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Perturbation theory in the exact linearized kinetic equation for a plasma

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An analysis is performed of the perturbation theory dealing with particle interactions in the exact linearized collision integral (CI) for a plasma in an electric field. It is shown that for a fully ionized plasma the usual results of kinetic theory correspond to a high-frequency expansion of the CI, whereas the static limit of the exact CI in a weak-interaction approximation is in disagreement with the usual kinetic theory.

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I. INTRODUCTION

The kinetic theory of plasmas is based on the statement that under certain conditions a closed system of kinetic equations (KE) for one-particle distribution functions (OPDF) $f_a(\mathbf{p},\mathbf{r},t)$ exists [1–4]. The derivation of closed kinetic equations is not only based on the weakness of interparticle interaction or low density of particles in the system considered, but also on statistical assumptions, such as weakening of correlations in the past, which cannot be proved strictly. In this connection the verification of the correctness of the assumptions is of essential interest. It is especially important with respect to attempts to develop a kinetic theory for strongly coupled plasmas; see, e.g., the recent paper [5] and references mentioned there. Evidently, such verification can be implemented by comparing consequences of the KE method with exact results from microscopic statistical theory.

In the present paper the generalized KE for the OPDF $f_a(\mathbf{p},\mathbf{r},t)$ follows from linear-response theory (LRT) for plasmas in an electric field. All effects of strong interaction are included in the collision integral (CI). As will be shown, the use of perturbation theory with respect to interactions between particles leads to different results for different sequences of limits, i.e., $\lambda\to 0$, $\omega\to 0$ or $\omega\to 0$, $\lambda\to 0$, where ω is the frequency of the electric field and λ is the interaction parameter. From a physical point of view it seems obvious that for the description of kinetic processes in a weakly nonideal plasma in a static situation, the sequence of limits $\omega\to 0$, $\lambda\to 0$ is appropriate. This cannot be realized, however, within the framework of the usual KE method which is restricted in principle to the sequence $\lambda\to 0$, $\omega\to 0$. Therefore, it is necessary to investigate the correctness of the usual results of the KE method in the static limit. To this end we consider in Sec. II the general form of the linearized KE and the linearized CI in the Lenard-Balescu approximation for $\omega\to 0$. In Sec. III we use the formalism of LRT to establish the generalized linear kinetic equation in closed form. This KE for the OPDF is exact for arbitrary strength of interaction and arbitrary frequency of the external field. We show that in the limit $\lambda\to 0$, $\omega\to 0$, the Lenard-Balescu CI is obtained. In Sec. IV we consider the opposite sequence of limits, $\omega\to 0$, $\lambda\to 0$, and discuss the structure of the CI in this case. In Sec. V some conclusions are formulated.

II. GENERAL LINEARIZED KINETIC EQUATION AND LENARD-BALESCU COLLISION INTEGRAL

One of the consequences of the KE method for a plasma is a closed integral equation for the OPDF $f_a(\mathbf{p},\mathbf{r},t)$ linearized with respect to the electric field in the limit of weak nonuniformity. The common form of this equation can be written in a convenient form by using Fourier variables $(\mathbf{k},\omega)$ instead of $(\mathbf{r},t)$:

$$-i\omega f_a^{(1)}(\mathbf{p},k\to 0,\omega)+Z_a e\frac{\partial f_a^{(0)}(p)}{\partial p_\alpha}E_\alpha(k\to 0,\omega)$$
$$=\sum_b\int\frac{d^3p_1}{(2\pi\hbar)^3}W_{ab}(\mathbf{p},\mathbf{p}_1,\omega)f_b^{(1)}(\mathbf{p}_1,k\to 0,\omega)\ . \quad (1)$$

Here, $\mathbf{E}(k\to 0,\omega)$ is the weakly inhomogeneous electric field and $f_a^{(0)}(p)$ is the exact equilibrium OPDF for particle species a (having charge $Z_a e$ and mass m_a). The kernels W_{ab} are fully determined by the equilibrium parameters of the plasma: temperature T and densities n_a for which the electric neutrality condition is valid:

$$\sum_a Z_a e n_a=0\ . \quad (2)$$

Within the framework of the KE method, the kernels W_{ab} can be calculated accurately to order λ^2, where λ is the interaction constant (proportional to e^2) [1–4]. In particular, for weakly nonideal two-component fully ionized plasmas, the functions $W_{ab}(\mathbf{p},\mathbf{p}_1,\omega)$ have the following form in the limit $\omega\to 0$:

$$W_{ab}^{\mathrm{BL}}(\mathbf{p},\mathbf{p}_1)=\lim_{\omega\to 0} W_{ab}(\mathbf{p},\mathbf{p}_1,\omega)$$

$$=-\left\{(2S_a+1)T\frac{\partial f_b^{(0)}(p_1)}{\partial\varepsilon_{p_1}^b}\right\}^{-1}$$

$$\times\sum_c\int\frac{d^3p_2}{(2\pi\hbar)^3}\int\frac{d^3q}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{d\omega_1}{2\pi}\sinh^{-2}(\hbar\omega_1/2T)|u_{ac}^S(q,\omega_1)|^2$$

$$\times\mathrm{Im}Q_{aa}^R(\mathbf{p}_1,\mathbf{p}+\hbar\mathbf{q},\omega_1)\mathrm{Im}Q_{cc}^R(\mathbf{p}_2-\hbar\mathbf{q},\mathbf{p}_2,\omega_1)$$

$$\times\{[\delta(\mathbf{p}-\mathbf{p}_1)-\delta(\mathbf{p}-\mathbf{p}_1+\hbar\mathbf{q})]\delta_{a,b}$$

$$+[\delta(\mathbf{p}-\mathbf{p}_2)-\delta(\mathbf{p}-\mathbf{p}_2+\hbar\mathbf{q})]\delta_{b,c}\}\ . \qquad (3)$$

Here, $f_a^{(0)}(p)$ is the Fermi-Dirac (or Bose-Einstein) distribution function with respect to energies $\varepsilon_p^a=p^2/2m_a$ for particles of species a characterized by spin S_a:

$$Q_{aa}^R(\mathbf{p},\mathbf{p}_1,\omega)=(2S_a+1)\frac{f_a^{(0)}(p)-f_a^{(0)}(p_1)}{\varepsilon_p^a-\varepsilon_{p_1}^a+\hbar\omega+i0}\ , \qquad (4)$$

$u_{ab}^S(q,\omega)$ is the screened Coulomb interaction potential for particles of species a and b:

$$u_{ab}^S(q,\omega)=u_{ab}(q)/\epsilon^R(q,\omega),\quad u_{ab}(q)=\frac{4\pi Z_aZ_be^2}{q^2}\ , \qquad (5)$$

$\epsilon^R(q,\omega)$ is the dielectric function in random-phase approximation (RPA):

$$\epsilon^R(q,\omega)=1-\sum_a u_{aa}(q)\Pi_{aa}^{\mathrm{RPA}}(q,\omega)\ , \qquad (6)$$

and $\Pi_{aa}^{\mathrm{RPA}}(q,\omega)$ is the polarization operator for particles of species a in RPA:

$$\Pi_{aa}^{\mathrm{RPA}}(q,\omega)=\int\frac{d^3p}{(2\pi\hbar)^3}Q_{aa}^R(\mathbf{p},\mathbf{p}+\hbar\mathbf{q},\omega)\ . \qquad (7)$$

The functions W_{ab}^{BL} of (3) determine a linearized CI corresponding to the quantum generalization of the Balescu-Lenard KE [2–4,6,7]. Up till now, there exists no exact solution of the linearized KE (1) with the W_{ab}^{BL} of (3). The only exception is the Lorentz plasma characterized by

$$Z_i\gg 1,\quad M_i\gg M_e\ . \qquad (8)$$

In this case and for frequencies much less than the plasma frequency, the linearized KE can be presented in the form of [8]:

$$-i\omega f_e^{(1)}(\mathbf{p},k\to 0,\omega)-e\frac{\partial f_e^{(0)}(p)}{\partial p_\alpha}E_\alpha(k\to 0,\omega)$$

$$=-\nu_{ei}^{\mathrm{BL}}(p)f_e^{(1)}(\mathbf{p},k\to 0,\omega)\ , \qquad (9)$$

where the static electron-ion collision frequency is given by

$$\nu_{ei}^{\mathrm{BL}}(p)=\frac{n_im_e}{4\pi p^3}\int_0^{2p/\hbar}dq\,q^3|u_{ei}(q)|^2\{\epsilon^R(q,o)\epsilon_{ee}^R(q,0)\}^{-1}\ , \qquad (10)$$

$$\epsilon_{ee}^R(q,\omega)=1-u_{ee}(q)\Pi_{ee}^{\mathrm{RPA}}(q,\omega)\ . \qquad (11)$$

At the same time, using relations (3)–(7), one can easily ascertain that in the limit case (8) one must also have

$$\nu_{ei}^{\mathrm{BL}}(p)=\frac{m_e}{ep^2}\sum_a\int\frac{d^3p_1}{(2\pi\hbar)^3}\frac{Z_aep_\alpha p_{1\alpha}}{m_a}W_{ae}^{\mathrm{BL}}(\mathbf{p}_1,\mathbf{p})\ . \qquad (12)$$

III. LINEAR-RESPONSE THEORY VERSUS KINETIC-EQUATION METHOD: THE LIMIT $\lambda\to 0$ FOLLOWED BY $\omega\to 0$

The above results have been checked and generalized on basis of LRT [4] allowing a linear (with respect to the electric field) approximation to account exactly for arbitrarily strong particle interactions. Particularly, an exact OPDF $f_a^{(1)}(p,k\to 0,\omega)$ has been defined by the following relation [9–11]:

$$-i\omega f_a^{(1)}(\mathbf{p},k\to 0,\omega)+Z_ae\frac{\partial f_a^{(0)}(p)}{\partial p_\alpha}E_\alpha(k\to 0,\omega)$$

$$=\langle\langle \hat N_{\mathbf{p}s}^a|\hat I^\alpha\rangle\rangle_\omega E_\alpha(k\to 0,\omega)\ , \qquad (13)$$

where $\hat N_{\mathbf{p},s}^a=\hat a_{\mathbf{p},s}^\dagger a_{\mathbf{p},s}$, $\hat I^\alpha=\sum_a\sum_{\mathbf{p}s}Z_aep_\alpha/m_a\hat N_{\mathbf{p}s}^a,\hat a_{\mathbf{p}s}^\dagger$, and $\hat a_{\mathbf{p}s}$ are the creation and annihilation operators for particles of species a characterized by momentum p and spin number s, $\langle\langle A|B\rangle\rangle_\omega$ is the retarded Green's function (GF) of operators $\hat A$ and $\hat B$:

$$\langle\langle\hat A|\hat B\rangle\rangle_\omega=-\frac{i}{\hbar}\lim_{\delta\searrow 0}\int_0^\infty dt\,\exp(i\omega t-\delta t)\langle[\hat A(t),\hat B(0)]\rangle\ ,$$

$\hat A(t)$ is the operator A in Heisenberg representation, and $\langle\ \rangle$ is the average value with respect to the Gibbs grand canonical ensemble containing the exact Hamiltonian of the plasma. From (13) the well-known Kubo formula [11] for the frequency-dependent plasma conductivity follows directly [15].

Direct substitution shows that the OPDF $f_a^{(1)}(\mathbf{p},k\to 0,\omega)$ from (13) satisfies Eq. (1), if the functions W_{ab} are defined by [12]

$$W_{ab}(\mathbf{p},\mathbf{p}_1,\omega)$$

$$=i\omega V\langle\langle\hat N_{\mathbf{p}s}^a|\hat N_{\mathbf{p}_1s_1}^b\rangle\rangle_\omega$$

$$\times\left\{\frac{\partial f_b^{(0)}(p_1)}{\partial\varepsilon_{p_1}^b}-\langle\langle\hat N_{\mathbf{p}_1s_1}^b|p_{1\alpha}\hat I^\alpha\rangle\rangle_\omega m_b/Z_bep_1^2\right\}^{-1}\ , \qquad (14)$$

where V is the volume of the system. Relation (14) is to be understood in the thermodynamic limit. The more general form of W_{ab} for the inhomogeneous case was found in [13]. So the use of LRT allows us to determine an exact form of the functions W_{ab} in Eq. (1). This form is such that the functions W_{ab} of (14) are expressed in terms of the exact equilibrium GF. It allows us to make use of the well-known methods for calculating the temperature GF on the basis of diagram techniques of perturbation theory [14].

According to [15,16], the calculation of functions like W_{ab} within the framework of the diagram technique for a plasma amounts to a series expansion with respect to three dimensionless parameters describing the interaction between particles: the thermodynamic parameter of nonideality γ, Born's scattering parameter α, and the dynamic parameter $\kappa \sim \bar{\nu}/\omega$, where $\bar{\nu}$ is a characteristic collision frequency. Thus, the corresponding series of perturbation theory for the kernels W_{ab} has the following structure:

$$W_{ab}(\mathbf{p},\mathbf{p}_1,\omega)=\sum_{n=1}^{\infty} W_{ab}^{(n)}(\mathbf{p},\mathbf{p}_1,\omega;\alpha,\gamma)\;, \tag{15}$$

where

$$W_{ab}^{(1)}(\mathbf{p},\mathbf{p}_1,\omega;\alpha,\gamma)=\left[\frac{\partial f_b^{(0)}}{\partial \varepsilon_{p_1}^b}\right]^{-1} i\omega V \langle\langle \hat{N}_{ps}^a | \hat{N}_{\mathbf{p}_1 s_1}^b \rangle\rangle_\omega^{(1)}\;. \tag{16}$$

The superscript for functions $W_{ab}^{(n)}$ in (15) indicates the order with respect to parameter κ. Some features of the series (15) require attention. The first term (16) of the expansion (15) is finite at $\omega \to 0$, while the remaining ones diverge [15,16]. The divergence problem connected with ω dependence is known in the theory of kinetic equations [17,18]. It arises from the attempt to go beyond the lowest order of perturbation theory in terms of the interaction between particles or their density.

Then the first term (16) in the expansion (15) is expected to comply with the usual results of kinetic theory for weakly nonideal plasma: $\alpha \ll 1$, $\gamma \ll 1$ [1–4]. Indeed, if we calculate the function W_{ab} and restrict ourselves to the diagrams shown in Fig. 1, we obtain [15,16] the known results of kinetic theory for a weakly nonideal plasma in a weak electric field including relation (3).

As for the remaining terms of series (15), there is the possibility of partial summation of the most diverging terms at $\omega \to 0$. This leads no doubt to an OPDF $f_a^{(1)}$ which is finite in the limit $\omega \to 0$.

However, the KE method implies that the result of summing diverging terms of series (15) corresponds to a higher order of magnitude with respect to the interaction between particles as compared with the first term (16). It is not possible to prove this statement within the framework of the KE method; in fact, as we will see from Sec. IV, it is incorrect.

IV. LINEAR-RESPONSE THEORY VERSUS KINETIC-EQUATION METHOD: THE LIMIT $\omega \to 0$ FOLLOWED BY $\lambda \to 0$

On the basis of relations (1), (13), and (14), it is possible to define the structure of the kernels W_{ab} in the static limit for the case of weak interaction between particles without using series expansions of perturbation theory. Let us first note that using diagram techniques one can easily verify that

$$\langle\langle \hat{N}_{ps}^a | \hat{N}_{\mathbf{p}_1 s_1}^b \rangle\rangle_\omega = \langle\langle \hat{N}_{\mathbf{p}_1 s_1}^b | \hat{N}_{ps}^a \rangle\rangle_\omega\;. \tag{17}$$

Next, generalizing relation (12), we consider the effective collision frequency $\nu_a(p,\omega)$ for particles of species a:

$$\nu_a(p,\omega)=-\frac{m_a}{Z_a e p^2}\sum_b \int \frac{d^3 p_1}{(2\pi\hbar)^3}\frac{Z_f e p_\alpha p_{1\alpha}}{m_b} W_{ba}(\mathbf{p}_1,\mathbf{p},\omega)\;. \tag{18}$$

Using (14), (17), and (18), we obtain

$$\nu_a(p,\omega)=i\omega\langle\langle \hat{N}_{ps}^a | p_\alpha \hat{I}^\alpha \rangle\rangle_\omega \left\{ -Z_a e p \frac{\partial f_a^{(0)}(p)}{\partial p} + \langle\langle \hat{N}_{ps}^a | p_\beta \hat{I}^\beta \rangle\rangle_\omega \right\}^{-1}\;. \tag{19}$$

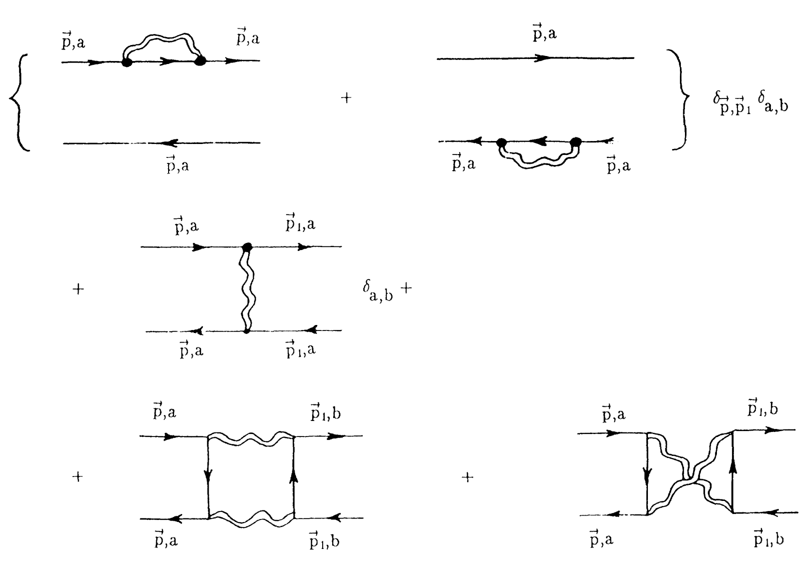

FIG. 1. Diagram expansion of the correlation function $\langle\langle N_p^a | N_{p'}^b \rangle\rangle_{i\Omega_n}$, which leads to a generalized Balescu-Lenard equation. The wavy lines are the screened potentials; the straight lines are the Green's functions.

Hence, for the OPDF $f_a^{(1)}(p,k\to 0,\omega)$, the following exact relation is valid [9,10]:

$$-i\omega f_a^{(1)}(\mathbf{p},k\to 0,\omega)+Z_a e\frac{\partial f_a^{(0)}(p)}{\partial p_\alpha}E_\alpha(k\to 0,\omega) = -\nu_a(p,\omega)f_a^{(1)}(\mathbf{p},k\to 0,\omega) , \quad (20)$$

where in the limit case of weak interaction we have $\nu_a \sim \lambda^2$ and

$$\lim_{\omega\to 0}\nu_a(p,\omega)=\nu_a(p)<\infty . \quad (21)$$

Within the framework of the usual KE method, the KE takes the form (20) only in the case of a Lorentz plasma [cf. (8)] and then only the static approximation (21) for the collision frequency $\nu_a(p,\omega)$ is used [see (9)–(12)].

The nonintegral form (20) of the exact KE and the integral form (1) with (14) lead, in weak interaction approximation, as shown in [15], to different results in the static limit $\omega\to 0$, whereas both coincide with results of the KE method at high frequencies [7,19]. The last statement is true for all frequencies, if e-e interactions are formally omitted. Next, taking into account (21), we find from (19) for the limit case of low frequencies ω:

$$\langle\langle \hat{N}^a_{\mathbf{p}s} | p_\alpha \hat{I}^\alpha \rangle\rangle_\omega \simeq Z_a e p \frac{\partial f_a^{(0)}}{\partial p}\left\{1+\frac{i\omega}{\nu_a(p)}+\cdots\right\} . \quad (22)$$

Substituting (22) into (14), we obtain

$$W_{ab}(\mathbf{p},\mathbf{p}_1)=\lim_{\omega\to 0}W_{ab}(\mathbf{p},\mathbf{p}_1,\omega) = -\nu_b(p_1)\left\{\frac{\partial f_b^{(0)}(p_1)}{\partial \varepsilon^b_{p_1}}\right\}^{-1} \times \lim_{\omega\to 0}V\langle\langle \hat{N}^a_{\mathbf{p}s}|\hat{N}^b_{\mathbf{p}_1 s_1}\rangle\rangle_\omega . \quad (23)$$

Hence, to determine the value of $W_{ab}(\mathbf{p},\mathbf{p}_1)$ up to order λ^2, it is sufficient to calculate the limit

$$\lim_{\omega\to 0}V\langle\langle \hat{N}^a_{\mathbf{p}s}|\hat{N}^b_{\mathbf{p}_1 s_1}\rangle\rangle_\omega$$

in the lowest approximation with respect to the interaction between particles, which can be done easily as follows [13]:

$$\lim_{\omega\to 0}V\langle\langle \hat{N}^a_{\mathbf{p}s}|\hat{N}^b_{\mathbf{p}_1 s_1}\rangle\rangle^{(0)}_\omega =(2\pi\hbar)^3\frac{\partial f_a^{(0)}(p)}{\partial \varepsilon^a_p}\delta_{a,b}\delta_{s,s_1}\delta(\mathbf{p}-\mathbf{p}_1) . \quad (24)$$

As a result, in the limit of weak interaction the functions $W_{ab}(\mathbf{p},\mathbf{p}_1)$ must have the following structure:

$$W_{ab}(\mathbf{p},\mathbf{p}_1)=-(2\pi\hbar)^3\nu_a(p)\delta_{a,b}\delta_{s,s_1}\delta(\mathbf{p}-\mathbf{p}_1) . \quad (25)$$

But this structure does not follow from the KE method; even the Lorentz model has a different structure.

V. CONCLUSIONS

Relation (25) means that for the calculation of kernels W_{ab} from the perturbation theory, one should not restrict oneself to the first term (16) of series (15), which leads to the usual results of kinetic theory including the Balescu-Lenard KE. It is necessary to carry out a partial summation of the divergent terms of series (15) in the limit $\omega\to 0$. Then a contribution of the same order with respect to the integration between particles as the first term (16) of the series (15) must appear. From this point of view, the usual results of the kinetic theory comply with the calculation of W_{ab} at the high-frequency limit: $\omega \gg \bar{\nu}$ only.

At the same time, there is the possibility of using the first term (16) of the series with respect to parameter $\kappa\sim\bar{\nu}/\omega$ for the calculation of the collision frequency $\nu_a(p,\omega)$ from (19) for a weakly nonideal plasma. In any case, the results of such calculations [9] do not contradict the results described above and expressed by (25).

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