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Signatures of quantum chaos in Wigner and Husimi representations

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In this paper, we study the quantum manifestations of classical chaos in phase space using Wigner and Husimi distribution functions. We test the claim that Husimi represents the correspondence better than Wigner does. The results show the claim is valid. We also use a quantum dissipation scheme empirically for classically damped motions often characterized by strange attractors. We believe quantum resemblance to classical distributions can be regarded as signatures of quantum chaos in phase space.

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With the lack of a universal definition of quantum chaos, a systematic measure of disorder in quantum systems is still open for further exploration. In fact, by employing the definition of classical chaos for quantum mechanics, never has its existence been reported [1] except in the semiclassical limit. More research in the context of the correspondence principle therefore seems to be necessary at present to detect any sign of quantum manifestations of classical chaos. It has been shown that uncertainty could be the measure [2,3], but generally its calculations are not simple to carry out since it requires expectation values of classical observables such as momentum. The calculation of a wave function from the time-dependent Schrödinger equation, however, is relatively easily accomplishable using well-known techniques. The dynamical information of this wave function can then be used to investigate quantum behavior in phase space.

To see the classical-like phase space, however, this wave function should be put into quantum distribution functions, such as the Wigner or Husimi functions, to identify noncommuting variables simultaneously. Studies containing the formulation of quantum mechanics in terms of both Wigner [4–7] and Husimi functions [8,9] have been carried out to some extent for various problems. These phase-space representations of quantum mechanics have been very useful to study quantum manifestations of classical chaos in recent years [10–12]. However, the Husimi function has been applied somewhat slower and has attracted less attention than the Wigner function. The immediate advantage of the Husimi function is the non-negativeness of the distribution so that its interpretation as a probability distribution in phase space is more convincing than for the Wigner distribution.

In this paper, we will study quantum resemblance of classical chaos in phase space using the Husimi distribution function as well as the Wigner function. This resemblance can be regarded as a signature of quantum chaos. A qualitative measure of the correspondence can be estimated by the degree of the resemblance. Our analysis is based on Schrödinger quantum mechanics, whereas some of the previous studies [10–12] involved the Heisenberg picture. We shall also use the quantum-mechanical scheme of the classical-like damping empirically [13] without exploring its details. Our model is a driven damped pendulum which is simple, but richly chaotic [14,15]. The main objective of this study is to focus on the quantum dynamics represented by the distribution functions in the classically chaotic regime.

To find the distribution functions, we need to solve the time-dependent Schrödinger equation. Then the solution is inserted into the appropriate form of the distribution functions that specify quantum conjugate variables (i.e., position and momentum) simultaneously. These phase-space distributions are directly compared with a classical distribution. A classical distribution simply represents a result of the evolved initial distribution that portrays many different initial conditions. An initial classical distribution should be taken according to the initial quantum probability. Then signatures of quantum chaos can be determined qualitatively from these distributions. We discuss the model briefly followed by the results of our calculations.

The time-dependent Hamiltonian, $\hat{H}_{\rm qm}=\hat{H}_{\rm qm}(\hat{p},\hat{q},t)$,

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for the driven pendulum of mass m and length l under the influence of the earth's gravity g is

$$\hat{H}_{\rm qm}=\frac{1}{2ml^2}[\hat{p}-\hat{A}]^2+\hat{V}\ , \tag{1}$$

where $\hat{A}=-ml^2f\sin(\omega t)/\omega$ and $\hat{V}=mgl(1-\cos\hat{q})$. The canonical conjugate variables of angular momentum and angular position are specified as p and q, respectively. The externally driving field amplitude and frequency are f and ω, respectively. An equivalent classical Hamiltonian is

$$H_{\rm cl}=\frac{p^2}{2ml^2}+mgl(1-\cos q)-ml^2fq\cos(\omega t)\ . \tag{2}$$

We recover the classical counterpart (2) from Eq. (1) after a gauge transformation with the following gauges [3]:

$$\hat{A}\to\hat{A}'=\hat{A}+\frac{\partial\hat{\chi}}{\partial\hat{q}}\ ,\quad \hat{V}\to\hat{V}'=\hat{V}-\frac{\partial\hat{\chi}}{\partial t}\ , \tag{3}$$

where $\hat{\chi}=ml^2f\hat{q}\sin(\omega t)/\omega$. These gauges are necessary to preserve the periodic nature of the potential. This gauge transformation is also beneficial to avoid a numerical problem at the periodic boundary for the Fourier transforms we use.

The time-dependent Schrödinger equation for the model in dimensionless form can be written as

$$i\frac{\partial\psi}{\partial\tilde{t}}=\frac{1}{2\mu}\left[-i\frac{\partial}{\partial q}+\mu\gamma\frac{\sin(\tilde{\omega}\tilde{t})}{\tilde{\omega}}\right]^2(1-i\beta)\psi +\mu(1-\cos q)\psi\ , \tag{4}$$

where $\hat{p}=-i\partial_q$, $\tilde{\omega}=\omega\sqrt{l/g}$, $\tilde{t}=t\sqrt{g/l}$, $\gamma=fl/g$, and $\mu=ml\sqrt{gl}/\hbar$. We will not use the sign $\sim$ hereafter. The inverse of μ is analogous to the usual semiclassical limit $\hbar\to 0$. This is evident from the expression μ for fixed values of the pendulum mass and length. Some words are in order concerning the part responsible for the damping in Eq. (4). The energy dissipation of a classical motion in a viscous medium is phenomenologically described by the Rayleigh term $\beta m\dot{q}^2/2$ [21]. This extra dissipating kinetic energy is artificially put into the Schrödinger equation via the term $(1-i\beta)$ in Eq. (4). It is not our intention to consider a detailed study and review existing research on this subject in this Brief Report. Empirically, however, this treatment provides valid results, specifically in the context of phase-space trajectory in several systems including the damped harmonic oscillator [13].

Now the solution $\psi(q,t)$ of Eq. (4) is obtained by using the split-operator method [16,17]. The initial Gaussian wave function is chosen because of its minimum uncertainty. Then the genuine solution corresponding to the Hamiltonian (2) becomes $\psi'(q,t)=\exp(i\hat{\chi})\psi(q,t)$ because of the gauge transformation. This solution is utilized to form the Wigner and Husimi distribution functions. The Wigner function is the Fourier transform of the spatial correlation function of $\psi'(q,t)$ [4–7]. The Husimi function can be obtained by a Gaussian smoothing method [18,19] and is given by the amplitude squared of the projection of ψ' onto the coherent state [8,9] whose uncertainty is minimum. As a final note, the classical distribution is taken according to the quantum probability using the Monte Carlo method.

TABLE I. Initial parameter values used in the calculations.

Case	x_0	p_0	γ	ω	β	μ	Δt
Fig. 1	0.5	0.0	1.25	4/3	0.05	10	0.0025
Fig. 2	0.5	0.0	1.5	2/3	0.25	10	0.0025
Fig. 3	0.0	−0.35	3.5	1.05	0.1	20	0.00125

Parameter values used in the calculations are listed in Table I. We use the same uncertainty value in the coherent state of the Husimi function as the one in the initial Gaussian wave function. Apparently all the figures have a periodic boundary in the position coordinate q at $\pm\pi$. Note that we use the quantity $[p-A(t)]/\mu$ for the vertical axes to compare with the classical velocities computed from the classical equations of motion using (2) by the fourth-order Runge-Kutta method. It should also be noted that contour plots of the Wigner distribution function are taken from the norm of the Wigner function to avoid possible negative values.

Figure 1 shows classical distributions, the contours of Husimi and Wigner distributions at two different times $2T$, $8T$, where $T=2\pi/\omega$. Distributions at $t=2T$ and $8T$ are shown in the left column and in the right column, re-

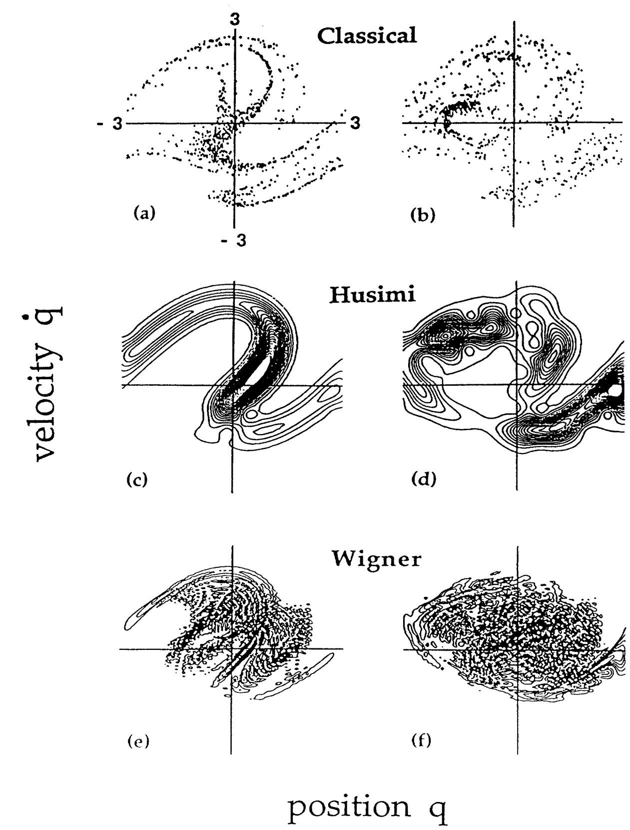

FIG. 1. Classical distributions are shown in (a), (b) and contour plots of the Husimi distributions in (c), (d). The Wigner distributions are shown in (e), (f). The pictures in the left column are picked at $t=2T$, whereas the ones in the right at $t=8T$.

spectively. The classical distributions are taken from 2000 points. It is easy to see in overall structures that Husimi distributions in Figs. 1(c) and 1(d) are closer to the classical ones in 1(a) and 1(b) than Wigner distributions in 1(e) and 1(f), respectively. This clearly supports the previous findings [10–12] that the Husimi distribution is a better representation of the correspondence. It is also interesting to observe that the Husimi distribution holds the correspondence somewhat longer than the Wigner distribution. This can be inferred through pictures Figs. 1(b), 1(d), and 1(f) on the right column for $t=8T$. In general, closer correspondence is found within the so-called break time [20] in the classically chaotic regime.

In the next case shown in Fig. 2, we observed similar phenomena to the previous case. In this case, the classical distributions are strange attractors. At times $t\sim 2.2T$, depicted at the left, and $t\sim 8.8T$, at the right, Husimi distributions certainly resemble classical distributions more than Wigner distributions. This time we use the larger value of β. The illustration in Fig. 2(f) does not seem to be close to the classical one in 2(b), whereas the Husimi distribution in 2(d) does. It is interesting to see that structures near the origin in Figs. 2(e) and 2(f) are not present both in 2(c) and 2(d) and in 2(a) and 2(b). Thus from these two figures, it becomes more evident that the Husimi distribution represents the correspondence better than the Wigner distribution.

The parts (a) and (b) in Fig. 3 also display typical strange attractors. The Lyapunov exponent in this case is positive, and self-similarity is apparent. However, the Husimi distributions in Figs. 3(c) and 3(d) do not expose the close correspondence. Even with the moderate value of β, the good correspondence fails to be seen in this case. But the Wigner distributions in Figs. 3(e) and 3(f) resemble the classical distributions less than the Husimi distributions do because of the appearance of distinctive islands around the middle of the Wigner distributions. It is still reasonable to assume that the general configuration in the Husimi case is better in the context of the correspondence. In this case distributions on the left are at $t=T$, and ones in the right at $t=3T$. Other Husimi cases of the strange attractor we tried also showed comparable patterns. But we do not have a good answer why islands appear in the Wigner case. We know that Wigner distribution contains more detailed information about the quantum dynamics since it has no such coarse-graining mechanism as the Husimi's Gaussian smoothing.

In summary, we have specifically focused on the phase-space behavior in various distributions including classical distributions. It has been demonstrated that the

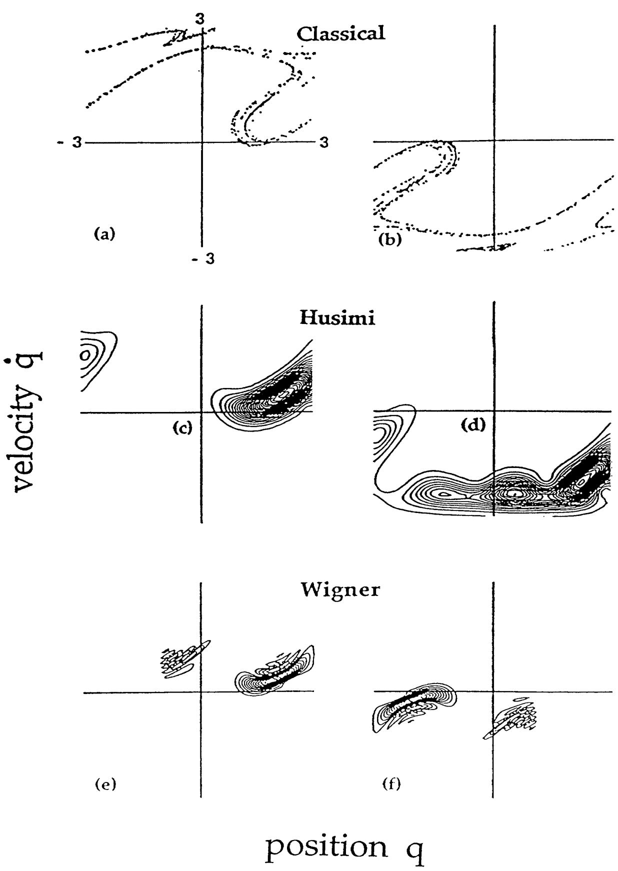

FIG. 2. Classical distributions with strange attractors are shown in (a), (b). Contour plots of the Husimi distributions and of the Wigner distributions are shown in (c), (d) and in (e), (f), respectively. The pictures in the left column are picked at $t\sim 2.2T$, whereas the ones in the right at $t\sim 8.8T$.

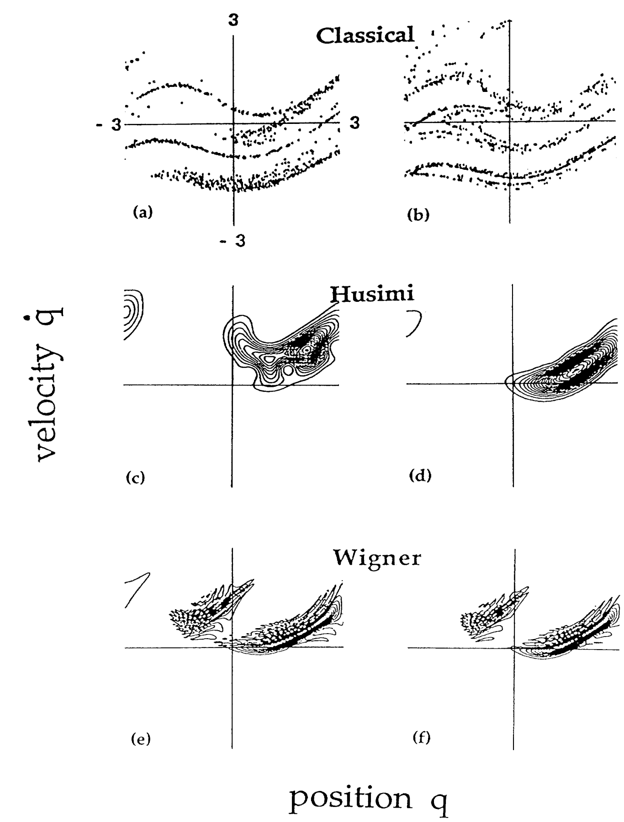

FIG. 3. All the distributions are shown in the same order as the previous two figures. The classical distributions in (a) and (b) have strange attractors in this case too.

Husimi representation is generally better than the Wigner representation of the correspondence. We found this to be identical to the previous claim [10]. Although earlier studies [2,3,22] reveal that the correspondence fails in the classically chaotic regime because of a large increase or fluctuation in the quantum uncertainty, these distribution functions can be used to detect signatures of quantum chaos. We also feel that the Husimi distribution function deserves more attention, not necessarily limited to the correspondence. For instance, the contour plots from these quantum distribution functions could be useful for the phase-space version of Heller's "scars" left by the quantum wave function [23,24] in a quantum system without a classical counterpart.

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