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Simple stochastic model for resonant activation

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We present three variants of an extremely simple stochastic model, one of which does not display a "resonant phenomenon" discussed recently [Ch. Doering and C. Gadoua, Phys. Rev. Lett. **69**, 2318 (1992)], while the other two variants do.

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I. INTRODUCTION

The phenomenon of stochastic resonance, whereby the effect of a small systematic periodic force can be amplified by noise in a nonlinear system, has attracted a lot of attention recently [1]. Another resonance phenomenon called resonant activation was reported in [3]. In this case, the mean first-passage time (MFPT) was calculated for the thermal escape out of a linear potential well, whose slope switches at random between two (high- and low-barrier) values. It was found that the MFPT goes though a minimum when the switching time is of the order of the escape time over the lowest barrier. The purpose of this paper is to present an extremely simple model which, under appropriate conditions, gives rise to a similar phenomenon. The advantage of such a model is that the calculations are technically easy, and can serve as a point of departure to answer more detailed questions about this type of resonance phenomenon.

II. VARIANT 1

A particle randomly switches at a rate γ between two "internal" states, called + and −, respectively. On the other hand, while being in the + or − state, the particle "exits" or "decays" at corresponding rates k_+ and k_-. Note that such a Markovian exit mechanism correctly represents the escape of a thermally excited particle over a high potential barrier. The question we are asking is how the MFPT to exit or decay is influenced by the switching mechanism. In particular, does there exist a value of the switching rate γ for which the MFPT is minimal?

A simple calculation shows that the survival probability $P(t)$, defined as the probability that the particle has not exited at time t, obeys the following equation:

$$\partial_t^2 P(t) = -(k_+ + k_- + 2\gamma)\partial_t P(t) - [\gamma(k_+ + k_-) + k_+ k_-]P(t) \ . \quad (1)$$

Assuming that both + and − states are equally probable at the initial time $t=0$, one finds for the average exit time or MFPT,

$$\langle \tau \rangle = \int_0^\infty d\tau\, P(\tau) = \frac{(k_+ + k_-)/2 + 2\gamma}{k_+ k_- + (k_+ + k_-)\gamma} \ . \quad (2)$$

We conclude that the MFPT decreases in a monotonic way from the maximum value $(1/k_+ + 1/k_-)/2$ for $\gamma=0$ to the minimum value $2/(k_+ + k_-)$ for $\gamma=\infty$. In [4], it is proven that this monotonic behavior and an analogous form of the two limiting values ($\langle 1/k \rangle$ and $1/\langle k \rangle$, respectively) hold for any number of internal states, provided the random walk between these states and the exit mechanism are all Markovian (i.e., characterized by exponential probability densities), and as long as the initial probability distribution is equal to the steady state that is reached when the decay or exit mechanism is switched off. We conclude that the resonance phenomenon does not occur in this type of models (for those specific but natural initial conditions).

III. VARIANT 2

We now turn to a non-Markovian variant of the above model and show that a resonance phenomenon becomes possible in this case. A particle is again switching between the + and − states at random time points and at a rate γ. On the other hand, the probability densities for exiting the + or − state after a residence time equal to t are now given by $\psi_+(t)$ and $\psi_-(t)$, respectively. The results of the preceding section are recovered in the particular case that these densities are exponential. We also assume that a new exit-time period for $\psi_+(t)$ and $\psi_-(t)$ sets in after each jump (i.e., the time used to calculate the exit probabilities is reset to zero). A simple calculation shows that the Laplace transform $\tilde{P}(s)$ of the survival probability $P(t)$ is given in terms of the Laplace transforms of the exit-time densities as follows (assuming that the initial probability to be in the + or − state is equal to $\frac{1}{2}$):

$$\tilde{P}(s) = \frac{2 + \tilde{\psi}_+(s+\gamma)\tilde{\psi}_-(s+\gamma) - \frac{3}{2}[\tilde{\psi}_+(s+\gamma) + \tilde{\psi}_-(s+\gamma)]}{\gamma[\tilde{\psi}_+(s+\gamma) + \tilde{\psi}_-(s+\gamma) - \tilde{\psi}_+(s+\gamma)\tilde{\psi}_-(s+\gamma)]} \ . \quad (3)$$

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The value of the mean exit time is found as $\langle \tau \rangle = \tilde{P}(s=0)$. To illustrate the possibility of a resonance phenomenon similar to the one discussed in [3], we consider the particular case $\psi_+(t)=\delta(t-k_+^{-1})$ and $\psi_-(t)=\delta(t-k_-^{-1})$. The following result for the mean exit time $\langle \tau \rangle$ is obtained:

$$\langle \tau \rangle = \frac{1+2e^{\gamma(1/k_+ + 1/k_-)} - 3/2(e^{\gamma/k_+} + e^{\gamma/k_-})}{\gamma(e^{\gamma/k_+} + e^{\gamma/k_-} - 1)} . \tag{4}$$

$\langle \tau \rangle$ diverges for $\gamma \to \infty$ since no exits take place for visits that last less than $\min(k_+^{-1}, k_-^{-1})$. On the other hand, $\langle \tau \rangle$ converges to $(k_+^{-1}+k_-^{-1})/2$ for $\gamma \to 0$, as expected. Moreover, a phenomenon of stochastic resonance occurs when the rates k_+ and k_- are sufficiently different. More precisely, $\langle \tau \rangle$ goes through a minimum in function of γ when the rates k_+ and k_- differ by a factor of more than ≈ 3.75. For the special case that $k_+ \ll k_-$, one finds that the minimum is attained for $\gamma \approx 0.58 k_-$ and is given by $\langle \tau \rangle \approx 3.6 k_-^{-1}$. The MFPT is thus of the order of the smallest of the two residence times.

IV. VARIANT 3

The purpose of this variant is to show that a minimum of the MFPT in function of the switching rate can also arise when the exit-time density is the same for the + and − state $\psi_+(t)=\psi_-(t)=\psi(t)$. Equation (3) simplifies as follows:

$$\tilde{P}(s) = \frac{1-\tilde{\psi}(s+\gamma)}{\gamma \tilde{\psi}(s+\gamma)} \tag{5}$$

and the value of the mean exit time is again obtained from $\langle \tau \rangle = \tilde{P}(s=0)$. The behavior of $\langle \tau \rangle$ in function of γ depends on the form of $\psi(t)$. For example, when $\psi(t)$ is an Erlang distribution,

$$\psi(t) = \frac{(nk)^n t^{n-1} e^{-nkt}}{(n-1)!} , \tag{6}$$

one finds

$$\langle \tau \rangle = \frac{1}{\gamma} \left[\left[1 + \frac{\gamma}{nk} \right]^n - 1 \right] . \tag{7}$$

In this case, the MFPT increases (for $n > 1$) with the switching rate γ, in contrast with the result obtained by the previously discussed variant 1. The origin of this behavior is clear: the probability for exit per unit time is smaller for short visits, while the duration of these visits does indeed become shorter as the switching rate increases. A more interesting situation arises when considering the following density which leads to fractal time properties with fractal dimension $\frac{1}{2}$,

$$\psi(t) = \frac{e^{-1/t}}{\pi^{1/2} t^{3/2}} . \tag{8}$$

Note that the first moment of this density is equal to infinity. However, the introduction of a random switching between two states characterized by this same density leads to an exit distribution with finite average exit time. The Laplace transform of $\psi(t)$ is given in [5], and one finds the following result from Eq. (5):

$$\langle \tau \rangle = \frac{e^{2\sqrt{\gamma}} - 1}{\gamma} . \tag{9}$$

This exit or MFPT diverges for both $\gamma \to 0$ [because of the fractal nature of exit times generated by the density $\psi(t)$] and $\gamma \to \infty$ (because exits are very unlikely during short visits) and attains a minimum at $\gamma \approx 0.635$.

V. DISCUSSION

Variant 1 of our model shows that the phenomenon reported in [3] does not occur if the states between which the switching takes place have an exponentially distributed first-passage-time density. This is in agreement with the observation that the resonance disappears in the low-temperature limit (whenever a positive barrier has to be crossed in both states). On the other hand, the coupling of states with nonexponential first-passage-time properties can lead to a wide variety of behavior, as is illustrated by variants 2 and 3 of our model. In particular we have proven the existence of a stochastic resonance phenomenon similar to the one discussed in [3]. We have not tried to give a physical meaning to the model that was discussed here. However, our model is closely related to the so-called continuous-time random walks, and the latter have been applied successfully to the description of a wide variety of physical problems [2]. For this reason, we believe that the stochastic resonance phenomenon presented here is not solely of academic interest, but could be observed in some of these situations.

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