}_\nu) = \Gamma_\nu(\mathcal{H}_\nu, \mathcal{U}_\nu, \mathcal{Y}_\nu) + 2\pi n_\nu . \tag{E23}$$

For swinging particles, Γ_ν is taken to have given values at $y = y_0$ and $y = y_0 + (2\pi/a)$, i.e.,

$$\begin{aligned} &\Gamma_\nu(\mathcal{H}_\nu, \mathcal{U}_\nu, \mathcal{Y}_\nu, y = y_0) = \Gamma_{\nu 0} , \\ &\Gamma_\nu \left[\mathcal{H}_\nu, \mathcal{U}_\nu, \mathcal{Y}_\nu, y = y_0 + \frac{2\pi}{a} \right] = \Gamma_{\nu 1} . \end{aligned} \tag{E24}$$

For gyrating particles Γ_ν is given by

$$(\Gamma_\nu)_{\rm GP} = \Gamma_{\nu h}(\mathcal{H}_\nu, \mathcal{U}_\nu, \mathcal{Y}_\nu) + \Gamma_{\nu i}(\mathcal{H}_\nu, \mathcal{U}_\nu, \phi, \mathcal{Y}_\nu) , \tag{E25}$$

$$\hat{d}_\nu(\Gamma_\nu)_{\rm GP} = (\hat{d}_\nu \phi)\frac{\partial(\Gamma_{\nu i})_{\rm GP}}{\partial \phi} , \tag{E26}$$

where $\Gamma_{\nu h}$ *is an arbitrary function of its arguments* and $\Gamma_{\nu i}$ is determined by integrating Eq. (E22) with respect to ϕ along the particles orbits. Γ_ν is not needed explicitly for the calculation of $\delta^2 H$; only C_ν is needed.

The corresponding expressions for swinging particles are

$$(\Gamma_\nu)_{\rm SP} = \Gamma_{\nu h}(\mathcal{H}_\nu, \mathcal{U}_\nu, \mathcal{Y}_\nu) + \Gamma_{\nu i}(\mathcal{H}_\nu, \mathcal{U}_\nu, \mathcal{Y}_\nu, y) \tag{E27}$$

and

$$\hat{d}_\nu(\Gamma_\nu)_{\rm SP} = (\hat{d}_\nu y)\frac{\partial(\Gamma_{\nu i})_{\rm SP}}{\partial y} . \tag{E28}$$

Dividing Eq. (E22) by $\hat{d}_\nu\phi$ and integrating along the particle orbits between $\phi=0$ and $\phi=2\pi$ yields the constant of the motion C_ν for gyrating particles (designated by $C_{\nu(\mathrm{GP})}$):

$$C_{\nu(\mathrm{GP})}\left\langle \frac{1}{\Psi_\nu^2}\right\rangle_{\tau_\phi}=k_\parallel(\mathcal{Y}_\nu)\frac{\Omega_\nu}{|\omega_\nu|}\frac{\partial f_\nu^{(0)}}{\partial \mathcal{U}_\nu} -2k_\parallel(\mathcal{Y}_\nu)\left\langle \mathbf{e}_B(\mathcal{Y}_\nu)\cdot\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\right\rangle_{\tau_\phi}(\mathcal{H}_\nu,\mathcal{U}_\nu) -2[\mathcal{D}f_\nu^{(0)}]\langle \hat{d}_\nu\phi\rangle_{\tau_\phi} n_\nu \;. \tag{E29}$$

Proceeding in a similar way for swinging particles, i.e., dividing Eq. (E22) by $\hat{d}_\nu y$ and integrating along the particle orbit between y_0 and $y_0+(2\pi/a)$, yields $C_{\nu(\mathrm{SP})}$, the constant of the motion C_ν for swinging particles:

$$C_{\nu(\mathrm{SP})}\left\langle \frac{1}{\Psi_\nu^2}\right\rangle_{\tau_y}=k_\parallel(\mathcal{Y}_\nu)\frac{\Omega_\nu}{|\omega_\nu|}\frac{\partial f_\nu^{(0)}}{\partial \mathcal{U}_\nu} -2k_\parallel(\mathcal{Y}_\nu)\left\langle \mathbf{e}_B(\mathcal{Y}_\nu)\cdot\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\right\rangle_{\tau_y}(\mathcal{H}_\nu,\mathcal{U}_\nu) -2[\mathcal{D}f_\nu^{(0)}]\langle \hat{d}_\nu y\rangle_{\tau_y}\Delta\Gamma_\nu\frac{a}{2\pi}\;, \tag{E30}$$

where

$$\Delta\Gamma_\nu=\Gamma_{\nu 1}-\Gamma_{\nu 0}\;. \tag{E31}$$

These expressions can be simplified if it is remembered that $\mathcal{D}f_\nu^{(0)}=m_\nu\partial f_\nu^{(0)}/\partial\mathcal{H}_\nu+(a/\omega_\nu)\partial f_\nu^{(0)}/\partial\mathcal{U}_\nu$, and the following definitions are introduced:

$$h_{\nu 1}(\mathcal{H}_\nu,\mathcal{U}_\nu,\mathcal{Y}_\nu)=\frac{k_\parallel(\mathcal{Y}_\nu)}{\langle \hat{d}_\nu\phi\rangle_{\tau_\phi}[\mathcal{D}f_\nu^{(0)}]}\left\langle \mathbf{e}_B(\mathcal{Y}_\nu)\cdot\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\right\rangle_{\tau_\phi}\;, \tag{E32}$$

$$h_{\nu 2}(\mathcal{H}_\nu,\mathcal{U}_\nu,\mathcal{Y}_\nu)=\frac{k_\parallel(\mathcal{Y}_\nu)}{\langle \hat{d}_\nu y\rangle_{\tau_y}[\mathcal{D}f_\nu^{(0)}]}\left\langle \mathbf{e}_B(\mathcal{Y}_\nu)\cdot\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\right\rangle_{\tau_y}\;, \tag{E33}$$

$$B_{\nu 1}(\mathcal{H}_\nu,\mathcal{U}_\nu,\mathcal{Y}_\nu)=k_\parallel(\mathcal{Y}_\nu)\frac{q_\parallel}{\langle \hat{d}_\nu\phi\rangle_{\tau_\phi}}\;, \tag{E34}$$

$$B_{\nu 2}(\mathcal{H}_\nu,\mathcal{U}_\nu,\mathcal{Y}_\nu)=k_\parallel(\mathcal{Y}_\nu)\frac{r_\parallel}{\langle \hat{d}_\nu y\rangle_{\tau_y}}\;. \tag{E35}$$

Note that the only dependence on $\mathcal{Y}_\nu$ *is given through* $k_\parallel(\mathcal{Y}_\nu)$. With these definitions one then obtains from Eqs. (E19), (E20), (E29), and (E30)

$$\left[C_{\nu(\mathrm{GP})}\left\langle \frac{1}{\Psi_\nu^2}\right\rangle_{\tau_\phi}\right]^2=\left[k_\parallel(\mathcal{Y}_\nu)\frac{\Omega_\nu}{|\omega_\nu|}\frac{\partial f_\nu^{(0)}}{\partial\mathcal{U}_\nu}\right]^2+\left[2[\mathcal{D}f_\nu^{(0)}]\langle\hat{d}_\nu\phi\rangle_{\tau_\phi}\right]^2[n_\nu^2+(h_{\nu 1}+B_{\nu 1})n_\nu+h_{\nu 1}B_{\nu 1}]$$
$$=(2[\mathcal{D}f_\nu^{(0)}]\langle\hat{d}_\nu\phi\rangle_{\tau_\phi})^2\left[n_\nu+\frac{(h_{\nu 1}+B_{\nu 1})}{2}\right]^2\;, \tag{E36}$$

$$\left[C_{\nu(\mathrm{SP})}\left\langle \frac{1}{\Psi_\nu^2}\right\rangle_{\tau_y}\right]^2=\left[k_\parallel(\mathcal{Y}_\nu)\frac{\Omega_\nu}{|\omega_\nu|}\frac{\partial f_\nu^{(0)}}{\partial\mathcal{U}_\nu}\right]^2+(2[\mathcal{D}f_\nu^{(0)}]\langle\hat{d}_\nu y\rangle_{\tau_y})^2\left[\left[\Delta\Gamma_\nu\frac{a}{2\pi}\right]^2+(h_{\nu 2}+B_{\nu 2})\left[\Delta\Gamma_\nu\frac{a}{2\pi}\right]+h_{\nu 2}B_{\nu 2}\right]$$
$$=(2[\mathcal{D}f_\nu^{(0)}]\langle\hat{d}_\nu y\rangle_{\tau_y})^2\left[\Delta\Gamma_\nu\frac{a}{2\pi}+\frac{(h_{\nu 2}+B_{\nu 2})}{2}\right]^2\;. \tag{E37}$$

APPENDIX F: NEGLECT OF THE ELECTROSTATIC ENERGY TERM

The contribution of the electrostatic energy term

$$\frac{1}{8\pi}\int d^3x\,\delta E^2 \tag{F1}$$

has been neglected. To justify this, let us consider the perturbed electric charge density $\delta\rho$. Generally, the change density is

$$\rho=\sum_\nu e_\nu\int f_\nu d^3v\;, \tag{F2}$$

and the perturbed charge density is

$$\delta\rho=\sum_\nu e_\nu\int\delta f_\nu d^3v\;. \tag{F3}$$

The perturbation in the distribution function is given by

$$\delta f_\nu=\left.\frac{\partial f_\nu^{(0)}}{\partial\mathbf{x}}\right|_{\mathbf{p}}\cdot\delta\mathbf{x}_\nu+\left.\frac{\partial f_\nu^{(0)}}{\partial\mathbf{p}_\nu}\right|_{\mathbf{x}}\cdot\delta\mathbf{p}_\nu\;, \tag{F4}$$

with $\mathbf{p}_\nu$ the canonical momentum of species ν, i.e.,

$$\mathbf{p}_\nu=m_\nu\mathbf{v}+\frac{e_\nu}{c}\mathbf{A}^{(0)}(\mathbf{x})\;. \tag{F5}$$

It therefore follows that

$$\left.\frac{\partial f_\nu^{(0)}}{\partial\mathbf{v}}\right|_{\mathbf{x}}=m_\nu\left.\frac{\partial f_\nu^{(0)}}{\partial\mathbf{p}_\nu}\right|_{\mathbf{x}}\;, \tag{F6}$$

$$\left.\frac{\partial f_\nu^{(0)}}{\partial\mathbf{x}}\right|_{\mathbf{v}}=\left.\frac{\partial f_\nu^{(0)}}{\partial\mathbf{x}}\right|_{\mathbf{p}}+\left.\frac{\partial(p_\nu)_i}{\partial\mathbf{x}}\right|_{\mathbf{v}}\left.\frac{\partial f_\nu^{(0)}}{\partial(p_\nu)_i}\right|_{\mathbf{x}}$$
$$=\left.\frac{\partial f_\nu^{(0)}}{\partial\mathbf{x}}\right|_{\mathbf{p}}+\frac{e_\nu}{c}\frac{\partial A_i^{(0)}}{\partial\mathbf{x}}\left.\frac{\partial f_\nu^{(0)}}{\partial(p_\nu)_i}\right|_{\mathbf{x}}\;, \tag{F7}$$

$$\left.\frac{\partial f_\nu^{(0)}}{\partial \mathbf{x}}\right|_{\mathbf{p}}=\left.\frac{\partial f_\nu^{(0)}}{\partial \mathbf{x}}\right|_{\mathbf{v}}-\left.\frac{e_\nu}{c}\frac{\partial A_i^{(0)}}{\partial \mathbf{x}}\frac{\partial f_\nu^{(0)}}{\partial (p_\nu)_i}\right|_{\mathbf{x}}$$

$$=\left.\frac{\partial f_\nu^{(0)}}{\partial \mathbf{x}}\right|_{\mathbf{v}}-\left.\frac{e_\nu}{m_\nu c}\frac{\partial A_i^{(0)}}{\partial \mathbf{x}}\frac{\partial f_\nu^{(0)}}{\partial v_i}\right|_{\mathbf{x}} . \tag{F8}$$

The perturbations $\delta \mathbf{x}_\nu$ and $\delta \mathbf{p}_\nu$ are given by

$$\delta \mathbf{x}_\nu=\left.\frac{\partial G_\nu}{\partial \mathbf{p}_\nu}\right|_{\mathbf{x}}=\left.\frac{1}{m_\nu}\frac{\partial G_\nu}{\partial \mathbf{v}}\right|_{\mathbf{x}} , \tag{F9}$$

$$\delta \mathbf{p}_\nu=-\left.\frac{\partial G_\nu}{\partial \mathbf{x}}\right|_{\mathbf{p}}=-\left.\frac{\partial G_\nu}{\partial \mathbf{x}}\right|_{\mathbf{v}}+\left.\frac{e_\nu}{m_\nu c}\frac{\partial A_i^{(0)}}{\partial \mathbf{x}}\frac{\partial G_\nu}{\partial v_i}\right|_{\mathbf{x}} . \tag{F10}$$

Employing the relations above, one obtains δf_ν as a function of $\mathbf{x}$ and $\mathbf{v}$:

$$\delta f_\nu=\frac{1}{m_\nu}\left[\frac{\partial f_\nu^{(0)}}{\partial \mathbf{x}}\cdot\frac{\partial G_\nu}{\partial \mathbf{v}}-\frac{e_\nu}{m_\nu c}\left[\mathbf{B}^{(0)}\times\frac{\partial G_\nu}{\partial \mathbf{v}}\right]\cdot\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}} -\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\cdot\frac{\partial G_\nu}{\partial \mathbf{x}}\right] . \tag{F11}$$

This expression can be transformed by taking Eqs. (20)–(23), (25)–(27), and the relation $d_\nu=\hat{d}_\nu+\dot{x}(\partial/\partial x)+\dot{z}(\partial/\partial z)$ into account, to obtain

$$\begin{aligned}\delta f_\nu&=\frac{1}{m_\nu}\left[-\omega_\nu[\mathcal{D}f_\nu^{(0)}]\mathbf{v}\times\mathbf{e}_B\cdot\frac{\partial G_\nu}{\partial \mathbf{v}}-\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\cdot\frac{\partial G_\nu}{\partial \mathbf{x}}\right]\\
&=-\frac{1}{m_\nu}\left[[\mathcal{D}f_\nu^{(0)}][d_\nu G_\nu]-[\mathcal{D}f_\nu^{(0)}]\left[\mathbf{v}\cdot\frac{\partial G_\nu}{\partial \mathbf{x}}\right]+\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\cdot\frac{\partial G_\nu}{\partial \mathbf{x}}\right]\\
&=-\frac{1}{m_\nu}\left[[\mathcal{D}f_\nu^{(0)}][\hat{d}_\nu G_\nu]+\left[-[\mathcal{D}f_\nu^{(0)}]\dot{y}\mathbf{e}_y+\frac{\partial f^{(0)}}{\partial \mathbf{v}}\right]\cdot\frac{\partial G_\nu}{\partial \mathbf{x}}\right]\\
&=-\frac{1}{m_\nu}\left\{[\mathcal{D}f_\nu^{(0)}][\hat{d}_\nu G_\nu]+\left[-\left[\mathbf{e}_y\cdot\frac{\partial f^{(0)}}{\partial \mathbf{v}}\right]\mathbf{e}_y+\frac{\partial f^{(0)}}{\partial \mathbf{v}}\right]\cdot\frac{\partial G_\nu}{\partial \mathbf{x}}\right\} .\end{aligned} \tag{F12}$$

Taking into account Eqs. (30), (31), and (36), one then obtains for the perturbed charge density

$$\delta\rho=-\sum_\nu\frac{e_\nu}{m_\nu}\int d^3v\left[[\mathcal{D}f_\nu^{(0)}][\hat{d}_\nu G_\nu]+\frac{i}{2}\left[\mathbf{k}_{xz}\cdot\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\right]\Psi_\nu(e^{i\Gamma_\nu+i\mathbf{k}_{xz}\cdot\mathbf{x}}-e^{-i\Gamma_\nu-\mathbf{k}_{zx}\cdot\mathbf{x}})\right] . \tag{F13}$$

Employing the relation

$$\hat{d}_\nu G_\nu=\tfrac{1}{2}[\hat{d}_\nu\Psi_\nu](e^{i\Gamma_\nu+i\mathbf{k}_{xz}\cdot\mathbf{x}}+e^{-i\Gamma_\nu-i\mathbf{k}_{xz}\cdot\mathbf{x}})+\frac{i}{2}\Psi_\nu[\hat{d}_\nu\Gamma_\nu](e^{i\Gamma_\nu+i\mathbf{k}_{zx}\cdot\mathbf{x}}-e^{-i\Gamma_\nu-i\mathbf{k}_{xz}\cdot\mathbf{x}}) , \tag{F14}$$

one then calculates

$$\delta\rho=-\sum_\nu\frac{e_\nu}{m_\nu}\int d^3v\left[\frac{[\mathcal{D}f_\nu^{(0)}]}{2}[\hat{d}_\nu\Psi_\nu](e^{i\Gamma_\nu+i\mathbf{k}_{xz}\cdot\mathbf{x}}+e^{-i\Gamma_\nu-i\mathbf{k}_{xz}\cdot\mathbf{x}})+\frac{i}{2}\left[[\mathcal{D}f_\nu^{(0)}]\hat{d}_\nu\Gamma_\nu+\mathbf{k}_{xz}\cdot\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\right]\times\Psi_\nu(e^{i\Gamma_\nu+i\mathbf{k}_{xz}\cdot\mathbf{x}}-e^{-i\Gamma_\nu-i\mathbf{k}_{xz}\cdot\mathbf{x}})\right] . \tag{F15}$$

Taking $\hat{d}_\nu\Psi_\nu=0$, i.e., $\Psi_\nu=\Psi_\nu(\mathcal{H}_\nu,\mathcal{U}_\nu,\mathcal{Y}_\nu)$, does not have any influence whatsoever on the results obtained in Sec. V. In this case, the perturbed charge density is

$$\delta\rho=-\sum_\nu\frac{e_\nu}{m_\nu}\int d^3v\left[\frac{i}{2}\left[[\mathcal{D}f_\nu^{(0)}]\hat{d}_\nu\Gamma_\nu+\mathbf{k}_{xz}\cdot\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\right]\Psi_\nu(e^{i\Gamma_\nu+i\mathbf{k}_{xz}\cdot\mathbf{x}}-e^{-i\Gamma_\nu-i\mathbf{k}_{xz}\cdot\mathbf{x}})\right] . \tag{F16}$$

The perturbed charge density $\delta\rho$ can be made zero, since the expressions for $\delta^2 H$ only contain Ψ_ν^2 and $(\hat{d}_\nu\Psi_\nu)^2$. Ψ_ν is chosen localized in $\mathcal{H}_\nu$ or $\mathcal{U}_\nu$. The distribution of *signs* in Ψ_ν is free. For instance, one can take Ψ_ν piecewise continuous in $\mathcal{H}_\nu$ or $\mathcal{U}_\nu$, with changing signs so that positive and negative contributions to $\delta\rho$ balance each other.

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