Evolution of patterns in the anisotropic complex Ginzburg-Landau equation: Modulational instability

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We investigate the instability of the anisotropic complex Ginzburg-Landau equation as a function of its parameters. We derive the conditions necessary for the instability of a homogeneous solution. In addition, the analytic geometry of the unstable solutions in wave-number space is investigated. This allows us to establish the most unstable wave (as a function of Reynolds number) whose evolution will eventually dominate the dynamics.

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I. INTRODUCTION

There are many papers devoted to the investigation of the complex Ginzburg-Landau equation (CGLE). This equation describes the evolution of unstable perturbations in a narrow spectral band [1–5]. However almost all of the previous research has concentrated on the isotropic version of the CGLE,

$$\frac{\partial a}{\partial t}=Ra-(1+i\beta)a|a|^2+(1+i\alpha)\Delta a ,$$

where a is the wave amplitude and R, β, and α are real parameters.

This equation has been used to describe wave dynamics in a variety of systems. For example, it has been used to describe chemical reactions in diffusive media [6], some types of waves in a nonequilibrium plasma [7], and some other nonequilibrium systems (see also the review [8]). The isotropic model supports a variety of dynamic and stochastic wave motions: vortices and spirals, complicated static patterns, periodic oscillations and spatiotemporal chaos (turbulence), and so on.

It should be noted, however, that the isotropic CGLE is only a particular (but rather typical) case of a more general equation. Investigation of nonlinear wave dynamics for many physical systems leads to the more general *anisotropic* model

$$\frac{\partial a}{\partial t}-ra-(1+i\beta)a|a|^2+\Delta a+i\left[\alpha_x\frac{\partial^2 a}{\partial x^2}+\alpha_y\frac{\partial^2 a}{\partial y^2}\right], \tag{1}$$

where $\alpha_x \neq \alpha_y$. Equation (1) describes in particular the amplitude of Tollmien-Schlichting waves in shear flows [9], wind waves on the surface of water [10], and so on. [Note that some other nonlinear evolution equations may be reduced to Eq. (1). For example, the well-known Newell-Whitehead equation transforms into Eq. (1) for long-scale perturbations [11,12].]

It is, in our opinion, quite accurate to think of the anisotropic CGLE, Eq. (1), as one of the fundamental equations in wave theory. This equation describes the nonlinear evolution of small disturbances in *arbitrary* nonlinear medium (see Sec. II). The word *arbitrary* is justification for our claim that Eq. (1) is fundamental. The scenario where Eq. (1) applies typically involves a nonlinear system that is known to be unstable for a narrow band of wave numbers near a particular value. If one wishes to investigate the evolution of perturbations that have components in this narrow band then one should generally use the anisotropic CGLE.

To our knowledge, no systematic investigation has been performed on the nonlinear dynamics of the *anisotropic* CGLE model [13]. A series of our presentations will be devoted to this very problem. Our results are described sequentially with increasing complexity of regimes under study. In this paper we will discuss, in detail, the most general form of anisotropic CGLE and investigate modulation instability of plane waves for anisotropic nonequilibrium media.

Our main nontrivial result is that even relatively small anisotropy drastically changes the behavior of solutions. We believe that this fact is of principal importance and is rather unexpected. In some cases anisotropy leads to phenomena not observed in the isotropic case.

II. COMPLEX PARABOLIC EQUATION

To substantiate our claims we will derive Eq. (1) for the general case of perturbations that develop in two space dimensions near the threshold where instability emerges [14]. A three-dimensional generalization is also possible. We begin by assuming that instabilities occur in the nonlinear medium as a parameter R increases. This parameter describes the deviation of the system from equilibrium (for example, the Reynolds number is such a parameter for shear hydrodynamic flows). Let the dispersion relation for wave motion in this nonlinear medium be given by $\tilde{\omega}=\tilde{\omega}(\mathbf{k})=\omega(\mathbf{k})+i\gamma(\mathbf{k})$, where $\omega(\mathbf{k})=\mathrm{Re}[\tilde{\omega}]$ and the instability increment is $\gamma(\mathbf{k})=\mathrm{Im}[\tilde{\omega}]$. Furthermore, assume $\gamma(\mathbf{k}_0)=0$ when $R=R_c$ and $\gamma(\mathbf{k})$ changes its sign in

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the neighborhood of $\mathbf{k}_0$ for $R > R_c$ (see Fig. 1). For small excess of the parameter R over the critical value R_c ($1 \gg R - R_c > 0$) the region of instability [the "top" of the surface $\gamma = \gamma(\mathbf{k}_0)$ in Fig. 1] is rather narrow. Under these conditions we can expand the real and imaginary parts of the dispersion relation in series near $\mathbf{k}_0$,

$$\begin{aligned} \omega &= \omega_0 + \mathbf{c}_g \cdot (\mathbf{k} - \mathbf{k}_0) \\ &\quad + \kappa_{ij} \cdot (k_i - k_{0i})(k_j - k_{0j}) + \cdots , \\ \gamma &= r + \nu_{ij} \cdot (k_i - k_{0i})(k_j - k_{0j}) + \cdots , \end{aligned} \tag{2}$$

where $\omega_0 = \omega(\mathbf{k}_0)$, $\mathbf{c}_g = [\nabla_{\mathbf{k}} \omega]_{\mathbf{k}=\mathbf{k}_0}$, and

$$r = \left[\frac{\partial \gamma}{\partial R} \right]_{\mathbf{k}=\mathbf{k}_0} (R - R_c) ,$$

$$\nu_{ij} = \left[\frac{1}{2} \frac{\partial^2 \gamma}{\partial k_i \partial k_j} \right]_{\mathbf{k}=\mathbf{k}_0} ,$$

$$\kappa_{ij} = \left[\frac{1}{2} \frac{\partial^2 \omega}{\partial k_i \partial k_j} \right]_{\mathbf{k}=\mathbf{k}_0} .$$

We are interested in wavelike motion in our nonlinear medium. Since the unstable disturbances are confined to a narrow region near $\mathbf{k}_0$ we consider a quasimonochromatic wave $\tilde{a}(\tilde{x}, \tilde{y}, t) \exp[i(\mathbf{k}_0 \cdot \mathbf{r} - \omega_0 t)]$, where $\mathbf{r} = (\tilde{x}, \tilde{y})$ and the amplitude $\tilde{a}$ is a slowly varying function of its arguments. The fact that the region of instability is narrow implies that the evolution of a disturbance can be described by the spatiotemporal dynamics of the complex amplitude $\tilde{a}$. From Eq. (2) we can obtain an evolution equation for the amplitude [15]. To this end it suffices to replace the vector $i(\mathbf{k} - \mathbf{k}_0)$ and the value $i(\omega - \omega_0)$ in Eq. (2) by the differential operators ∇ and $\partial / \partial t$ and let them act on $\tilde{a}$. We must also take into account the nonlinear corrections to the frequency and the instability increment [16]

$$\begin{aligned} \omega_{\mathrm{nl}} &= \omega(\mathbf{k}) + \beta |a|^2 + \cdots , \\ \gamma_{\mathrm{nl}} &= \gamma(\mathbf{k}) - \rho |a|^2 + \cdots . \end{aligned} \tag{3}$$

By restricting ourselves to second-order terms in the expansion given by Eq. (2) (parabolic approximation) and to the first nonvanishing terms found in Eq. (3) we can

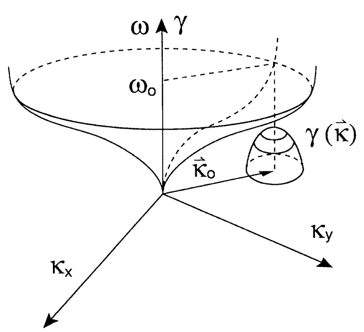

FIG. 1. Dispersion surface $\omega(\mathbf{k})$ and increment function $\gamma(\mathbf{k})$ for a system slightly above the instability threshold.

obtain the following complex parabolic evolution equation for the amplitude $\tilde{a}(\tilde{x}, \tilde{y})$:

$$\frac{\partial \tilde{a}}{\partial t} = r\tilde{a} + \mathbf{c}_g \cdot \nabla \tilde{a} + (i\delta - \rho)\tilde{a} |\tilde{a}|^2 + (i\kappa_{ij} + \nu_{ij}) \frac{\partial^2 \tilde{a}}{\partial x_i \partial x_j} . \tag{4}$$

Note that Eq. (4) gives the most general description of the evolution of disturbances near the threshold of instability.

Some special cases are also possible. Take, for example, media where quadratic and cubic nonlinearities are absent, or the wave frequency has a particular form which produces the condition $\partial^2 \omega / \partial k_i^2 = 0$ for $i = 1$ and/or 2. The latter case corresponds to an inflection point on the dispersion curve $\omega(k)$ in the one-dimensional case. Under these circumstances Eq. (4) is an invalid description of modulational waves, even close to the threshold of instability. The correct equation will have to take into account nonlinearities or dispersions of higher order. Terms such as $a|a|^2$, $\partial_{xxx} a /$, $a \partial_x |a|^2$, and the like must be added to the equation. Equations of this type, with purely imaginary coefficients—generalizations of the nonlinear Schrödinger equation—were studied earlier (see, for example, the paper by Johnson [17]). We will now show that Eq. (4) can be transformed into Eq. (1) by means of some simple changes in variables and parameters.

The series expansion of $\gamma(\mathbf{k})$ was made near the maximum of the surface described by $\gamma = \gamma(\mathbf{k})$ in (k_x, k_y, γ) space. Therefore the expression $\nu_{ij} \cdot (k_i - k_{0i})(k_j - k_{0j})$ has a positive-definite quadratic form near $\mathbf{k} = \mathbf{k}_0$. It can be shown that the change of variables

$$\begin{aligned} \tilde{x}' &= \tilde{x} \cos\theta + \tilde{y} \sin\theta , \\ \tilde{y}' &= -\tilde{x} \sin\theta + \tilde{y} \cos\theta , \end{aligned}$$

where θ, defined by $\tan(2\theta) = 2\nu_{xy}(\nu_{xx} - \nu_{yy})$, will transform $\nu_{ij}(k_i - k_{0i})(k_j - k_{0j})$ into a diagonal quadratic form. Making this transformation, and passing over to a coordinate system moving with a group velocity, $\mathbf{c}_g$ via $\mathbf{r}' = \mathbf{r} - \mathbf{c}_g t$ results in

$$\begin{aligned} \frac{\partial \tilde{a}}{\partial t} &= ra + (i\delta - \rho)\tilde{a} |\tilde{a}|^2 + \nu'_{xx} \frac{\partial^2 \tilde{a}}{\partial \tilde{x}'^2} + \nu'_{yy} \frac{\partial^2 \tilde{a}}{\partial \tilde{y}'^2} \\ &\quad + i\kappa'_{ij} \frac{\partial^2 \tilde{a}}{\partial \tilde{x}'_i \partial \tilde{x}'_j} , \end{aligned} \tag{5}$$

where

$$\begin{aligned} \nu'_{xx} &= \nu_{xx} \cos^2\theta + 2\nu_{xy} \sin\theta \cos\theta + \nu_{yy} \sin^2\theta , \\ \nu'_{yy} &= \nu_{xx} \sin^2\theta - 2\nu_{xy} \sin\theta \cos\theta + \nu_{yy} \cos^2\theta , \\ \kappa'_{xx} &= \kappa_{xx} \cos^2\theta + 2\kappa_{xy} \sin\theta \cos\theta + \kappa_{yy} \sin^2\theta , \\ \kappa'_{yy} &= \kappa_{xx} \sin^2\theta - 2\kappa_{xy} \sin\theta \cos\theta + \kappa_{yy} \cos^2\theta , \\ \kappa'_{xy} &= (\kappa_{yy} - \kappa_{xx}) \sin(2\theta) + 2\kappa_{xy} \cos(2\theta) . \end{aligned}$$

We can eliminate the cross terms involving κ' in Eq. (5) by employing the coordinate transformation

$$x = A(\tilde{x}' + W\tilde{y}') ,$$
$$y = B(\tilde{x}' + Z\tilde{y}') , \tag{6}$$

where $A = (\nu'_{xx} + W^2\nu'_{yy})^{-1/2}$, $B = (\nu'_{xx} + Z^2\nu'_{yy})^{-1/2}$ and $W = G + (\nu'_{xx}/\nu'_{yy} + G^2)^{1/2}$, $Z = G - (\nu'_{xx}/\nu'_{yy} + G^2)^{1/2}$, $G = (\nu'_{xx}\kappa'_{yy} - \nu'_{yy}\kappa'_{xx})/(2\kappa'_{xy}\nu'_{yy})$. We remark that this transform is possible for arbitrary values of the coefficients κ'_{ij} because the square roots include only positive values.

After applying Eq. (6) to Eq. (5) we acquire Eq. (1) by identifying

$$\beta = -\frac{\delta}{p} ,$$
$$a = \tilde{a}\rho^{1/2} ,$$
$$\alpha_x = A^2(\kappa'_{xx} + 2W\kappa'_{xy} + W^2\kappa'_{yy}) ,$$
$$\alpha_y = B^2(\kappa'_{xx} + 2Z\kappa'_{xy} + Z^2\kappa'_{yy}) .$$

These manipulations prove that given a proper choice of variables, Eq. (1) is the most general equation that describes the appearance of waves in an anisotropic medium near the threshold of instability.

We note, in passing, that one only needs to consider the case $\beta > 0$. The opposite case, $\beta < 0$, can be found by taking the complex conjugate of Eq. (1).

III. ANISOTROPIC MODULATION INSTABILITY

In this section we begin our discussion of the stability of solutions to the anisotropic CGLE, Eq. (1). (Before beginning our discussion we note that modulational instability has been investigated by other researchers such as Stuart and DiPrima [18], as well as Newell and Whitehead [19].) Our initial observation is that the trival solution of Eq. (1), $a = 0$, is linearly unstable to small perturbations. As a perturbation to this solution grows the nonlinear oscillations that are established can have various forms depending on parameters, as well as initial and boundary conditions.

The solution to Eq. (1) that is of interest to us is given by

$$a = r^{1/2}\exp[-i(\beta r)t] . \tag{7}$$

This solution is homogeneous in space and oscillates with a constant frequency βr. It describes a monochromatic wave (of infinite wavelength) with a constant amplitude. The remainder of this paper will be devoted to investigating the appearance and characteristics of modulational instability of this solution as a function of the parameters α_x, α_y, β, and r.

To begin, imagine a small perturbation to the basic solution, Eq. (7), which we write as $a = r^{1/2}\exp[-i(\beta r)t](1+\xi)$, where $\xi(x,y,t)$ is the small perturbation. Inserting this solution into Eq. (1) results in

$$\frac{\partial\xi}{\partial t} = -(1+i\beta)r[\xi+\xi^*] + (1+i\alpha_x)\frac{\partial^2\xi}{\partial x^2} + (1+i\alpha_y)\frac{\partial^2\xi}{\partial y^2} - (1+i\beta)r[2|\xi|^2 + \xi^2 + \xi|\xi|^2] . \tag{8}$$

Investigating the behavior of this equation will be one of the major tasks of this and our subsequent paper [20]. It is logical to begin our investigation with a discussion of the linear stability of the solution given by Eq. (7). To do so we assume that ξ can be written as $\xi \sim \exp[i(k_x x + k_y y)]$. After separating ξ into real and imaginary parts, $\xi = \eta + i\zeta$, we find that Eq. (8) may be written in the form

$$\frac{d}{dt}\begin{bmatrix}\eta\\ \zeta\end{bmatrix} = \mathbf{L}\begin{bmatrix}\eta\\ \zeta\end{bmatrix} + \begin{bmatrix}\mathrm{Re}[N]\\ \mathrm{Im}[N]\end{bmatrix} ,$$

where $N = -(1+i\beta)r[2|\xi|^2 + \xi^2 + \xi|\xi|^2]$ is the nonlinear part, the linear matrix is

$$\mathbf{L} = \begin{bmatrix} -2r - k^2 & \Phi \\ -2r\beta - \Phi & -k^2 \end{bmatrix} ,$$

and we have defined $k^2 = k_x^2 + k_y^2$ and $\Phi = \alpha_x k_x^2 + \alpha_y k_y^2$.

The linear stability of the perturbation ξ is determined by the eigenvalues of $\mathbf{L}$. The characteristic equation for the eigenvalues of $\mathbf{L}$ (which we denote as Λ) is

$$\det[\mathbf{L} - \Lambda\mathbf{I}] = \Lambda^2 + 2(r+k^2)\Lambda + k^2(2r+k^2) + \Phi(2r\beta + \Phi) = 0 . \tag{9}$$

This equation is easily solved for its eigenvalues

$$\Lambda_\pm = -(r+k^2) \pm [r^2 - \Phi(2r\beta + \Phi)]^{1/2} . \tag{10}$$

ξ is linearly unstable if one or more of the Λ's is positive. An examination of Eq. (10) indicates that unstable perturbations can exist only for $\Lambda = \Lambda_+$, and if $r + k^2 < [r^2 - \Phi(2r\beta + \Phi)]^{1/2}$. This condition can be transformed into the quadratic form

$$(X+Y)(X+Y+2r) + (\alpha_x X + \alpha_y Y) \times (\alpha_x X + \alpha_y Y + 2r\beta) < 0 \tag{11}$$

for new variables $X = k_x^2$, $Y = k_y^2$.

The simple quadratic form given by Eq. (11) can be written in a more recognizable form by a few simple changes of variables. We first translate the origin of the (X,Y) plane to $\mathbf{d} = (d_x, d_y)$, where

$$d_x = r\frac{\beta - \alpha_y}{\alpha_y - \alpha_x} ,$$
$$d_y = -r\frac{\beta - \alpha_x}{\alpha_y - \alpha_x} . \tag{12}$$

In terms of new variables $u = X - d_x$ and $v = Y - d_y$ Eq. (11) becomes

$$(u+v)^2 + (\alpha_x u + \alpha_y v)^2 < r^2(1+\beta^2) . \tag{13}$$

We now rotate (u,v) into new coordinates (Z,W) via $u = Z\cos\theta + W\sin\theta$ and $v = -Z\sin\theta + W\cos\theta$. The angle θ is defined by

$$\tan(2\theta) = \frac{2(1+\alpha_x\alpha_y)}{\alpha_y^2 - \alpha_x^2} . \tag{14}$$

After translation and rotation Eq. (11) acquires the standard form

$$\frac{Z^2}{Z_0^2}+\frac{W^2}{W_0^2}<1\ , \tag{15}$$

where

$$Z_0^2=\frac{r^2(1+\beta^2)}{(\cos\theta-\sin\theta)^2+(\alpha_x\cos\theta-\alpha_y\sin\theta)^2}\ ,$$
$$W_0^2=\frac{r^2(1+\beta^2)}{(\cos\theta+\sin\theta)^2+(\alpha_x\cos\theta+\alpha_y\sin\theta)^2}\ . \tag{16}$$

The inequality given by Eq. (15) specifies the interior of the ellipse given by

$$F(X,Y)=(X+Y)(X+Y+2r)+(\alpha_x X+\alpha_y Y)(\alpha_x X+\alpha_y Y+2r\beta)=0\ . \tag{17}$$

The ellipse is centered at $\mathbf{d}$ and has axes inclined at the angle θ. It should be understood that only the quadrant $X=k_x^2>0$, $Y=k_y^2>0$ has a physical sense. So the problem of existence of instability reduces to a question of whether or not the first quadrant in the (X,Y) plane includes some part of the ellipse's interior.

Having established the analytic geometry of the instability condition the remaining task is to investigate all possible cases for various parameters α_x, α_y, β, and r. We note first of all that the origin $(X=0, Y=0)$ is always on the ellipse. We conclude that the ellipse must cross the X or Y axis if a portion of its interior is to appear in the first quadrant. The ellipse crosses the X axis if $F(X_0,0)=0$ has a positive root. This condition results in

$$X_0=-2r\frac{1+\beta\alpha_x}{1+\alpha_x^2}>0\ , \tag{18}$$

which is possible only if

$$1+\beta\alpha_x<0\ . \tag{19}$$

In the same manner we can establish that the ellipse crosses the Y axis if

$$Y_0=-2r\frac{1+\beta\alpha_y}{1+\alpha_y^2}>0\ , \tag{20}$$

which is possible only if

$$1+\beta\alpha_y<0\ . \tag{21}$$

Therefore all possible cases of instability can be determined by Eqs. (12), (19), and (21) and are as follows [21].

(A) $\alpha_x<-\beta^{-1}<0$, $\alpha_y>0$.

(i) $\beta<\alpha_y$,

(ii) $\beta>\alpha_y$.

(B) $\alpha_x>0$, $\alpha_y<-\beta^{-1}<0$.

(i) $\beta<\alpha_x$,

(ii) $\beta>\alpha_x$.

(C) $\alpha_x<-\beta^{-1}<0$, $\alpha_y<-\beta^{-1}<0$.

(i) $\alpha_x<\alpha_y$,

(ii) $\alpha_x>\alpha_y$.

There is an obvious symmetry between cases A and B, as well as cases C(i) and C(ii). The geometry of the ellipses corresponding to some of these cases are shown in Fig. 2.

As the parameter r increases, the portion of the ellipse that extends into the first quadrant of the (X,Y) plane increases and more waves become unstable. The anisotropic nature of Eq. (1) sometimes permits these instabilities to arise in a nontrivial order. (We are interested in periodic boundary conditions; hence k_x and k_y can only take discrete jumps of a particular size.) For example, when $\alpha_x=\alpha_y$ (the isotropic case) the first unstable waves that occur are $(k_x=0,\ k_y\neq 0)$ and $(k_x\neq 0,\ k_y=0)$. The symmetry of the problem insures that both waves become unstable simultaneously. Only *after* these waves have become unstable is it possible for a wave with wave numbers $(k_x\neq 0,\ k_y\neq 0)$ to become unstable. This orderly progression of instability need not occur when the system is anisotropic. Indeed for the anisotropic case it is possible for a wave with wave number $(k_x\neq 0,\ k_y\neq 0)$ to become unstable *before* a wave with wave number $(k_x=0,\ k_y\neq 0)$ or $(k_x\neq 0,\ k_y=0)$.

Anomalous ordering in the instability is determined by the inclination of the ellipse as it crosses either the X or the Y axis. (In what follows we are interested in the crossings that occur away from the origin.) If the function $Y(X)$ [determined by Eq. (17)] increases with X as it crosses either the X or the Y axis then it is possible for anomalous ordering of the instability to occur. When anomalous ordering occurs the region of instability in the (k_x,k_y) plane has a "tongue" that is stretched into the first quadrant (see Sec. IV and Fig. 2).

We now seek a condition for the appearance of this tongue in case A. Differentiating Eq. (17) and substituting $X=X_0$, $Y=0$ from Eq. (18) yields the following inclination of the ellipse near the X axis:

$$\left.\frac{dY}{dX}\right|_{Y=0}=\frac{(1+\beta\alpha_x)(1+\alpha_x^2)}{\alpha_x^2-2\beta\alpha_x-1-\alpha_y(\beta\alpha_x^2+2\alpha_x-\beta)}>0\ . \tag{22}$$

Taking into account that for the case considered here $1+\beta\alpha_x<0$ and $\alpha_x^2-2\beta\alpha_x-1=1+\alpha_x^2-2(1+\beta\alpha_x)>0$, Eq. (22) reduces to

$$\alpha_y(\beta\alpha_x^2+2\alpha_x-\beta)>\alpha_x^2-2\beta\alpha_x-1>0\ . \tag{23}$$

This inequality can be valid only if $\beta\alpha_x^2+2\alpha_x-\beta>0$. Therefore a tongue can be realized in case A only if

$$\alpha_x<-\beta^{-1}-[1+\beta^{-2}]^{1/2}$$

and

$$\alpha_y>\frac{\alpha_x^2-2\beta\alpha_x-1}{\beta\alpha_x^2+2\alpha_x-\beta}\ .$$

By interchanging α_x and α_y we obtain the condition for the existence of a tongue near the Y axis for case B.

Finally, the condition for the existence of a tongue in case C can be found. In general a tongue exists near the X axis when Eq. (23) is valid. But for case C we know $\alpha_y < 0$ and so Eq. (23) yields the following conditions for case C:

$$-\beta^{-1}+[1+\beta^{-2}]^{1/2} > \alpha_x > -\beta^{-1}-[1+\beta^{-2}]^{1/2}$$

and

$$\alpha_y < \frac{\alpha_x^2 - 2\beta\alpha_x - 1}{\beta\alpha_x^2 + 2\alpha_x - \beta} < 0 \ .$$

Once again symmetry allows us to interchange α_x and α_y to obtain the condition for the existence of a tongue near the Y axis.

In addition it should be emphasized that it is impossible for both tongues to simultaneously exist in case C. Such a situation could be realized only if the center of the ellipse $F(X,Y)=0$ is inside the first quadrant on the (X,Y) plane. However, under the conditions $\alpha_x < 0$, $\alpha_y < 0$ (case C) the coordinates d_x and d_y have opposite sign [cf. Eq. (12)].

We devote the remainder of this section to the discussion of various interesting limiting cases. The isotropic limit occurs when $\alpha_x = \alpha_y = \alpha$. Equations (14) and (16) indicate that the angle of rotation for the ellipse is $\theta = 45°$ in the $\alpha_y^2 - \alpha_x^2 \to 0$ limit, while the long and short axes of the ellipse become $Z_0^2 \to \infty$ and $W_0^2 \to 2r^2(1+\beta^2)/(1+\alpha^2)$, respectively. Under these conditions Eq. (17) reduces to a pair of straight lines,

$$X+Y=0 \ ,$$
$$X+Y=-2r\frac{1+\alpha\beta}{1+\alpha^2} \ . \tag{24}$$

In the (k_x,k_y) plane Eqs. (24) denote the arc of a circle centered at the origin with a radius equal to $[-2r(1+\alpha\beta)/(1+\alpha^2)]^{1/2}$. The condition of instability comes from the second of Eqs. (24) and is $k_x^2+k_y^2 < -2r(1+\alpha\beta)/(1+\alpha^2)$. We see that $1+\alpha\beta<0$ is required for unstable waves. Although this condition was known before [22] and thus is not a surprise, it sheds

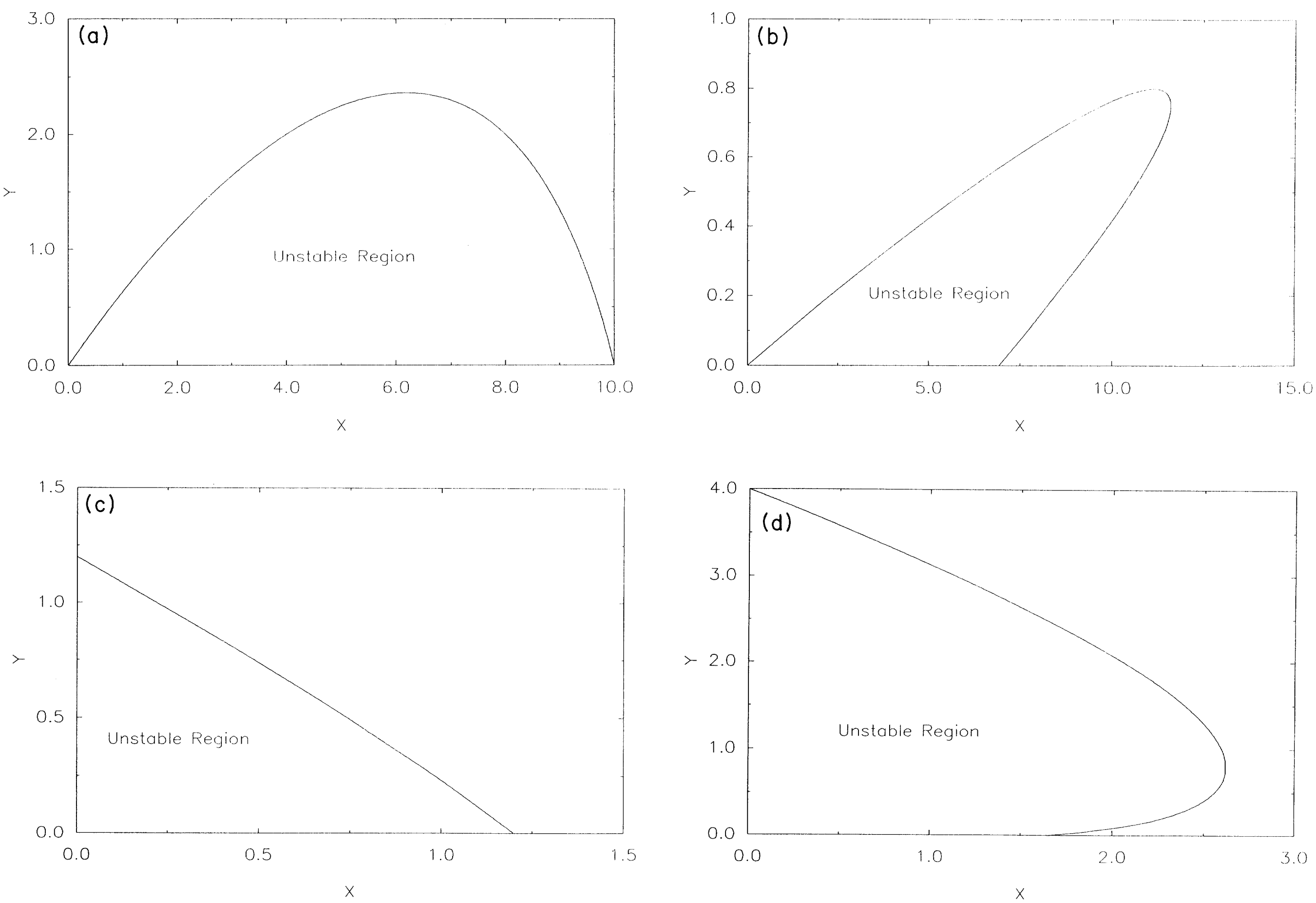

FIG. 2. Instability boundary types in the (X,Y) plane. The presence of a "tongue" indicates that anomalous ordering is possible. The cases that are now shown produce figures similar to those shown. (a) Case A(i) without a tongue. (b) Case A(i) with a tongue. (c) Case C(ii) without a tongue. (d) Case C(ii) with a tongue.

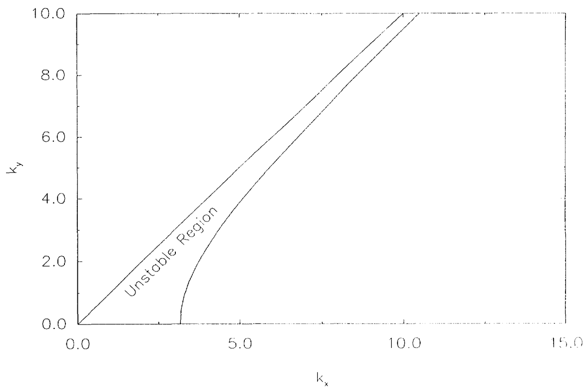

FIG. 3. Modulation instability region for the nonlinear Schrödinger equation.

light on the conditions for instability in the anisotropic case, $1+\beta\alpha_x<0$ and $1+\beta\alpha_y<0$.

The conservative limit is given by $\alpha_x\to\infty$, $\alpha_y\to\infty$, and $\beta\to\infty$. In this limit Eq. (1) is transformed into the nonlinear Schrödinger equation and Eq. (14) becomes $\tan(\theta)\to\alpha_x/\alpha_y$. We also find, from Eq. (16), that the ellipse transforms into the two lines

$$\begin{aligned}&\alpha_x X+\alpha_y Y=0\ ,\\&\alpha_x X+\alpha_y Y=-2r\beta\ .\end{aligned}\tag{25}$$

In the (k_x,k_y) plane Eqs. (25) determine curves that are shown in Fig. 3. This instability region was previously known (see, for example, Yuen and Lake [23]).

As a final pair of the limiting case we mention the Benjamin-Feir and Eckhaus limits. The Benjamin-Feir limit occurs when the parameter in front of $\partial_{x_i x_j}$ is purely imaginary. This corresponds to an instability that results purely from dispersion, without diffusion. The Eckhaus limit occurs when this parameter is purely real. This corresponds to an instability that results purely from diffusion, without dispersion. These complimentary limits have been investigated previously (see, for example, Brand and Deissler [11]).

In conclusion of this section we note that the instability boundary has a shape that does not depend on the value of parameter r. This fact can be deduced by regarding Eq. (17) as an equation for the variables (X/r) and (Y/r). This fact will become important in Sec. V.

IV. SEQUENCE OF UNSTABLE MODES

Our results from the preceding section make it possible to classify the order and type of all modulational components that may appear in our two-dimensional nonequilibrium system when the threshold of instability is crossed. Imposing periodic boundary conditions on the general problem of modulational instability, Eq. (1), superimposes a grid of permissible wave numbers onto both the (k_x,k_y) and (X,Y) planes. We will write these boundary conditions as [24]

$$a(x+2\pi,y)=a(x,y+2\pi)=a(x,y)\ .$$

Thus the solution to Eq. (8) can be expanded into a Fourier series of harmonics, $\exp[i(k_x x+k_y y)]$, where $k_x=1,2,\ldots$, and $k_y=1,2,\ldots$.

For a given set of coefficients α_x, α_y, β, and r, Eq. (17) defines a curve in the (k_x,k_y) plane whose perimeter defines the wave vectors that are neutrally stable. Not every possible wave vector in (k_x,k_y) space is admissible in our boundary value problem. If the nonequilibrium parameter (Reynolds number) r is too small, then all admissible wave vectors on the grid will lie outside of the neutral curve. As r increases the neural curve keeps its shape but increases its characteristic size in a manner that is proportional to r. When r reaches the critical value r_c the first admissible wave vectors cross to the interior of the neutral curve.

Figure 4 shows various possible neutral curves just after the threshold of instability when additional one or two pairs of wave vectors become unstable. Note that the wave vectors may change their stability by quartets if the parameter values permit tongues, while they change their stability in pairs when the parameter values do not permit tongues.

There are two interesting exceptions to these rules. The first is the isotropic case, when two pairs of wave vectors $k_x=\pm1$ and $k_y=\pm1$ become unstable for the same critical value r_c. (We have discussed this case above.) The second exception exists for anisotropic systems and can be found by examining Eqs. (18) and (20). Let $k_x=k_y=1$ and rewrite Eq. (18) as

$$r>r_{cx}=-\frac{1+\alpha_x^2}{2(1+\beta\alpha_x)}>0\ .$$

A symmetric equation can be found for $r>r_{cy}$. For case C it is possible to have $r_{cx}=r_{cy}$ *without* $\alpha_x=\alpha_y$. Define $r_{cx}=r_{cy}=T/2$, where T can only depend on β. The parametric conditions

$$\begin{aligned}\alpha_x&=-\frac{T\beta}{2}\pm\tfrac{1}{2}[T^2\beta^2-4(T+1)]^{1/2}\ ,\\ \alpha_y&=-\frac{T\beta}{2}\mp\tfrac{1}{2}[T^2\beta^2-4(T+1)]^{1/2}\end{aligned}\tag{26}$$

may be fulfilled for any T such that

$$T>\frac{2}{\beta}[1+(1+\beta^2)^{1/2}]\ .\tag{27}$$

If the conditions given by Eqs. (26) and (27) are valid we have $r_{cx}=r_{cy}=T/2$, while $\alpha_x\neq\alpha_y$. To be sure this is a very special case and small deviations of the parameters destroys the conditions making $r_{cx}\neq r_{cy}$.

Regardless of the special exceptions just discussed, an analysis of Figs. 4 and Eqs. (18) and (20) permits us to determine the exact order in which the modes become unstable.

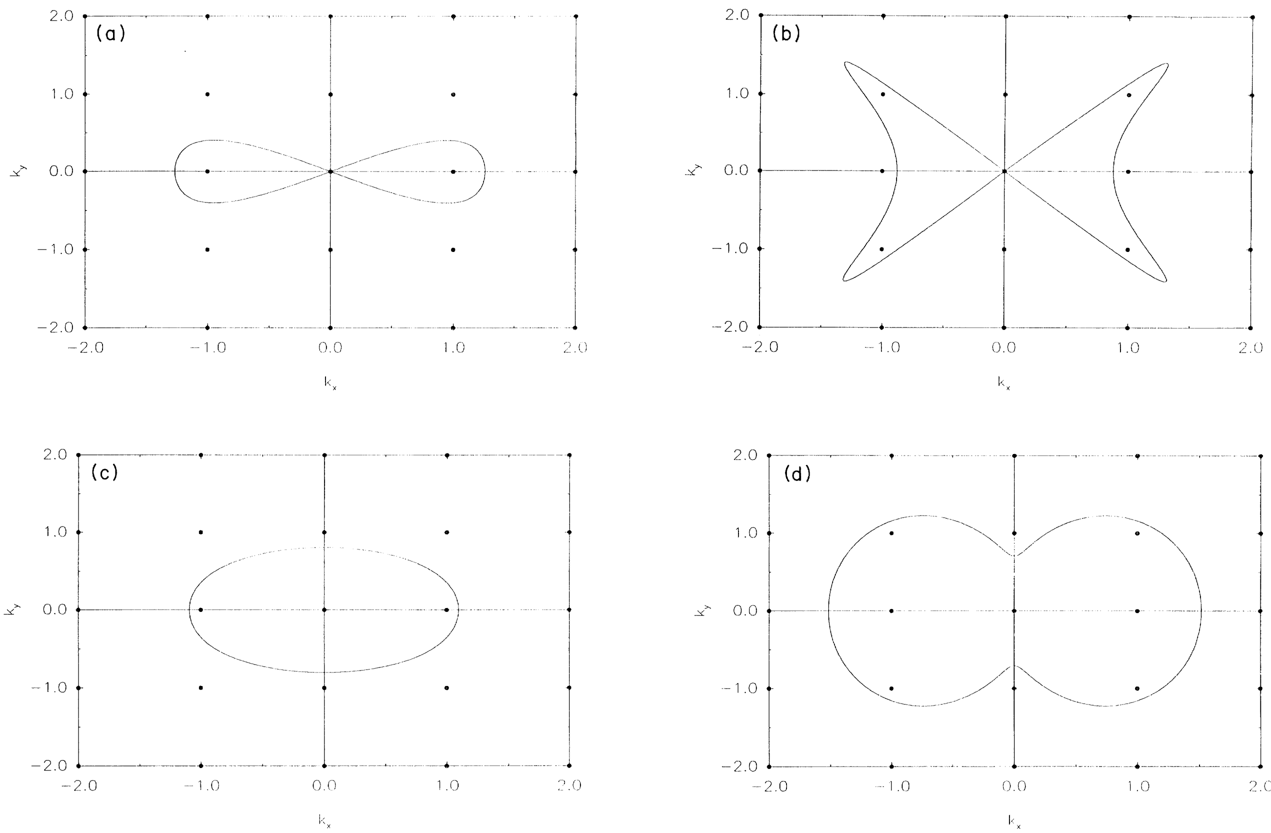

FIG. 4. Possible sets of modes excited due to modulation instability in anisotropic media. The dots indicate wave modes consistent with the boundary conditions. The cases that are now shown produce figures similar to those shown. (a) Case **A**(i) without anomalous ordering. (b) Case A(i) with anomalous ordering. (c) Case C(ii) without anomalous ordering. (d) Case C(i) with anomalous ordering.

V. INCREMENT OF MODULATION INSTABILITY

In this section of the paper our primary purpose is to determine which wave has the largest value of Λ. We conjecture that since the instability associated with this wave will grow the fastest it will dominate the dynamics. Our analysis will assume that r is sufficiently large that $\Lambda>0$ for many different values of $\mathbf{k}$, and we will ignore the grid of acceptable values of $\mathbf{k}$ in favor of the continuum. The first task is to discuss the surface given by $\Lambda=\Lambda(X,Y)$ in the three-dimensional space (X,Y,Λ).

Return to the characteristic equation, Eq. (9), relating X, Y, and Λ, which we rewrite in the form

$$\Lambda^2+2(r+X+Y)\Lambda+(X+Y)(2r+X+Y) +(\alpha_x X+\alpha_y Y)(2\beta r+\alpha_x X+\alpha_y Y)=0 . \quad (28)$$

Now consider an arbitrary fixed value of Λ. In the fashion of our discussion in Sec. III we define new variables $u=X-d_x(\Lambda)$ and $v=Y-d_y(\Lambda)$ where $\mathbf{d}(\Lambda)=(d_x(\Lambda),d_y(\Lambda))$ is

$$d_x(\Lambda)=\frac{r(\beta-\alpha_y)-\alpha_y\Lambda}{\alpha_y-\alpha_x}=d_x-\frac{\alpha_y}{\alpha_y-\alpha_x}\Lambda ,$$
$$d_y(\Lambda)=\frac{-r(\beta-\alpha_x)+\alpha_x\Lambda}{\alpha_y-\alpha_x}=d_y+\frac{\alpha_x}{\alpha_y-\alpha_x}\Lambda . \quad (29)$$

In Eqs. (29) we have used the formulas for d_x and d_y first defined in Eq. (12).

In terms of (u,v), Eq. (28) becomes

$$(u+v)^2+(\alpha_x u+\alpha_y v)^2=r^2(1+\beta^2) . \quad (30)$$

This equation describes the same curve as Eq. (13) but is valid *for all values of* Λ.

For $\Lambda=\text{const}\neq 0$, Eq. (28) defines the cross section of the surface $\Lambda=\Lambda(X,Y)$ in the three-dimensional (X,Y,Λ) space. Equation (30) indicates that for all value of Λ this cross section is an ellipse whose center is given by $\mathbf{d}$. From Eq. (29) we see that $\mathbf{d}$ changes linearly with Λ. Thus in the three-dimension space (X,Y,Λ) the instability boundary is a "tube" with ellipsoidal cross section in the (X,Y) plane. The axis of the tube is a straight line in (X,Y,Λ) space along which centers of ellipses for various $\Lambda=\text{const}$ are situated. The line forms some nontrivial angle with respect to the (X,Y) plane. The projection of the axis of the tube down onto the (X,Y) plane can be obtained from Eq. (29), which we recognize as the parametric equation for the line

$$Y=-\frac{\alpha_x}{\alpha_y}X-\frac{r\beta}{\alpha_y}\ . \tag{31}$$

Therefore changing the value of instability increment Λ has a simple interpretation. As Λ increases the center of the ellipse of Eq. (30) moves in the (X,Y) plane along the line given by Eq. (31). Isolines of Λ are shown in Fig. 5.

We are now in a position to answer an important question: Where is the point $(X_{\max},Y_{\max})$ that corresponds to the maximum value of Λ $(\Lambda_{\max})$? For case A this point is on the X axis. To find $X_{\max}$ we calculate the coordinates $X_\pm$ where the isolines $\Lambda=\Lambda_0$, a constant, crosses the X axis. Using Eq. (28), with $Y=0$, we find

$$X_\pm=\frac{-\Lambda_0-r(1+\beta\alpha_x)\pm[r^2(1+\beta\alpha_x)^2+2r\alpha_x(\beta-\alpha_x)\Lambda_0-\Lambda_0^2\alpha_x^2]^{1/2}}{1+\alpha_x^2}\ .$$

From this equation it is rather easy to determine that the points $X_\pm$ monotonically close on each other as Λ_0 increases. When X_+ and X_- coincide Λ has the maximum value

$$\Lambda=\Lambda_{\max}^{(1)}=\frac{r}{\alpha_x}\{\beta-\alpha_x-[(1+\beta^2)(1+\alpha_x^2)]^{1/2}\} \tag{32}$$

and is achieved at the point

$$X_{\max}=\frac{r}{\alpha_x}\left[-\beta+\left[\frac{1+\beta^2}{1+\alpha_x^2}\right]^{1/2}\right]\ ,$$
$$Y_{\max}=0\ . \tag{33}$$

Case B leads, not surprisingly, to the symmetrical result, which can be obtained from Eqs. (32) and (33) by interchanging α_x and α_y.

Case C is only slightly more complicated. For case C the direction of motion of the ellipse depends on the sign of $(\alpha_y-\alpha_x)$ [cf. Eq. (29)]. If $|\alpha_x|>|\alpha_y|$ then the ellipse moves in the direction of increasing X (see Fig. 5) and $\Lambda_{\max}$ occurs on the X axis. In this case Eqs. (32) and (33) are valid. Otherwise, if $|\alpha_x|<|\alpha_y|$ then the ellipse moves in the direction of decreasing X and $\Lambda_{\max}$ occurs on the Y axis.

To study the isotropic limit $(\alpha_x=\alpha_y=\alpha)$ we return to Eq. (28). The isolines of $\Lambda=\Lambda_0$ are

$$X+Y=\frac{-\Lambda_0-r(1+\beta\alpha)\pm[r^2(1+\beta\alpha)^2-2r\alpha(\beta-\alpha)\Lambda_0-\Lambda_0^2\alpha^2]^{1/2}}{1+\alpha^2}\ .$$

We may conclude from this equation that the isolines of Λ are two parallel straight lines inclined at the angle 45°. The lines move closer to each other as Λ_0 increases. The maximum value of the increment, $\Lambda_{\max}$, can be found from Eq. (32) for the isotropic case. However, the location in the (X,Y) plane of the maximal increment for the isotropic case differs from the location for the anisotropic case. For the anisotropic case the maximal increment is achieved at a specific point in (X,Y) space. For the isotropic case the maximal increment is achieved, simultaneously, on a continuum set of points along the line

$$X+Y=-\frac{r}{\alpha}\left[-\beta+\left[\frac{1+\beta^2}{1+\alpha^2}\right]^{1/2}\right]\ .$$

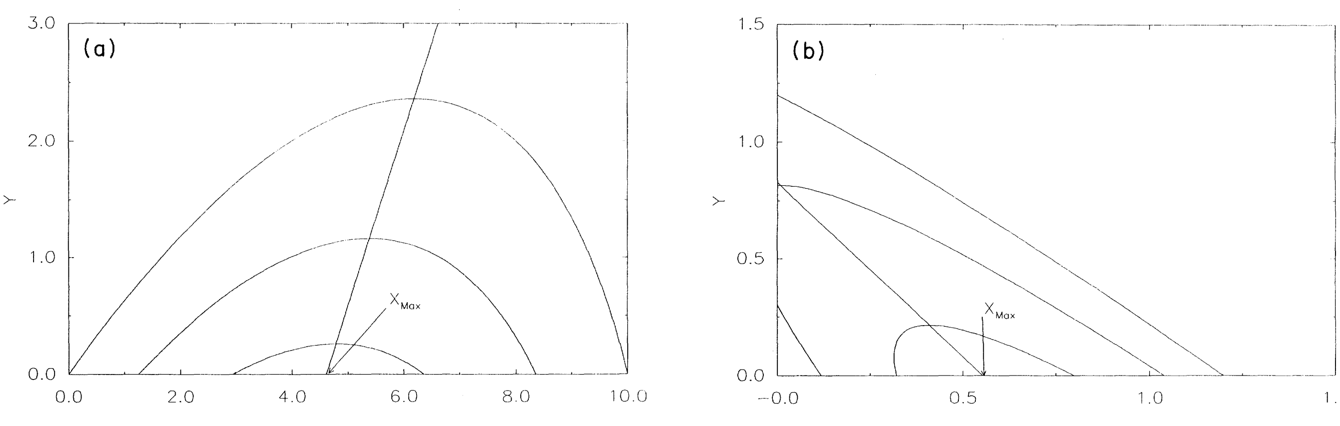

FIG. 5. Isolines Λ=const for increasing values of Λ. The line is parallel to Eq. (31) and indicates the motion of the ellipse. (a) Case A(i). (b) Case C(i).

VI. CONCLUSION

In this paper we have discussed the first of a series of results based on our investigation of the anisotropic CGLE. This part of our research has been devoted to a study of the analytic geometry in (k_x,k_y) space (or similar spaces) that gives rise to instabilities of a particular type of general solution. The base solution we have studied is one whose amplitude is homogeneous in space, and has simple sinusoidal oscillations in time. Next to the trival solution, this is the simplest type of basic solution available to the CGLE. We have presented a brief derivation of the anisotropic CGLE. Our analysis indicates that the form of the CGLE we have studied, Eq. (1), is the most general form of the CGLE. We also present the coordinate transformations necessary for transforming an arbitrary version of the CGLE to our form.

Our investigations reveal that the condition for instability of perturbations is given by $1+\beta\alpha_x<0$ and/or $1+\beta\alpha_y<0$, a condition that is similar to the isotropic case, namely, $1+\beta\alpha<0$. We have determined that the region in the (k_x^2,k_y^2) plane that corresponds to unstable waves is the interior of an ellipse. The ellipse's center and the angle of inclination to the k_x^2 axis have been determined as a function of the parameters α_x, α_y, β, and r of the CGLE. Since the physically realizable waves correspond only to the first quadrant of this space, the question of the stability of the base solution reduces to determining whether the ellipse extends into the first quadrant.

An interesting and unexpected result of our investigations is the anomalous ordering in which unstable waves are excited as the Reynolds number is increased. For the isotropic case the order in which various wave numbers become unstable with increasing Reynolds number is quite orderly and easily understood. In contrast, for the anisotropic case the order in which various wave numbers become unstable is intimately connected with the values of the parameters. In some cases the order in the anisotropic case will differ drastically from the order in the isotropic case. We have provided a simple explanation for this anomalous ordering as well as conditions that are necessary for the anomalous ordering to occur.

When the Reynolds number is large one expects many unstable waves to be excited after the occurrence of an arbitrary perturbation to the base solution. These waves will grow and their competition will result in complicated spatial and temporal patterns. In an effort to understand these patterns one would need to know which unstable waves dominate the dynamics. We have determined which wave number(s) correspond(s) to the largest unstable eigenvalue(s). Our belief is that these wave number(s) will eventually dominate the dynamics since they will eventually correspond to the waves with the largest amplitudes.

In all of our discussions we have examined the behavior of the system in the isotropic limit ($\alpha_x=\alpha_y$) as well as the conservative limit ($\alpha_x\to\infty$, $\alpha_y\to\infty$, and $\beta\to\infty$). We find that for the isotropic limit the shape of the unstable region changes from the interior of an ellipse to the region between two parallel lines. In the conservative limit, the unstable region is again between two parallel lines, but the equation for the lines are different. We also find that in the isotropic limit there is a continuum of wave numbers that all correspond to the largest eigenvalue. This is in contrast to the anisotropic case where the largest eigenvalue always corresponds to only one wave number.

We will use the remainder of this conclusion to introduce the second part of our analysis [20]. In our second paper we will attempt to answer a simple question: What happens when r is large enough to allow a pair (or a quartet) of unstable modes? To answer this question we must investigate the *nonlinear* evolution for the unstable waves discussed in this paper. The solution of Eq. (8) may be approximately written as a sum of excited modes that arise due to the instability. A pair of excited waves can be written as

$$\xi=C(t)\exp(i\mathbf{k}\cdot\mathbf{r})+C^*(t)\exp(-i\mathbf{k}\cdot\mathbf{r}) , \tag{34}$$

where $C(t)$ is a complex amplitude. We will allow these amplitudes to slowly change over time due to the nonlinearities in Eq. (8). For now we state that the evolution in space and time of the amplitudes of the excited waves can be quite complicated [25,26,20].

Investigation of this type of nonlinear problems, in particular, types of patterns and their dependence on parameters, their stability, and various types of time evolution, will be the subject of our second paper.

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