Extensive numerical study of spectral statistics for rational and irrational polygonal billiards

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An extensive numerical study of level-spacing properties of rational and irrational polygonal billiard systems is carried out. It is found that level statistics of both rational and irrational polygonal billiards deviate from a Gaussian-orthogonal-ensemble-type fluctuation. It is also explored in detail whether the polygonal billiards with the infinite genus number provide different level-spacing characteristics from those with the finite genus number. Some delicate problems in dealing with several types of pseudointegrable systems are also discussed.

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I. INTRODUCTION

Chaos in classical mechanics is characterized by sensitive dependence of orbits on their initial conditions, and its degree is measured by the Kolmogorov-Sinai entropy or the Lyapunov exponent. Due to the loss of an initial memory at an exponential rate, one is unable to make long-time predictions in principle. In this sense, entropy is a very important concept which draws a border between the predictable and unpredictable world. One of the central subjects in quantum chaos is to clarify how positive entropy influences various kinds of quantum-mechanical properties. Alternatively stated, can one judge correctly whether or not the corresponding classical system exhibits chaotic behavior solely from knowledge of some quantum-mechanical quantities? This setting of the problem is regarded as one aspect of Berry's proposal of *quantum chaology* [1].

To our knowledge, there exists no convincing theoretical explanation that leads to the universality in quantum mechanics as a direct consequence of positive classical entropy. The universality of quadratic long-range level statistics such as spectral rigidity or number variance, which measures the degree of fluctuation in the energy interval of the order $\hbar$, has been derived by a semiclassical argument [2]. On the basis of the uniformity principle of classical periodic orbits [3], it predicts a Gaussian-orthogonal-ensemble (GOE) -type fluctuation. However, classical entropy does not appear explicitly in the semiclassical formula for quadratic level statistics because the decreasing rate of the square of the amplitude factors in the periodic orbit expansion cancels the increasing rate of the number of periodic orbits. Moreover, the uniformity principle is derived on the assumption that a system has the ergodicity and all periodic orbits are unstable, but it is not yet known what kind of modification should be done if only ergodicity or the mixing property is assumed. Concerning the nearest-neighbor level-spacing distribution, one cannot develop a semiclassical argument because of the divergence of Gutzwiller's trace formula caused by the exponential proliferation law of the number of periodic orbits [4]. Another implication of the role of the classical entropy is mentioned in the billiard problem on a compact surface of constant negative curvature, in which the bottom of the spectrum of the Laplace-Beltrami operator has an expression directly in terms of the classical topological entropy [5]. However, the answer to the converse question is obscure because one cannot judge the chaoticity of a system only from the absolute value of the ground-state energy. Despite these partial theoretical understandings, a large number of numerical experiments strongly suggest the universality of GOE-type fluctuations in completely chaotic systems [6].

On the other hand, from recent analyses for polygonal billiard systems which are called pseudointegrable systems in the literature [7], we also know that a positive Lyapunov exponent is not a necessary condition for level repulsion [7,8]. Yet, it has not been clarified whether the level-spacing distribution of pseudointegrable systems completely coincides with the GOE-type fluctuation and whether one can discriminate the underlying classical chaoticity only by means of the signature of the level-spacing distribution. Hence to get a promising hint to understand the role of the positive classical entropy in quantum mechanics, a detailed study of pseudointegrable systems becomes particularly important. Ordinary approaches, which give a necessary condition for the manifestation of the classical chaoticity to the corresponding quantum systems, are to study quantum systems whose classical counterparts have positive metric entropy. On the other hand, our present approach is, in a sense, devoted to attaining the same aim from the opposite direction, and would give a sufficient condition. However, as actually shown in the present paper, in order to draw a convincing conclusion to this rather delicate problem, one must collect as many examples as possible and treat them with care. The delicate nature of the problem is that the peculiarity of the pseudointegrable billiards lies in the presence of vertices which are only singular points having negative curvature. Even though such singularities have zero measure in classical phase space, they might play a significant role in determining the structure of the energy-level sequence. A drastic role of such a zero-measure singularity is indeed rigorously shown in the billiard with a point obstacle which is the most ideal version of the pseudointegrable billiard system [9].

II. MODEL AND CLASSICAL DYNAMICS

As mentioned in the introduction, our motives in this article are to explore whether or not the GOE characteristic is a property owned only by classically chaotic systems, and to examine what governs the level structure of pseudointegrable systems. Toward this end, we shall take a rhombus-shaped polygonal billiard whose boundary is schematically demonstrated in Fig. 1. We choose the vertex angle α as a system parameter. An interesting point in the polygonal billiard system is that classical dynamics shows an intermediate property between completely integrable and chaotic systems, because it is not integrable in the sense of Liouville and Arnold but it has null metric entropy [10]. In addition, from the ergodic theoretical viewpoint, there are two important classes in polygonal billiard systems. One is a type of polygon whose vertex angles are all rational multiples of π (for abbreviation, we refer to this type of polygon as the rational polygon), and the other is a polygon where at least one of the vertex angles is an irrational multiple of π (the irrational polygon, for abbreviation). In the case of the rational polygon, its classical phase space becomes a multihandled sphere with a finite genus number, which means that the number of possible directions for classical trajectories is finite. On the other hand, since the genus number for irrational polygons is infinite, it is expected that typical orbits bounce in all directions. Up to now, we do not have any rigorous mathematical proof, but it is believed that almost all irrational polygons possess ergodicity [11]. Moreover, it is also conjectured that they possess a weak mixing property which is a minimum condition for equilibrium statistical mechanics [12].

TABLE I. Genus number of the rational polygon with the vertex angle $(p/q)\pi$, where p and q are relatively prime integers. In this table, $\frac{1}{4}$, $\frac{1}{2}$, and 1 mean the cases of the quarter rhombus, the upper half of the rhombus, and the full rhombus, respectively.

p,q	$\frac{1}{4}$	$\frac{1}{2}$	1
odd,odd	$\frac{q-1}{2}$	$\frac{q-1}{2}$	$q-1$
even,odd	$\frac{q-1}{2}$	$q-1$	$q-1$
odd,even	$\frac{q}{2}$	$q-1$	$q-1$

The prescription to compute the genus number for the rational polygon is given as [7,13]

$$g = 1 + \frac{N}{2}\sum_{k}\left\{\frac{m_k - 1}{n_k}\right\}, \tag{1}$$

where $(m_k/n_k)\pi$ are interior vertex angles of a polygon, N is the least common multiple of the integer set n_k, and the sum is taken over all polygonal vertices. For our model, the genus number in the case of the rational polygon is given in Table I. In the following study of quantum spectral statistics, we shall restrict ourselves to odd-odd parity states, which means that Dirichlet boundary condition is imposed on an oblique side and two symmetry axes. Therefore, we shall treat a right-angled triangle substantially. From the rule given by Table I, in the case of a quarter of the rhombus or right-angled triangle, the reciprocal of the genus numer as a function of the vertex angle exhibits a self-similar structure as shown in Fig. 2.

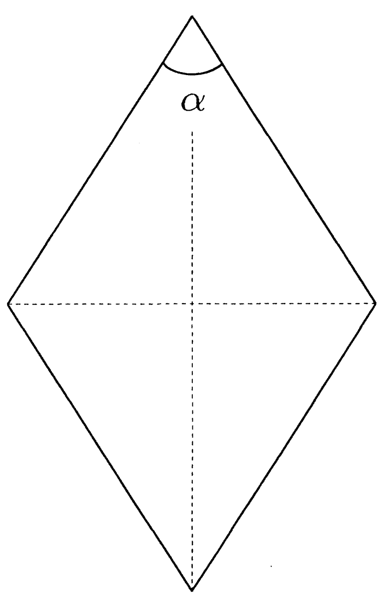

FIG. 1. Schematic picture of the rhombus. The system parameter is an interior vertex angle α $[0<\alpha\leq(\pi/2)]$. The dashed lines represent symmetry axes.

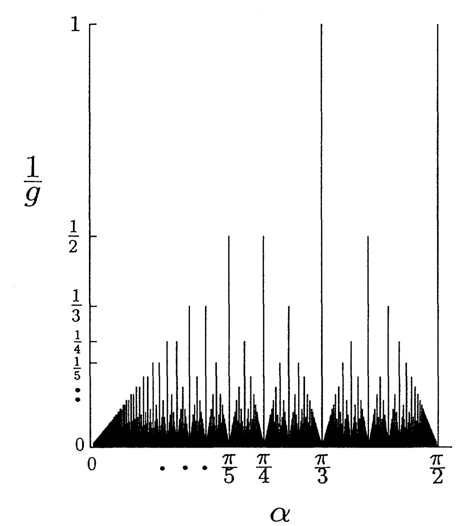

FIG. 2. The reciprocal of the genus number for the quarter rhombus as a function of the system parameter α. The quarter rhombus is integrable only when α is equal to $\pi/2$ and $\pi/3$. The genus number for irrational α is infinite and the reciprocal is equal to zero.

When the system parameter α is an irrational multiple of π, the reciprocal of the genus number is equal to zero.

Before investigating quantum level statistics, it is instructive to observe the classical dynamics of the present billiard system. The classical dynamics for billiard problems is appropriately described by the mapping on the Birkhoff coordinate which is composed of a pair of the arc length and the sine of the reflection angle. As a typical example of an irrational rhombus, we present in Fig. 3 the phase-space plot starting from a single initial point. In this example, the vertex angle α is chosen to be $(1-r_1)\pi$, where $r_1=(\sqrt{5}-1)/2$. This is an irrational number for which the rate of approximation by the continuous fractional expansion is the slowest. Though the vertex angle is such a typical irrational number and a large number of iterations is performed, the rate at which a single trajectory fills the phase space is considerably slow. To see such a slow diffusion, we divide the phase space of Birkhoff coordinates into sufficiently small cells and count the number of cells in which a trajectory passes. Figure 4(a) is a plot of the occupation rate as a function of the time step. For the sake of contrast and in order to compare it with a typically chaotic system with positive metric entropy, we present in Fig. 4(b) the same plot for the stadium billiard system. Although the length of straight segments of the stadium boundary is very small, its trajectory rapidly fills the phase space and the difference in diffusion processes between these two cases is obvious. In particular, while an orbit in the stadium billiard spreads over the whole phase space at an approximately constant rate, the polygonal billiard with an irrational angle does not show such a uniform diffusion, rather an orbit is frequently trapped in very narrow regions for a sufficiently long time, and after that it spreads at a relatively fast rate. This graph for the occupied area resembles the so-called devil's staircase. The observed fractal structure is closely related to the power-law correlation decay found in other types of pseudointegrable systems [14,15]. This implies that the time required for an individual orbit to wander the entire phase space is exceedingly large, though polygonal billiards with irrational angles are expected to possess the ergodicity as mentioned above and its symptom is shown even in the present numerical experiment. Such a slow diffusion is mainly attributed to the fact that the irrational polygon is always sandwiched between rational polygons with a small genus number as illustrated in Fig. 2. An alternative interpretation is that numerous cantori filling densely all over the phase space strongly disturbs the diffusion of an orbit [16].

From the behavior observed in the classical mapping, the following prediction is possible for the property of eigenstates in the corresponding quantum system. Suppose an initially localized wave packet which is composed of the superposition of a large number of eigenstates in the semiclassical regime. Owing to considerable slow diffusion of a classical trajectory in the irrational polygon, such a localized quantum wave packet also spreads

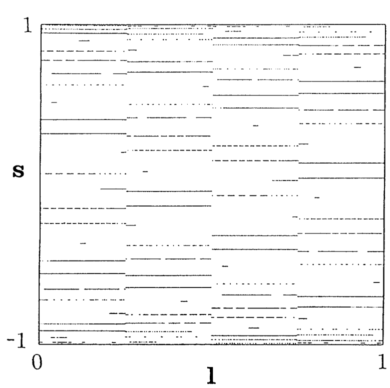

FIG. 3. The classical mapping on the Birkhoff coordinate. The horizontal axis corresponds to the arc length from an origin, and the vertical axis to the sine of the angle between the direction of a trajectory and that of the inner normal to its side. The number of iterations is one million.

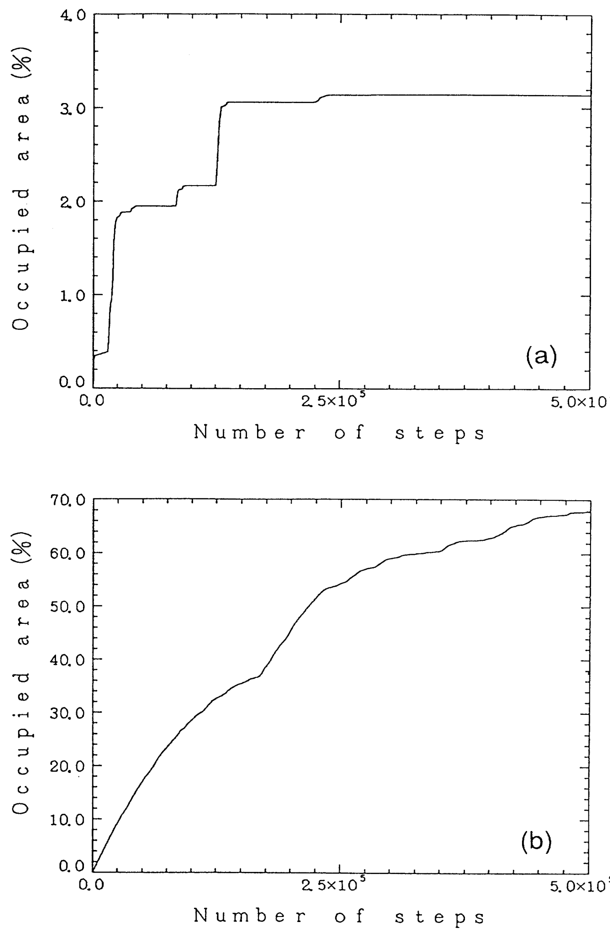

FIG. 4. The area occupied in the phase space of the Birkhoff coordinate by the iteration of the classical mapping in the case of (a) the rhombus billiard with $\alpha=(1-r_1)$, where $r_1=(\sqrt{5}-1)/2$, and (b) the stadium billiard with $\gamma=0.1$ where γ is the ratio of the length of straight segments and the radius of the circle. The whole phase space is divided into 25 000 cells, and the number of cells in which a trajectory resides is counted. The horizontal axis represents the number of steps, and the vertical one is the percentage of the occupied cells.

out at a very slow rate, which is a consequence of the most naive quantum-classical correspondence principle. It leads us to a speculation that a fairly large part of eigenstates superposed to construct an initial wave packet are localized ones. Therefore, if one wishes to judge the precise asymptotic behavior of the level-spacing property of a system with slow diffusion, one must enter into a deep semiclassical regime, below which a purely quantum effect dominates before the wave packet exhibits very slow spreading reflecting the corresponding slow diffusion of classical dynamics. Alternatively stated, in order to see whether level statistics of a typically irrational polygon approach GOE-type fluctuation, a sufficiently large number of levels must be prepared.

III. QUANTUM SPECTRAL STATISTICS

Taking account of the preliminary speculation from classical dynamics, we now examine the level-spacing distribution of the present polygonal billiards. Our strategy to seek a primary factor controlling the struture of the level-spacing distribution is twofold. The first is to explore whether rational polygons with different genus number yield different level-spacing characteristics. The second focus is directed to a question of whether the behavior of level-spacing properties can be discriminated by the rationality of vertex angles. From these two directions, we intend to get a unified view about the key ingredient determining the structure of the level-spacing distribution in polygonal billiard systems, and to confirm how spectral statistics certainly reflect the underlying classical chaoticity.

To obtain eigenvalue sequences numerically, we employed the boundary element method, the details of which are described in Ref. [17]. We checked numerical accuracy and convergency by computing eigenvalues of the rectangular and a circular billiard whose exact eigenvalues are already known. Due to the roughness of the discretized interval of the wave number, some of the eigenvalues might be missed, which is inevitable in these kind of numerical or experimental searching procedures. However, by referring to Weyl's asymptotic rule including both the boundary and corner terms, the percentage of missing eigenvalues can be estimated and the error was less than 2%, which did not crucially affect the essential features of level statistics.

As representatives of rational polygons we examined the cases with $g=2$ and 3. From the rule given by Eq. (1) or Table I, vertex angles $\frac{1}{2}\alpha=\frac{1}{5}\pi$, $\frac{1}{8}\pi$, and $\frac{1}{10}\pi$ only yield the cases of $g=2$, which could be regarded as the simplest possible pseudointegrable billiard systems. For these three cases, splitting angles of nearby orbits which bounce at vertices are all different (among them the case of $\frac{1}{2}\alpha=\frac{1}{5}\pi$ shows the maximum instantaneous separation). However, their phase spaces have the same topological structure with genus 2. Other rational polygons examined are the cases with vertex angles $\frac{1}{2}\alpha=\frac{3}{14}\pi$, $\frac{1}{7}\pi$, $\frac{1}{12}\pi$, and $\frac{1}{14}\pi$, all of which classically give the multihandled sphere with genus 3.

Strictly speaking, the concept of level-spacing distribution makes sense in the infinite limit of level number ($N\rightarrow\infty$), and it is difficult to draw a definite conclusion only from a finite number of numerical data, especially when one treats a delicate problem as posed here. Nevertheless, if the number of levels is quite large, one is able to predict its asymptotic behavior and obtain valuable information as to its limiting distribution from finite numerical samples. To make the asymptotic behavior evident, we see how the feature of the nearest-neighbor level-spacing distribution changes as a function of the level number. To quantify it, the resulting histograms are fitted to the Brody distribution which is a semiempirical interpolation formula between Poisson and Wigner distributions [18], and its explicit form, which is parametrized by the Brody parameter β, is given as

$$P_\beta(s)=As^\beta\exp(-\alpha s^{1+\beta}) ,$$
$$A=(1+\beta)\alpha, \quad \alpha=\left[\Gamma\left[\frac{2+\beta}{1+\beta}\right]\right]^{1+\beta} , \tag{2}$$

where $\beta=0$ and 1 yield Poisson and Wigner distributions, respectively [18]. Although several alternative fitting formulas have been devised [19], the present concern is not to confirm the validity of those formulas but to clarify the degree of level repulsion and the extent of deviation of empirical histograms from the Wigner spacing distribution.

Figures 5(a) and 5(b) show the variation of the Brody parameter as a function of the level number for the cases of $g=2$ and 3, respectively. A common feature found in the two figures is that the Brody parameter for the lower-energy region takes a relatively higher value than that of the higher-energy regime, and that the Brody parameter gradually decreases for first hundreds of levels. Then it seems to saturate or oscillate without a secular increasing or decreasing trend. A possible intuitive interpretation for this behavior is that the influence of singularities with zero measure is relatively large for lower eigenstates in comparison with higher-energy ones. Such a large effect caused by the presence of the vertices can be gradually weakened as the energy goes to the very short wavelength regime. A similar behavior has been reported in the case of the billiard with a point obstacle, in which the first hundreds of levels appear to approach the Poisson characteristic [20], but the limiting distribution certainly shows the level repulsion as mentioned before [9]. Furthermore, the same plots of the variation of the Brody parameter for the stadium billiard system do not yield such a high value in the lower-energy regime [21]. Accordingly, the behavior found in the first lower levels must be regarded as a transient one, and an asymptotic feature should be judged in the sufficiently high-energy regime, in which stationary behavior is confirmed.

From the comparison of Figs. 5(a) and 5(b), it would be rather difficult to conclude that there exists a definite difference between a set of vertex angles giving $g=2$ and those giving $g=3$, because the final values for the $g=3$

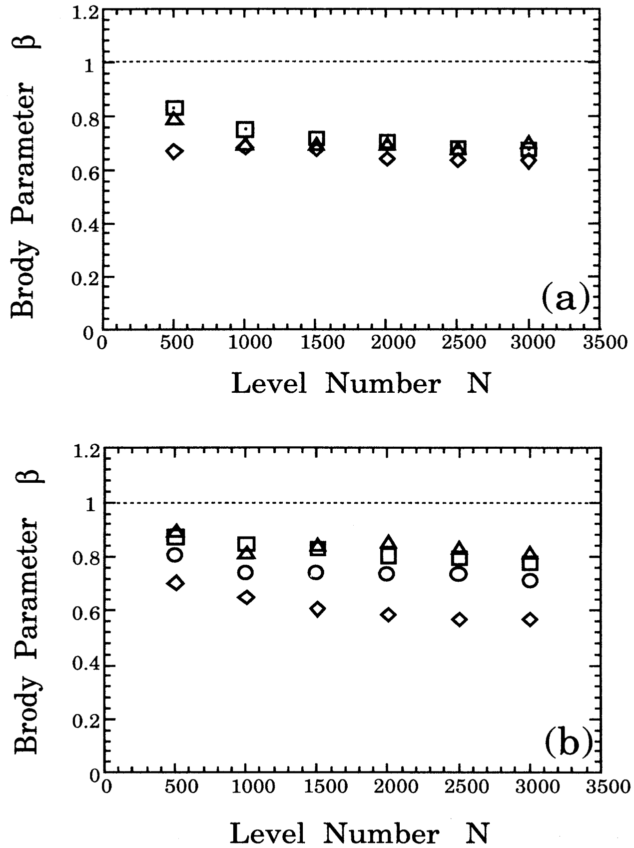

FIG. 5. The values of the Brody parameter as a function of the level number. Figure (a) corresponds to the cases of $g=2$, where each mark represents the case of $\frac{1}{2}\alpha=\frac{1}{5}\pi$ (△), $\frac{1}{8}\pi$ (◇), and $\frac{1}{10}\pi$ (□). Figure (b) corresponds to the cases of $g=3$, where each mark represents the case of $\frac{1}{2}\alpha=\frac{3}{14}\pi$ (○), $\frac{1}{7}\pi$ (△), $\frac{1}{12}\pi$ (◇), and $\frac{1}{14}\pi$ (□).

cases spread more widely than those of $g=2$, and the value for the case of $\frac{1}{2}\alpha=\frac{1}{12}\pi$ is smaller than either value for the $g=2$ cases. Though one may expect that the genus number is an important factor determining the energy-level statistics, it is still only a matter of conjecture within the present result. A similar situation also holds in the behavior of the quadratic long-range correlation. In Figs. 6(a) and 6(b), we present the results of the Δ_3 statistic [22] for the cases of $g=2$ and 3, respectively. Of course as L becomes larger, the difference or the system specificity in the plots of Δ_3 becomes more evident, which should be due to a natural consequence that the short-time behavior of classical dynamics strongly depends on the vertex angles. However, resulting curves including the small L regime, which is less influenced by such a system specificity, spread over a certain region, which makes it difficult to conclude that only the genus number determines the complete characteristic of spectral statistics. In order to give a reliable statement, more detailed study of other types of pseudointegrable billiards must be made, the reason for which is discussed in the final section. Despite these unsettled ambiguities, an important and clear finding obtained at least within the present calculation is that level-spacing distributions for all cases with $g=2$ and 3 significantly deviate from that derived by the GOE assumption. Furthermore, the degree of deviation is very similar to the result of the billiard with a point obstacle [23].

Next, we shall focus on the difference between rational and irrational polygons. As irrational-angled cases, we choose the vertex angles $\frac{1}{2}\alpha=\frac{1}{2}(1-r_1)\pi$, $\frac{1}{2}r_2\pi$, $\frac{1}{2}r_3\pi$ where $r_1=(\sqrt{5}-1)/2$, $r_2=\sqrt{3}-1$, $r_3=(-3+\sqrt{21})/2$, and $\frac{1}{2}\alpha=(1/\sqrt{5})\pi$, $(1/\sqrt{7})\pi$, $(1/\sqrt{11})\pi$, $(1/\sqrt{19})\pi$. The numbers r_1, r_2, and r_3 are typical Diophantine numbers which are very hard to be approximated by any rational numbers, and all denominators and numerators of

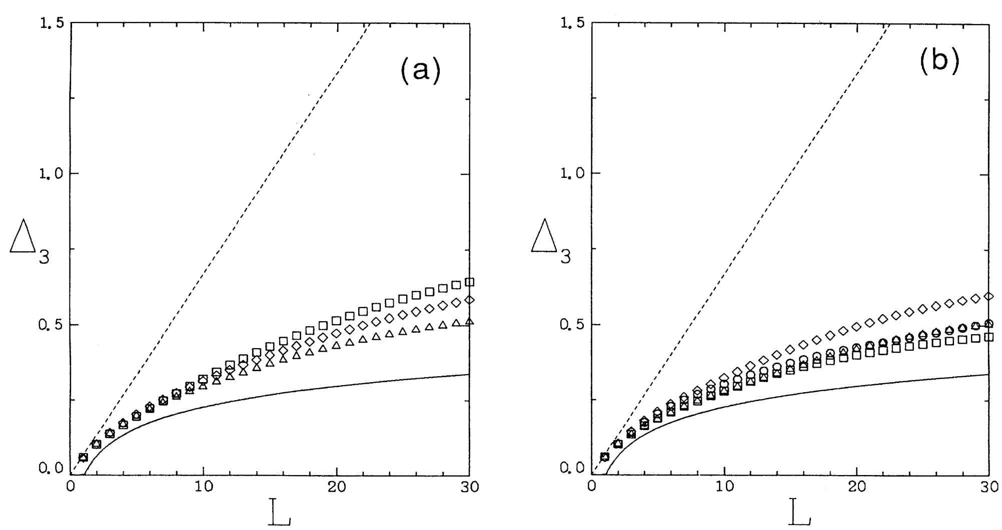

FIG. 6. Δ_3 statistics in the cases of (a) $g=2$, where each mark represents the case of $\frac{1}{2}\alpha=\frac{1}{5}\pi$ (△), $\frac{1}{8}\pi$ (◇), and $\frac{1}{10}\pi$ (□), and (b) $g=3$, where each mark represents the case of $\frac{1}{2}\alpha=\frac{3}{14}\pi$ (○), $\frac{1}{7}\pi$ (△), $\frac{1}{12}\pi$ (◇), and $\frac{1}{14}\pi$ (□). The dashed line represents the straight line for the purely Poisson sequence, and the solid curve the logarithmic dependence for the GOE sequence.

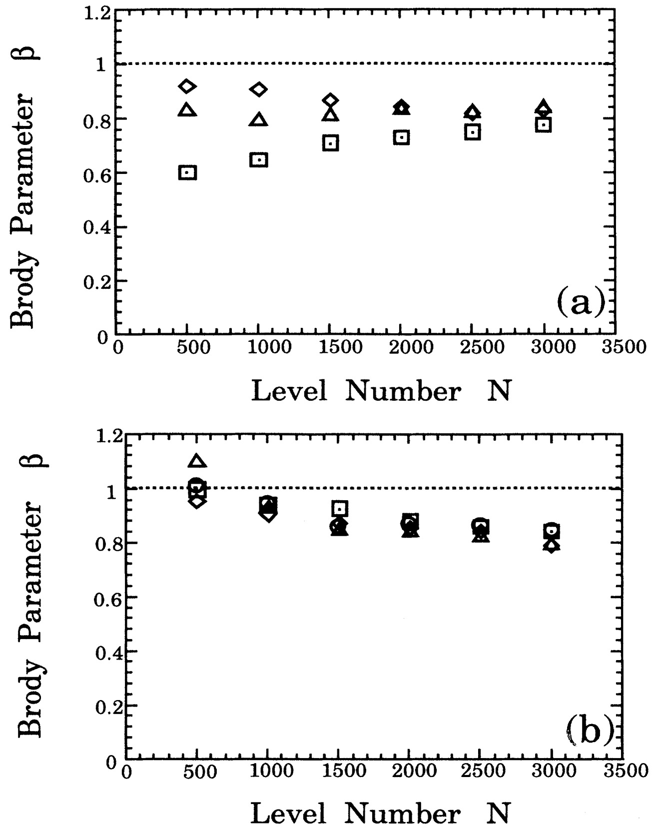

FIG. 7. The values of the Brody parameter as a function of the level number. Figure (a) represents the cases of the genus number g being typical Diophantine numbers, in which $\frac{1}{2}\alpha=\frac{1}{2}(1-r_1)\pi$ (□), $\frac{1}{2}r_2\pi$ (◇), and $\frac{1}{2}r_3\pi$ (△) where $r_1=(\sqrt{5}-1)/2$, $r_2=\sqrt{3}-1$, $r_3=(-3+\sqrt{21})/2$. Figure (b) represents the cases for the genus number g being typical quadratic irrational numbers where $\frac{1}{2}\alpha=(1/\sqrt{5})\pi$ (□), $(1/\sqrt{7})\pi$ (◇), $(1/\sqrt{11})\pi$ (△), and $(1/\sqrt{19})\pi$ (○).

the continued-fraction expansion are given as 1 for r_1, 2 for r_2, and 3 for r_3. The latter four are typical quadratic irrational numbers. In Figs. 7(a) and 7(b), we also plotted the Brody parameter of level-spacing distributions of these irrational polygons as a function of the level number. As is found in rational cases, the values of the Brody parameter take relatively high values for the first hundred of levels and gradually decrease, though the case of $\frac{1}{2}\pi=\frac{1}{10}\pi$ is exceptional. For this phenomenon, the same intepretation as done in rational-angles cases would be possible. For most cases, final values of the Brody parameter for irrational cases are large compared with those for the rational polygons, but several cases of $g=3$ exceed some irrational cases and the differences observed are very minute. Thus, only from the data of nearest-neighbor level-spacing distributions, it would be too speculative to conclude that the rationality of vertex angles divides the nature of the level statistics. However, the long-range correlation property, results of which are demonstrated in Figs. 8(a) and 8(b), reveals the possibility of the rationality of vertex angle being an important ingredient determining the signature of the level-spacing distribution, because all level sequences for irrational cases exhibit stronger rigidity than those for rational cases.

If we put the working hypothesis that the level statistics reflect the finiteness of genus, which determines the ergodicity of the underlying classical dynamics, it is meaningful to see the degree of difference for these statistics. For this purpose, we compare the combined data, which is a common means to treat experimental data in nuclear physics, and is referred to as nuclear data ensemble [6]. Histograms shown in Fig. 9 are ensemble-averaged level-spacing distributions for rational and irrational cases, respectively. The total number of levels to yield these histograms is 21 000 for both cases. It is apparent that both distributions exhibit level repulsion but

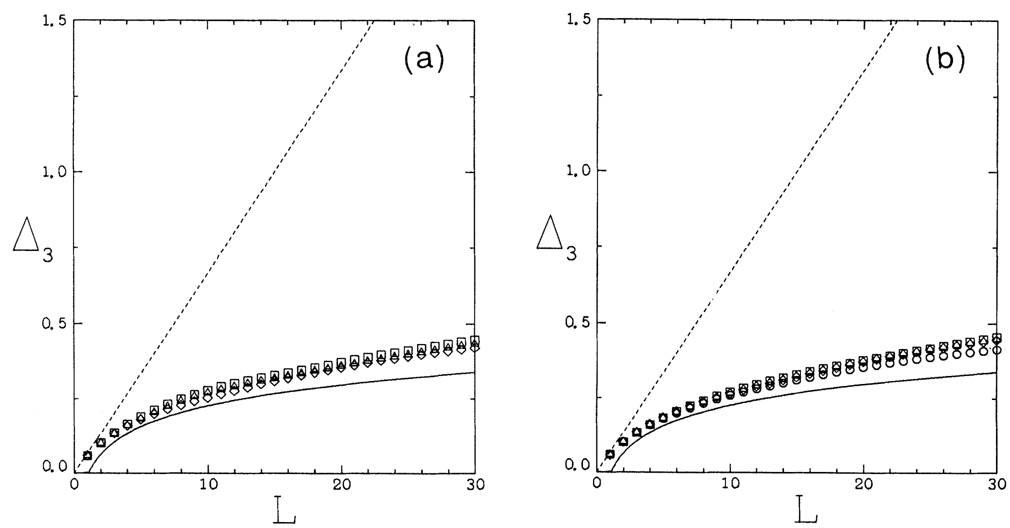

FIG. 8. Δ_3 statistics in the cases for (a) the genus number g being typical Diophantine numbers, in which each mark represents the case of $\frac{1}{2}\alpha=\frac{1}{2}(1-r_1)\pi$ (□), $\frac{1}{2}r_2\pi$ (◇), and $\frac{1}{2}r_3\pi$ (△) where $r_1=(\sqrt{5}-1)/2$, $r_2=\sqrt{3}-1$, $r_3=(-3+\sqrt{21})/2$, and (b) the genus number g being the typical quadratic irrational numbers, in which each mark represents the case of $\frac{1}{2}\alpha=(1\sqrt{5})\pi$ (□), $(1/\sqrt{7})\pi$ (◇), $(1/\sqrt{11})\pi$ (△), and $(1/\sqrt{19})\pi$ (○).

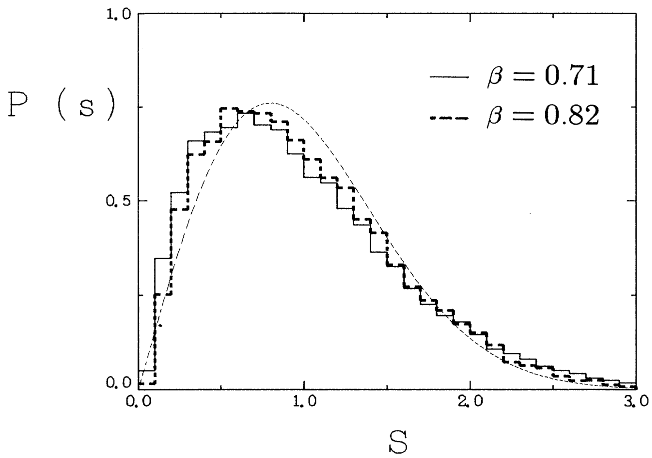

FIG. 9. The nearest-neighbor level-spacing distribution for the combined data. The solid line represents the case with finite genus, and the dotted line the case with infinite genus. The broken curve inserted in the figure is the Wigner distribution.

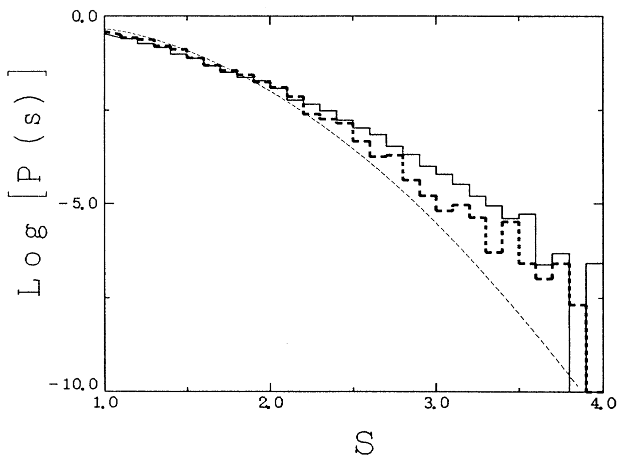

FIG. 11. The nearest-neighbor level-spacing distribution for the combined data. The vertical axis is drawn on a logarithmic scale. The solid line represents the case with finite genus, and the dotted line the case with infinite genus. The broken curve inserted in the figure is the Wigner distribution.

do not completely coincide with GOE characteristic. The Brody parameter for combined data of rational polygons is equal to 0.71, and 0.82 for irrational polygons. Ensemble averaged Δ_3 statistics are presented in Fig. 10, which reveal, under the working hypothesis mentioned above, the difference between the finite and infinite genus case of the long-range correlation property.

Though our selection of the vertex angles of both rational and irrational polygons are not biased in any particular sharp- or broad-angled regime, and the result for

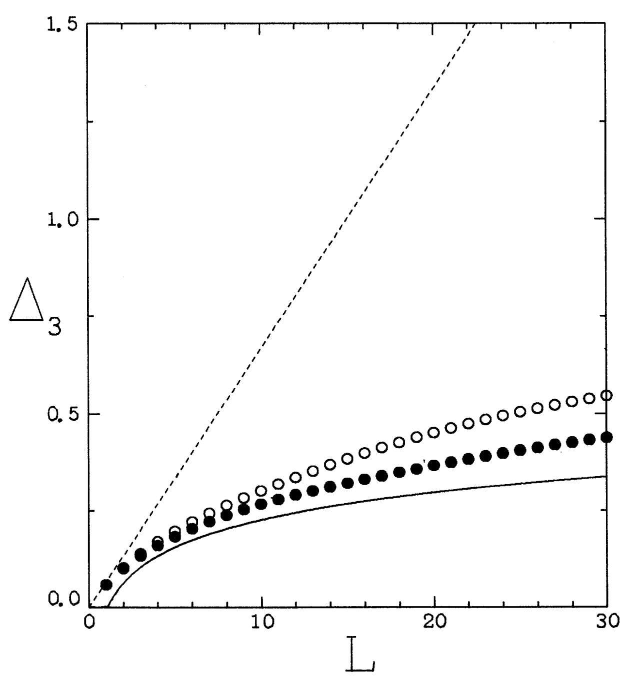

FIG. 10. Δ_3 statistics for the combined data. The open circle denotes the case with finite genus, and the solid circle the case with infinite genus.

the spectral rigidity seems to support our working hypothesis, it is dangerous to form a hasty conclusion to this delicate problem, because we have ignored all other factors which control the underlying classical dynamics. Furthermore, the number of levels might be too small to infer the limiting distribution [24], and within the present examples it is still unclear that the level-spacing distribution of irrational polygons approaches the Wigner characteristic. Even if it reaches the Wigner one in the semiclassical limit, its rate of convergence is certain to be exceedingly slow. In any case, it has not been clarified so far that the observation of a fairly large number of levels is required to judge the precise asymptotic behavior of spectral statistics of polygonal billiard systems.

A large number of levels enables us to check another detailed property of the level-spacing distribution which has not been examined so far. As is proved rigorously [9], in the case of the billiard with a point obstacle, the signature of the level-spacing distribution in the large spacing region differs from that of the ordinary GOE prediction. While GOE gives $P(s) \sim e^{-s^2}$ for $s \to \infty$, the estimation $P(s) \sim e^{-s}$ holds for the billiard with a point obstacle. We here check the behavior of the tail of the level-spacing distribution of the present rhombus billiard model. To see the functional form of s dependence clearly, we present histograms on a logarithmic scale, the results of which are given in Fig. 11. Although both empirical histograms deviate from the GOE curve, the distributions for large s values do not appear to decay as $P(s) \sim e^{-s}$ [25]. This result suggests a possibility of the behavior of the tail dividing the universality class of the pseudointegrable system, though a theoretical origin of this difference has not been understood yet.

IV. SUMMARY AND CONCLUDING REMARKS

Concluding this article, we summarize our results and discuss future unsolved questions in this direction. Our main aim in the present study is to see to what extent the quantum level statistics sensitively reflect the nature of the underlying classical dynamics, and to verify whether the level statistics of polygonal billiard systems truly show the GOE-type universality, which has not been clarified in studies prior to ours. Our results show that level statistics of both rational and irrational polygons significantly deviate from the distribution derived from the Wigner surmise. Moreover, on the working hypothesis of level statistics correlating with the underlying classical ergodicity, we concentrated on the difference between rational and irrational polygons, and the possibility that the rationality of vertex angles plays a role to determine level statistics is suggested especially from the observation of the long-range level correlation. Although a final conclusion concerning this problem must be drawn carefully, the present result poses an interesting question of what is the theoretical background of the difference of level statistics in pseudointegrable systems. Of course, toward a definite answer, much more numerous levels are needed, but at present it is beyond our computational reach.

To understand the results obtained in this study theoretically, the semiclassical analysis is desired. However, as carried out in the case of $\alpha=\frac{1}{3}\pi$, the presence of singularities caused by vertices which are peculiar to pseudointegrable billiard systems might play an important role to determine the structure of energy-level sequences [26]. Hence, to obtain a full semiclassical understanding, the contribution from the diffraction caused by the scattering at vertices would be taken into account in the usual periodic-orbit expansion. In this sense, the semiclassical origin of the level repulsion found in the present pseudointegrable system somewhat differs from that in chaotic systems in which the asymptotic balance of the number and amplitude of periodic orbits mainly governs the semiclassical behavior of quadratic long-range level statistics.

As mentioned in the introduction, our final goal is to know the precise reflection of the classical positive entropy to the structure of quantum energy sequences, which is analogous to the famous question posed by Kac: "Can one hear the shape of a drum?" [27]. If the limiting distribution of the irrational polygon becomes the Wigner characteristic, it seems that one must abandon an attempt to infer the classical chaoticity only from the statistical property of energy sequences. However, as is also discovered in the present calculation, the level-spacing distribution of irrational polygons takes a fairly large number of levels to approach the Wigner one, even if the limiting distribution is the Wigner one. The shape of the limiting form, therefore, contains important information and at the same time one possibility to discriminate properties owned by chaotic systems from those by nonchaotic ones is to see the rate of convergency toward its limiting form. One more important piece of information discarded so far is the fluctuation of the Brody parameter around its mean value. To see it, our only task is to examine the Brody parameter of each energy regime separately, not the cumulated one. However, within the data obtained in the present study, we have not succeeded in detecting a meaningful discrepancy of such fluctuation features between the irrational polygon and the stadium billiard. Likewise, it has not been possible to verify the difference of Planck-constant dependence of the saturated value of the quadratic level correlation predicted by Berry [2].

Finally, we wish to remark on the treatment of pseudointegrable systems. To fix an idea that the number of genus completely determines the universality of quantum spectral statistics, the following ambiguous questions should be fully resolved. The numerical study of even-parity states of the rhombus billiard with $\alpha=\frac{1}{3}\pi$ yields a somewhat anomalous nature concerning the nearest-neighbor level-spacing distribution, though the genus number is also two [26]. This anomaly is due to the presence of degeneracy which is regarded as remnant of the nontypical character of the corresponding odd-odd parity states with $g=1$. An alternative interpretation is that the classical billiard problem corresponding to the even-parity states of the $\frac{1}{3}\pi$ rhombus belongs to the *almost integrable billiard,* the concept of which is first proposed by Gutkin [28]. Its billiard plane is composed of several pieces of completely integrable billiards with the same shape. The almost integrable billiard is of course a subset of generic pseudointegrable billiards with the finite genus number. The truncated triangle billiard analyzed by Richens is one example of such a class, and the energy sequence of the truncated triangle and that of the corresponding integrable billiard have several eigenvalues in common [29]. Hence it is expected that a class of almost integrable billiards is nongeneric, and must be treated separately.

Another remark we wish to make is that there are other types of pseudointegrable billiards with a finite genus number. The generalized Sinai billiard studied by Cheon and Cohen is an example [8]. It also has the finite genus number, but the difference from the present rational rhombus lies in that the possible direction of classical trajectories in the generalized Sinai billiard is the same as those of the ordinary rectangular billiard but there are convex corners splitting the banded trajectories, whereas the present rhombus billiard has no such corners but the number of possible directions of trajectories is proportional to the genus number. We have not yet known whether or not these two types of pseudointegrable systems with the same genus number exhibit the same statistical property of the energy-level sequence. Accordingly, one must carefully examine the difference among them and, toward the full understanding of pseudointegrable systems, these unsolved questions must be dealt with in the future.

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