Modifying the onset of homoclinic chaos: Application to a bistable potential

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We analyze, by means of the Melnikov method, the possibility of modifying the threshold of homoclinic chaos in general one-dimensional problems, by introducing small periodic resonant modulations. We indicate in particular a prescription in order to increase the threshold (i.e., to prevent chaos), and consider then its application to the bistable Duffing-Holmes potential. All results are confirmed both by numerical and by analog simulations, showing that small modulations can in fact sensibly influence the onset of chaos.

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The problem of "controlling chaos" has received great attention in recent years [1–4]. Even if, for many authors, this term means in general the stabilization of unstable orbits, we are concerned here more specifically with the modification of the threshold for the onset of chaos. One of the simplest and most interesting methods used in order to modify (possibly to increase) the threshold of chaos, which appears in the presence of homoclinic orbits, is the introduction of a periodic modulation of the parameters describing the unperturbed potential. This possibility has been analyzed for both the Duffing-Holmes and the Josephson-junction potential [2,3].

In the present Brief Report, we want to propose some generalizations of this idea: we will consider first a generic one-dimensional problem by means of Melnikov theory [5,6], and obtain a simple criterion about the "correct" choice of this modulation; next, we will apply this indication to the bistable Duffing-Holmes potential, introducing *two* independent modulations: this will allow us to confirm both the general results and the possibility of modifying the onset of chaos. All the theoretical discussion is well supported both by numerical and by analog simulations.

Let us start by considering the general case of a one-dimensional "equation of motion" for the real variable $x = x(t)$:

$$\ddot{x} = f(x) - \delta\, \dot{x} + \gamma \cos\omega t + \epsilon\, g(x)\cos(\Omega t + \theta), \qquad (1)$$

where for $\epsilon = 0$ we have the standard problem of a periodically forced and linearly damped motion, whereas the additional term with $\epsilon \neq 0$ takes into account the presence of general modulating terms. We assume as usual $f = -dV/dx$, where the unperturbed potential $V = V(x)$ has a maximum point at $x = x_0$ and a homoclinic orbit, which we indicate by

$$q = q(t) \qquad (2)$$

doubly asymptotic to x_0 for $t \to \pm\infty$. In order to obtain a theoretical estimate of the effect of the last term of Eq. (1) to the threshold of chaos, let us write down the Melnikov function $M(t_0)$ for the problem (1): taking into account that $q(t) = q(-t)$, it is easily seen that $M(t_0)$ acquires the form

$$\begin{aligned} -M(t_0) &= \delta \int_{-\infty}^{+\infty} \dot{q}^2(t)\, dt + \gamma \sin\omega t_0 \int_{-\infty}^{+\infty} \dot{q}(t) \sin\omega t\, dt \\ &\quad + \epsilon \sin(\Omega t_0 + \theta) \int_{-\infty}^{+\infty} \dot{q}(t)\, g(q(t)) \sin\Omega t\, dt \\ &\equiv \delta J_\delta + \gamma J_\gamma \sin\omega t_0 + \epsilon J_\epsilon \sin(\Omega t_0 + \theta). \end{aligned} \qquad (3)$$

At the resonant case, i.e., when $\omega = \Omega$, this can be written

$$-M(t_0) = \delta J_\delta + \gamma K \sin(\omega t_0 + \alpha) \qquad (4)$$

with

$$K = \left| J_\gamma^2 + \left(\frac{\epsilon J_\epsilon}{\gamma}\right)^2 + 2\left(\frac{\epsilon}{\gamma}\right) J_\gamma J_\epsilon \cos\theta \right|^{1/2}. \qquad (4')$$

The condition for the appearance of chaos according to Melnikov criterion, i.e., that $M(t_0)$ has simple zeros, becomes now (it is not restrictive to assume $\gamma > 0$, whereas $\delta > 0$ for physical reasons, and clearly $J_\delta > 0$)

$$\gamma K > \delta J_\delta. \qquad (5)$$

With fixed damping δ, we can then use the modulation term in order to modify the threshold of chaos: here, this amounts to modifying the range of the forcing amplitudes γ which do not produce chaotic responses. Now, according to Eqs. (4)–(5), the maximum increase of this threshold is obtained by choosing $\theta = 0$ and the sign of ϵ according to the following prescription:

$$\begin{aligned} &\epsilon > 0 \quad \text{if } J_\gamma J_\epsilon < 0, \\ &\epsilon < 0 \quad \text{if } J_\gamma J_\epsilon > 0 \end{aligned} \qquad (6)$$

(or, which is the same, $\epsilon > 0$ in any case, $\theta = 0$ if $J_\gamma J_\epsilon < 0$, and $\theta = \pi$ if $J_\gamma J_\epsilon > 0$).

With this choice for ϵ, and observing that the amplitude $|\epsilon|$ of the modulation is usually very small (here we only require $|\epsilon| < \gamma |J_\gamma / J_\epsilon|$), the above condition (5) becomes

$$\gamma > \frac{\delta J_\delta}{|J_\gamma|} + \left| \epsilon \frac{J_\epsilon}{J_\gamma} \right| \qquad (7)$$

or also

$$\frac{\gamma}{\delta} > R(\epsilon) = R_0 + |\epsilon| \frac{|J_\epsilon|}{\delta |J_\gamma|}, \tag{8}$$

where

$$R_0 = \frac{J_\delta}{|J_\gamma|} \tag{8'}$$

is the "Melnikov ratio" in the case $\epsilon = 0$ of no modulation.

Then Melnikov theory predicts that the presence of modulation produces, if the phase of modulation is chosen according to the above rule, an increase (proportional to the quantity $|J_\epsilon / J_\gamma|$) of the threshold of chaos.

Before further discussing this result, let us remark incidentally that in the case of nonresonant modulations, $\omega \neq \Omega$, Eq. (3) suggests that one may expect [at most after some time delay of the order $\sim 1/(\omega - \Omega)$] an "in-phase" contribution of both forcing and modulating terms $\sim \gamma |J_\gamma| + |\epsilon J_\epsilon|$. Then, the above arguments indicate that a lowering of the threshold, favoring the onset of chaos, is to be expected in this case. A careful analysis of nonresonant modulations in the case of the Duffing-Holmes potential can be found in Ref. [2].

We now want to provide a precise test of the above results by checking them in the case of the Duffing-Holmes potential. We find it convenient to introduce a small generalization of the above discussion, by choosing the perturbation $g(x)$ in (1) in the form of *two* independent terms modulating *both* the linear and the cubic components of the force: this requires the presence of *two* modulation parameters ϵ_1, ϵ_3, as made in [4], where a Duffing-Holmes-type potential was obtained by means of magnetoelastic equipment. Precisely, we consider the equation

$$\begin{aligned} \ddot{x} = & -Ax^3(1 + \epsilon_3 \cos \Omega_3 t) + Bx(1 + \epsilon_1 \cos \Omega_1 t) \\ & - \delta\, \dot{x} + \gamma \cos \omega t, \end{aligned} \tag{9}$$

where $A, B, \gamma, \delta > 0$, and the signs of ϵ_1, ϵ_3 are for the moment undetermined. As is well known, the unperturbed Duffing-Holmes potential possesses two homoclinic orbits, given by

$$q^{(\pm)} = \pm \left(\frac{2B}{A}\right)^{1/2} \operatorname{sech} t\sqrt{B}, \tag{10}$$

where the signs + and − denote, respectively, the orbit surrounding the minimum of the potential at $x^{(+)} = +\sqrt{B/A}$ and at $x^{(-)} = -\sqrt{B/A}$. All integrals appearing in the Melnikov function can be evaluated exactly; in particular, the integrals $J_{\epsilon_1}^{(+)}$ and $J_{\epsilon_1}^{(-)}$ take the same value when evaluated, respectively, along the orbit $q^{(+)}(t)$ and along $q^{(-)}(t)$, the same is true for $J_{\epsilon_3}^{(+)}$ and $J_{\epsilon_3}^{(-)}$: We have

$$\begin{aligned} J_{\epsilon_1}^{(\pm)} &= -2\frac{B^{3/2}}{A} \int_{-\infty}^{+\infty} \sinh t\sqrt{B} \operatorname{sech}^3 t\sqrt{B} \sin \Omega_1 t \, dt \\ &= -\frac{\pi B \Omega_1^2}{A} \operatorname{csch} \frac{\pi \Omega_1}{2\sqrt{B}} < 0, \end{aligned} \tag{11}$$

$$\begin{aligned} J_{\epsilon_3}^{(\pm)} &= 4\frac{B^{5/2}}{A} \int_{-\infty}^{+\infty} \sinh t\sqrt{B} \operatorname{sech}^5 t\sqrt{B} \sin \Omega_3 t \, dt \\ &= \frac{\pi \Omega_3^2}{6A}(\Omega_3^2 + 4B) \operatorname{csch} \frac{\pi \Omega_3}{2\sqrt{B}} > 0 \end{aligned} \tag{11'}$$

[notice that the results in Eq. (9) of Ref. [2] and Eq. (4) of Ref. [4] are incorrect]. One also has $J_\delta^{(+)} = J_\delta^{(-)}$, whereas $J_\gamma^{(+)} = -J_\gamma^{(-)}$:

$$J_\delta^{(\pm)} = \frac{4}{3}\frac{B^{3/2}}{A} > 0, \quad J_\gamma^{(+)} = -\pi\omega \sqrt{\frac{2}{A}} \operatorname{sech} \frac{\pi\omega}{2\sqrt{B}} < 0. \tag{12}$$

Thus, the Melnikov condition for the appearance of chaos, in the resonant case $\omega = \Omega_1 = \Omega_3$, can be finally written in the form

$$\frac{\gamma}{\delta} > R^{(\pm)}(\epsilon_1, \epsilon_3) = R_0 \mp \epsilon_1 r_1 \pm \epsilon_3 r_3, \tag{13}$$

where again ± distinguishes between the two homoclinic orbits $q^{(\pm)}(t)$, and

$$R_0 = \frac{2\sqrt{2} B^{3/2}}{3\sqrt{A}\pi\omega} \cosh \frac{\pi\omega}{2\sqrt{B}} \tag{14}$$

is the well-known ratio for the unperturbed Duffing-Holmes potential [6], and

$$r_1 = \frac{\omega B}{\delta\sqrt{2A}} \coth \frac{\pi\omega}{2\sqrt{B}}, \quad r_3 = \frac{\omega(\omega^2 + 4B)}{6\delta\sqrt{2A}} \coth \frac{\pi\omega}{2\sqrt{B}}. \tag{14'}$$

Therefore, we can conclude: If the motion occurs near the homoclinic orbit $q^{(+)}(t)$, the best choice in order to prevent chaos (by increasing the threshold) is

$$\epsilon_1 < 0 \quad \text{and} \quad \epsilon_3 > 0. \tag{15}$$

This choice, however, favors the onset of chaos when the motion is in the potential well around $x^{(-)} = -\sqrt{B/A}$. The opposite choice $\epsilon_1 > 0$, $\epsilon_3 < 0$ would produce exactly opposite effects. All these results agree with our above discussion (cf. the signs of $J_\gamma^{(\pm)}$ and $J_{\epsilon_1}^{(\pm)}, J_{\epsilon_3}^{(\pm)}$).

Numerically, with, e.g., $A = B = \omega = 1$, $\delta = 0.25$, we obtain

$$r_1 = 3.08\,, \qquad r_3 = 2.57. \tag{16}$$

The relatively large numerical values of these coefficients (compared with $R_0 = 0.753$) show that the role of modulation in the Duffing-Holmes potential is really important: according to Eqs. (13) and (16), one may in fact expect that very small ϵ_1, ϵ_3 may considerably influence the onset of chaos. Another interesting remark is that the introduction of the modulation (clearly the effect is present also when choosing one of the two ϵ_1, ϵ_3 equal to zero) produces a sort of "dynamical asymmetry" between the two potential wells. Let us emphasize that it has already been noticed [7] that a small "geometrical" asymmetry in the double-well potential sensibly modi-

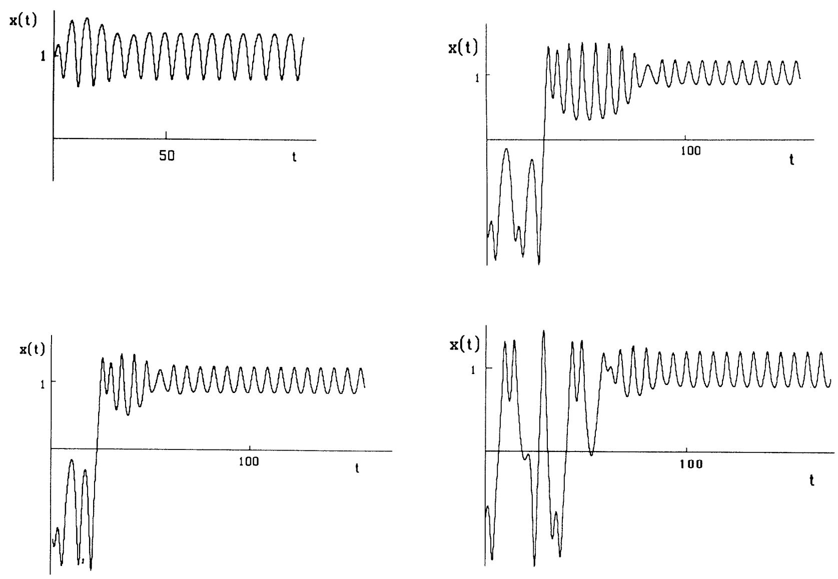

FIG. 1. Numerical solution of Eq. (9) with different initial conditions, and $A = B = \omega = 1$; $\delta = \gamma = 0.2$. The modulation terms $\epsilon_1 = -0.05$, $\epsilon_3 = +0.04$ are chosen according to the prescription (15) in order to remove chaos: indeed, after some short transient, the solution is periodic.

fies the thresholds of chaos in the two wells. Precisely, considering an asymmetric potential

$$V(x) = \frac{A}{4}x^4 - \frac{B}{2}x^2 + \beta x, \tag{17}$$

it can be shown that the Melnikov ratios $R^{(\pm)}(\beta)$, at the first order in the asymmetry parameter β, are given by

$$R^{(\pm)}(\beta) = R_0 \pm \beta\rho, \tag{18}$$

where

$$\rho = R_0 \frac{\sqrt{2A}}{B^{3/2}}\left(\frac{3\pi}{4} - \frac{\omega}{\sqrt{B}}\coth\frac{\pi\omega}{2\sqrt{B}}\right). \tag{19}$$

With $A = B = \omega = 1$ as before, one has $\rho = 1.35$, showing that the two effects are in fact comparable.

We have tested the above theoretical results for the Duffing-Holmes potential by means of both numerical and analog simulations. The agreement is globally good enough. For instance, choosing $A = B = \omega = 1$, and $\delta = 0.2, \gamma = 0.2$ (i.e., largely within the chaotic region if no modulation would be present, cf. [6]), it is sufficient to insert a modulation with $\epsilon_1 = -0.05, \epsilon_3 = +0.04$ in order to obtain a periodic response. Figure 1 shows some of these periodic motions, oscillating around $x^{(+)} = 1$, obtained by numerical integration, starting from different initial conditions. Analog simulation is another very convenient and known method (see [9] and references therein, and [8,10,3]) to examine nonlinear systems. The analog device we use in this case is essentially similar to others already used and discussed elsewhere (see especially [9,3]). In particular, the modulation is obtained by means of multipliers operating in the reaction loop, in a similar way to the case of the modulated Josephson-junction device, discussed in detail in [3]. The values of the parameters are obtained by direct measurements on the experimental circuit. In particular, the damping term δ is deduced from the resonance bandwidth at the limit of small amplitudes; therefore, this measure is actually subject to some uncertainties: We obtain $\delta = 0.25$ with an estimated error of $\pm 20\%$. The other parameters (in dimensionless units, as usual) are $B = 1$, $1/\sqrt{A} = 2.83$, $\omega = 1.42 \simeq \sqrt{2B}$ (i.e., the frequency of the small oscillations around the equilibrium points $x^{\pm}$ of the unperturbed potential). Then, by increasing the forcing amplitude γ, we look for the threshold values of γ which produce the appearance of chaotic responses, for different values of the amplitudes ϵ_1, ϵ_3 of the modulating perturbation. Figure 2 shows the values we obtain for the ratio $R = \gamma/\delta$ (where γ are these threshold values), as a function of ϵ_1, with $\epsilon_3 = 0$. In agreement with Eq. (13), we obtain a completely similar behavior with $\epsilon_1 = 0$ and varying ϵ_3. From our measurements, we get the following results:

$$R_0 = 4.3\ , \qquad r_1 = 10.1\ , \qquad r_3 = 11.2, \tag{20}$$

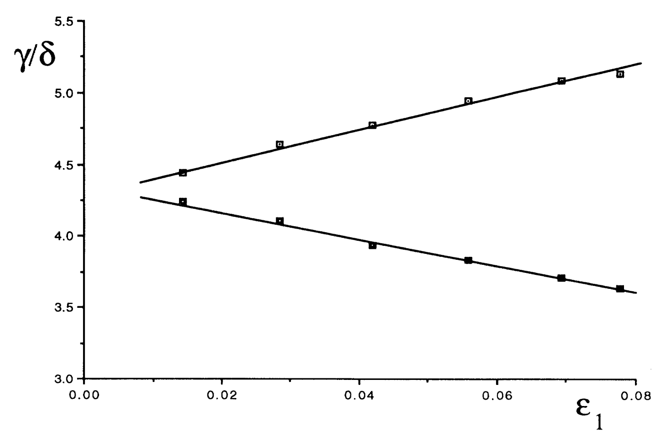

FIG. 2. Threshold of chaos in the bistable potential [Eq. (9)] as a function of modulating perturbation. Here, γ is the experimental value (via analog simulation) of the threshold; the ratio $R = \gamma/\delta$ is plotted vs the amplitude ϵ_1 of modulation, with $\epsilon_3 = 0$. The results with $\epsilon_1 = 0$ and varying ϵ_3 are completely similar, see Eq. (13).

to be compared with the theoretical values deduced from Eqs. (14) and (14′)

$$R_0 = 2.82\ , \qquad r_1 = 11.63\ , \qquad r_3 = 11.66. \qquad (21)$$

The numerical agreement is not perfect; let us stress, however, that—apart from the uncertainties and unavoidable errors in the experimental determinations of the various quantities (see [3] for some short comments on this point)—in any case one cannot hope that the Melnikov method is able to give a precise determination of the threshold of chaos; rather, a common and expected result is actually that the Melnikov theory indicates a somewhat smaller value than the threshold experimentally detected (see [6,3]). Let us point out, on the other hand, the better agreement we find for the values of r_1, r_3, and in particular the result $r_1 \simeq r_3$ [according to Eq. (14′), $r_1 = r_3$ for $\omega = \sqrt{2B}$], and finally the agreement shown by Fig. 2 with Eq. (13). With these observations, we believe that all the facts discussed up to now may be considered globally as a rather good test for both theory and analog experiments.

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