PHYSICAL REVIEW E VOLUME 47, NUMBER 6 JUNE 1993

Relaxation and stationary properties of a nonlinear system driven by white shot noise: An exactly solvable model

R. Zygadło
Institute of Physics, Jagiellonian University, Reymonta 4, PL-30059 Kraków, Poland
(Received 15 December 1992)

The relaxation and stationary properties of the Verhulst-type model with linearly coupled white shot noise are considered. It is found that in spite of the appearance of the noise-induced transitions in the stationary probability density distribution, the moments change continuously. The autocorrelation function is calculated and the relaxation rates are examined. It is shown that the presence of the noise slows down the relaxation. It is found for the true Verhulst model that the integral characteristic, the relaxation time T, is simply proportional to the square of the noise-controlling parameter. The Gaussian-white-noise limit is discussed.

PACS number(s): 05.40.+j, 02.50.−r, 05.70.Ln

I. INTRODUCTION

The evolution of nonlinear systems is strongly affected by the presence of fluctuations (noise). It may be reflected both quantitatively by changes of values of different observed quantities, such as the transition or relaxation rates or the time-dependent or stationary averages, and qualitatively as the stabilization or destabilization of the process, the appearance of noise-induced transitions, among others [1–12]. Until the fluctuations are described simply by the Gaussian white noise (GWN) there exists a convenient and general method of theoretical examination, namely by the use of the Fokker-Planck equation [3, 10]. However, GWN—which results as a rather artificial limit of more realistic processes [3,6–8]—should be considered only as a "first approximation" of the physical fluctuations. Moreover, if for some reasons the fluctuating parameters have to be bounded, GWN cannot be used at all. Therefore, for some time past, there is an interest in analyzing the influence of other noises on a macroscopic kinetics [6–9,11,12]. It seems that one of the best theoretical descriptions of the real (at least external) noise is provided by the (compound) Campbell process (shot noise) [13]

$$\xi_t = -c(t) + \sum_i z_i h(t-t_i)\,, \qquad (1)$$

where $h(t)$ is a certain response function, t_i are random times given by Poisson process (with parameter λ), z_i are independent random weights with the same probability distribution, and $c(t)$ is the so-called deterministic compensator defined by the condition $\langle \xi_t \rangle = 0$. If the support ("width") of $h(t)$ is small compared to λ^{-1} and to the characteristic time of the system's response, one can simplify the further discussions assuming $h(t) = \delta(t)$, i.e., by considering a (generalized) white shot noise (WSN) [9].

Many theoretical and numerical investigations concerning the influence of different noises (colored Gaussian noise [12]; dichotomic Markov process [7, 8]; WSN [9]) have been done during the last decade, and the effect of such noises on nonlinear kinetics is generally well understood. However, it is still difficult to go with exact analytical calculations beyond the "GWN approximation" (especially when the time-dependent problems are treated). The reason is that appropriate equations for the probability density distribution usually turn out to be nonlocal or operator, difficult to handle, equations [14].

Nevertheless, for some classes of Langevin equations the stochastic solution can be found. Then, provided the averaging can be effectively carried out, it is possible to obtain several quantities of interest (moments, correlations) directly, without finding the probability distribution. In a recent paper [15] such an approach has been applied to the stochastic Bernoulli equation (Verhulst-type model) [16]

$$\dot{x}_t = (a + A\xi_t)x - bx^{\mu+1}\,, \qquad (2)$$

where ξ_t describes the noise, $A \neq 0$ and $b\mu > 0$. The formal expressions for transient moments in the presence of an arbitrary white noise, and their rigorous analytical representation for the particular case of WSN with exponentially distributed weights, have been found.

In this paper we want to discuss the stationary properties of the model. In Sec. II we briefly recall the asymptotic behavior of the system in different regions of parameter space, in Sec. III the noise-induced transitions are examined, and in Sec. IV the critical case ($a = 0$) is considered. The discussion of the relaxation is done in Sec. V, and Sec. VI contains some conclusions. In Appendix A the *explicit* form of stationary autocovariance is provided, and in Appendix B the so-called relaxation time T is calculated.

II. ASYMPTOTIC BEHAVIOR

Consider, without loss of generality, the "standardized" WSN with random weights $z \in [0, +\infty)$ given

1063-651X/93/47(6)/4067(9)/$06.00 47 4067

by exponential distribution $p(z) = e^{-z}$, $h(t) = \delta(t)$, $c(t) = 1$, and $\lambda = 1$ (which means that t, a, and b are considered as dimensionless variables). In a number of formulas we will write *explicitly* the λ dependence, in order to discuss the GWN limit. Because of the "stability" condition $b\mu > 0$ there are four cases with respect to the signs of the system parameters a, b, μ.

It follows from the *explicit* forms of $\langle x_t^\omega \rangle$ [15] that in cases $b, \mu > 0$, $a < 0$, and $b, \mu < 0$, $a > 0$ [i.e., when both linear and nonlinear term in Eq. (2) describe either annihilation or creation processes, respectively] the presence of WSN cannot change the deterministic trend. The mean value either decays exponentially in the former, or growths infinitely in the latter case.

In the remaining cases: $b, \mu, a > 0$ and $b, \mu, a < 0$, which correspond to the presence of the opposite processes in Eq. (2), the situation is more interesting. The moments (if they exist) relax to the finite (stationary) values, which are given by three different analytical formulas; which one is to be used depends on the value of the parameter A (to which the noise is coupled),

$$m_1(\omega) = Q^{-\omega\nu} \frac{\Gamma(-\frac{\nu}{A})\Gamma(\omega\nu + \frac{\nu a}{A(A-a)})}{\Gamma(\omega\nu - \frac{\nu}{A})\Gamma(\frac{\nu a}{A(A-a)})}, \qquad \omega\nu - \nu/A > 0, \quad (3a)$$

$$m_2(\omega) = Q^{-\omega\nu} \frac{\Gamma(1 - \omega\nu + \frac{\nu}{A})\Gamma(1 + \frac{\nu a}{A(a-A)})}{\Gamma(1 - \omega\nu + \frac{\nu a}{A(a-A)})\Gamma(1 + \frac{\nu}{A})}, \qquad 1 + \nu/A - \omega\nu > 0, \quad (3b)$$

$$m_3(\omega) = |Q|^{-\omega\nu} \frac{\Gamma(1 - \omega\nu + \frac{\nu}{A})\Gamma(\omega\nu + \frac{\nu a}{A(A-a)})}{\Gamma(1 + \frac{\nu}{A})\Gamma(\frac{\nu a}{A(A-a)})}, \qquad 1 + \nu/A - \omega\nu > 0, \quad (3c)$$

where

$$\nu = 1/\mu, \quad Q = \frac{b}{a - A}, \quad \omega > 0, \quad \text{and } m(\omega) \equiv \langle x^\omega \rangle_{\rm st}.$$

For the positive a, b, μ, expressions m_1, m_2, or m_3 are valid provided A belongs to the intervals $(-\infty, 0)$, $(0, a)$, or $(a, +\infty)$, respectively. For the negative system parameters, m_3 [if, moreover, $\omega\nu + \nu a/A(A-a) > 0$], m_2 or m_1 correspond to $A \in (-\infty, a)$, $(a, 0)$, or $(0, +\infty)$, respectively. Such behavior of moments suggests an appearance of the so-called noise-induced transitions in the stationary probability density distribution.

III. NOISE-INDUCED TRANSITIONS

The stationary probability density distribution can be found as the inverse Mellin transforms of $m(\omega)$ [15]

$$P_{\rm st}(x) = \chi(x)\, x^{-1-a/(a-A)A}\, |x^\mu - Q^{-1}|^{-1+\nu/(a-A)}, \quad (4)$$

where

$$\chi(x) = \begin{cases} \frac{1}{N_\pm}\Theta(Q^{-\nu} - x) & \text{if } A < 0 < a \text{ or } a < A < 0 \\ \frac{1}{N_\mp}\Theta(x - Q^{-\nu}) & \text{if } 0 < A < a \text{ or } a < 0 < A \\ \frac{1}{N}\Theta(x) & \text{if } 0 < a < A \text{ or } A < a < 0. \end{cases}$$

The upper signs and the middle column correspond to the case of $\mu > 0$; the lower signs and the right column correspond to $\mu < 0$. Θ is the unit step function, and the normalization factors are

$$N_+ = |\nu| Q^{1+\nu/A} B\Big(\frac{\nu}{a-A}, \frac{\nu a}{A(A-a)}\Big),$$

$$N_- = |\nu| Q^{1+\nu/A} B\Big(\frac{\nu}{a-A}, 1 + \frac{\nu}{A}\Big),$$

$$N = |\nu| |Q|^{1+\nu/A} B\Big(\frac{\nu a}{A(A-a)}, 1 + \frac{\nu}{A}\Big),$$

respectively. (The arguments of Euler B functions have to be positive.)

Note that for WSN considered here the general form of stationary probability distribution is known. Namely, Van Den Broeck [6] has shown that for the stochastic equation involving this type of WSN

$$\dot{x}_t = f(x) + g(x)\xi_t, \quad (5)$$

the stationary probability distribution is given by

$$P_{\rm st}(x) \sim \frac{1}{f(x) - g(x)} \exp\Big\{ \int^x \frac{f(y)dy}{g(y)[g(y) - f(y)]} \Big\}. \quad (6)$$

The problems of boundaries of the process (5) and of the existence of $P_{\rm st}(x)$ have been studied by Sancho *et al.* [17]. Using the fact that trajectories of WSN (with exponentially distributed weights) are bounded from below, they provided some general arguments how to determine the domain of probability distribution. As an example (among others) the true Verhulst model ($\mu = 1$) with linearly coupled noise has been examined. Let us note that in the present approach the above-mentioned problems never appear explicitly, and both the form and the support of stationary probability distribution are uniquely determined by the method.

The "phase space" available in the stationary state depends on the noise-controlling parameter A according to the following schema:

$$(0,q) \underset{A<0}{\longrightarrow} (0, x_{\rm st}^-) \underset{A=0^-}{\longrightarrow} \{x_{\rm st}\} \underset{A=0^+}{\longrightarrow} (x_{\rm st}^+, \infty) \underset{A<a}{\longrightarrow} (q, \infty) \underset{A=a}{\longrightarrow} (0, \infty) \underset{A>a}{\longrightarrow} (0, \infty),$$

for the positive a, b, μ, and

$$(0,\infty) \underset{A<a}{\longrightarrow} (0, \infty) \underset{A=a}{\longrightarrow} (0, q) \underset{A>a}{\longrightarrow} (0, x_{\rm st}^-) \underset{A=0^-}{\longrightarrow} \{x_{\rm st}\} \underset{A=0^+}{\longrightarrow} (x_{\rm st}^+, \infty) \underset{A>0}{\longrightarrow} (q, \infty),$$

for a, b, $\mu < 0$, respectively, where $q = (\frac{b}{a-A})^{-\nu}$ and $x_{\rm st} = (b/a)^{-\nu}$ is the deterministic stationary state. The examples of $\tilde{P}_{\rm st}(\tilde{x})$ (where $\tilde{x} = x/q$ is a rescaled variable, in order to keep the finite boundary in the unity) are presented in Fig. 1 (compare Ref. [17]). They show that besides the two transitions at $A = 0$ ("near"

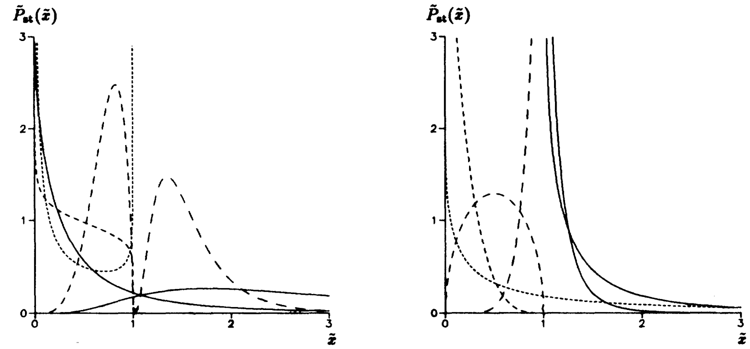


FIG. 1. The shapes of the (normalized) stationary probability density distribution for different values of parameters. The variable $\tilde{x}$ is scaled to keep the possible finite boundary in the unity, and consequently $\tilde{P}_{st}$ does not depend on b. The remaining deterministic parameters are $\mu = 1$, $a = 0.4$ on the left graph, and $\mu = -0.6$, $a = -2.1$ on the right graph. The successive curves correspond to $A = -1, -0.5, -0.15, 0.15, 0.5, 0.9$ or $A = -3, -1.7, -1, -0.15, 0.15, 0.5$, respectively. The growth of A is marked by increasing the length of dashes, so the continuous lines correspond to the greatest values of A.

the deterministic case) and $A = a$ (when $P_{st}(x) \propto x^{-1-\mu-1/a}\exp[-1/(b\mu x^{\mu})]$ takes the exponential form)—which are related to the change of the formula describing moments: $m_1(\omega) \leftrightarrow m_2(\omega)$ and $m_2(\omega) \leftrightarrow m_3(\omega)$, respectively—a few others may appear, which correspond to the change of a character of the boundaries (from repulsive to attractive, or vice versa). Let us consider, e.g., the case shown on the left graph ($\mu = 1$, $a = 0.4$, b is positive). For large negative A the domain of $P_{st}(x)$ is wide (but finite) and both the left (at $x = 0$) and the right [at $x = q = (a - A)/b$] boundary are attractive. When $A < 0$ is increased (i.e., $|A|$ tends to zero) the right and left boundaries become successively repulsive. This is accompanied by the changes in the shape of the stationary probability distribution. The phase space is contracted to the interval $(0, x_{st})$. When A becomes positive the support of $P_{st}(x)$ changes "discontinuously," and the values of x greater than q become allowed. At $A = a$ the whole positive semiaxis becomes available, and the boundary is still repulsive. Finally, for sufficiently large A, the point $x = 0$ becomes attractive. In a similar way the graphs appropriate for others values of a and μ may be discussed.

It should, however, be noted that the averages (observable quantities) change continuously during these transitions. In fact, using the well-known formula

$$\lim_{|z|\to\infty} \frac{\Gamma(z+\gamma)}{z^{\gamma}\Gamma(z)} = 1\,, \tag{7}$$

it is easy to obtain that

$$\lim_{A\to 0} m_1(\omega) = \lim_{A\to 0} m_2(\omega) = x_{st}^{\omega}$$

(which means that the deterministic result is approached continuously), and that

$$\lim_{A\to a} m_2(\omega) = \lim_{A\to a} m_3(\omega) = \frac{\Gamma(1-\omega\nu+\nu/a)}{(b\mu)^{\omega\nu}\Gamma(1+\nu/a)}\,.$$

More detailed calculations (involving the asymptotic expansion of the Γ functions) show that no discontinuities are present in the derivatives. The situation is thus analogous to the case of the presence of GWN, where no transitions in moments were observed, see Graham and Schenzle [4]. The plots of $\langle x\rangle_{st}$ against A are shown in Fig. 2, and actually look like smooth functions. For $0 \neq \mu < 1$ the stationary mean value (as a function of A) has a local minimum, and for $\mu > 1$ a local maximum, at $A = 0$. For the true Verhulst model the stationary mean value is not affected by the noise.

IV. THE CRITICAL CASE

There is another interesting limit, namely for $a = 0$, which corresponds to the absence of the linear term in the deterministic version of Eq. (2). Then the determin-

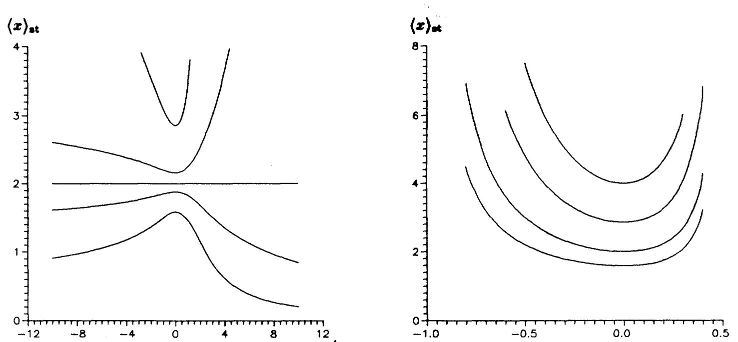


FIG. 2. Plots of $\langle x\rangle_{st}$ vs A for $b = 1$, $a = 2$, and $\mu = 2$ (the lowest), 1.1, 1, 0.9, 0.66 (left), or $b = -1$, $a = -0.5$, and $\mu = -1.5, -1, -0.66, -0.5$ (right). The curves, although given piecewise by different analytical formulas, Eqs. (3), are continuous (and even C^{∞}).

istic solution $x(t) = (x_0^\mu + b\mu t)^{-1/\mu}$ decays (if μ, $b > 0$) or diverges (for μ, $b < 0$) according to the power law. In the presence of noise, the moments of x_t growth exponentially with time in the latter, and relax to zero in the former case. The relaxation is governed by the continuous spectrum (Eq. (4.9a) of Ref. [15]). The calculation of a dominant contribution, using the saddle-point method, yields ($\lambda = 1$)

$$\langle x_t^\omega \rangle \sim \frac{\Gamma(\omega\nu)}{|b/\lambda A|^{\omega\nu}} \frac{1}{\sqrt{\pi\lambda t}} \times \begin{cases} \frac{\Gamma(1-\omega\nu+\nu/A)}{\Gamma(\nu/A)} & \text{if } \nu/A > 0 \\ \frac{\Gamma(1-\nu/A)}{\Gamma(\omega\nu-\nu/A)} & \text{if } \nu/A < 0 \end{cases} \tag{8}$$

so all the existing moments decay by the common law, like $t^{-1/2}$. It reflects the so-called critical slowing down, and supports the general considerations of Ref. [18]. Note that by taking the limit

$$A \to 0, \quad \lambda \to \infty, \quad \lambda A^2 = D \tag{9}$$

in (8) we correctly recover the appropriate result for the GWN (with the strength D) [4,19].

V. AUTOCORRELATION FUNCTION, RELAXATION

The stationary autocorrelation function (ACF) $K(\tau) = C(\tau) - \langle x \rangle_{\rm st}^2 = \lim_{t\to\infty} \langle x_{t+\tau} x_t \rangle - \langle x \rangle_{\rm st}^2$ is related to transient moments by the following expression [20]:

$$C(\tau) = \int_0^\infty dy\, P_{\rm st}(y) y \langle x_\tau(y) \rangle \,, \tag{10}$$

where $P_{\rm st}$ is defined in Eq. (4), and the time-dependent mean value $\langle x_\tau(y) \rangle$ is given—in a form of spectral decomposition involving Gauss hypergeometric functions (of the variable y)—by appropriate equations of Ref. [15]. The integration in Eq. (10) can be done by the use of a "generic" formula [21]

$$\int_0^1 du\, u^{p-1} (1-u)^{r-1} {}_2F_1(\alpha, \beta; \gamma; u) = \frac{\Gamma(p)\Gamma(r)}{\Gamma(p+r)} {}_3F_2(\alpha, \beta, p; \gamma, p+r; 1) \,, \tag{11}$$

$\mathrm{Re}(p)$, $\mathrm{Re}(r)$, $\mathrm{Re}(\gamma + r - \alpha - \beta) > 0$.

The resulting explicit form of the autocovariance function $C(\tau)$ is provided in Appendix A, Eqs. (A1), (A3b), and (A3c). These formulas have been used to calculate the autocorrelation coefficient. The comparison with the results of digital simulation is presented in Fig. 3 (the left graph).

The analysis of asymptotic properties of $K(\tau)$ for large τ becomes now elementary. In the case of $0 < \mu A \le \mu a$ [see Eq. (A1)] ACF relaxes exponentially with the rate

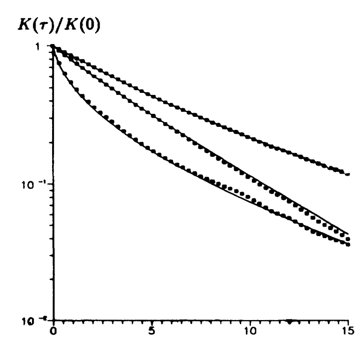
$$\alpha_{\rm LT} = -\eta(1) = \mu a - \frac{(\mu A)^2}{1 + \mu A}. \tag{12}$$

In cases (b) and (c) (see Appendix A), the relaxation is governed by the slowest convergent from the possible terms A_1, B_0, C_0, or—if they are all excluded from summation in Eqs. (A3b) or (A3c)—by the continuous spectrum $K_c(\tau)$. More detailed considerations show the following.

(i) For $\nu > 0$ only the term A_1 may appear, and it happens provided the distance between $|\nu/A|$ and $R = |\nu/A|/\sqrt{1 - a/A}$ exceeds unity (i.e., for sufficiently small $|A|$). Then, the long-time relaxation rate is given by Eq. (12). Otherwise, the relaxation is governed by the continuous spectrum and follows according to the non-purely exponential law $\tau^{-3/2}\exp(-\alpha_{\rm LT}\tau)$, where

$$\alpha_{\rm LT} = 2 - a/A - 2\sqrt{1 - a/A}. \tag{13}$$

In the critical case $a = 0$ ($\mu > 0$), $K(\tau)$ has the long-time tail $\tau^{-1/2}$.



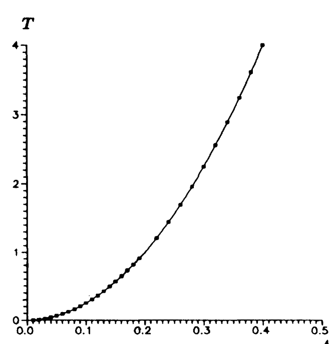


FIG. 3. Numerical results for the true Verhulst model $\mu = 1$, $a = b = 0.2$. The time-dependence of the (stationary) autocorrelation coefficient is shown on the left graph. The upper, middle, and lower solid lines, which correspond to $A = -1$, 0.1, 0.4, are calculated from Eqs. (A3c), (A1), and (A3b), respectively. Marks represent appropriate values obtained by digital simulation [22]. The right graph shows the relation (15). The data points are computed by the use of appropriate series (or series + integral) representation of T. The (power) best-fit line, $T = 25A^2$, is presented. The quadratic dependence was obtained also for negative values of A (however, for numerical accuracy reasons, $|A|$ and a should be of the order unity, or greater, in such case).

(ii) For $\nu \in (-1,0)$ and for sufficiently small $|A|$ the conclusions of (i) remain still valid. However, increasing $|A|$ successively we find such conditions that B_0 and/or C_0 are no longer excluded from summation. Because $K_c(\tau)$ vanishes faster, the relaxation becomes dominated by the discrete spectrum again (see Fig. 4).

(iii) The remaining case $\nu \leq -1$ covers in particular the special cases of negative integers ν, for which ACF is simply given by a finite sum (as well as all moments of an integer order). Therefore, one can expect that the asymptotic behavior should not depend on properties of $K_c(\tau)$. In fact, the relaxation turns out to be governed either by A_1, or possibly by B_0 for sufficiently large $|A|$.

Note that considering the long-time relaxation of $\langle x_t \rangle$ (to its stationary value) we obtain the same conclusions (i)–(iii), with the only exception that the term C_0 is not allowed at all [there are no terms corresponding to C_K in the spectral decomposition of the transient mean value; the additional poles appear in (A3a) just as a result of the integration in Eq. (10)]. In all the considerations above the values of A should be (additionally) restricted according to the proper conditions of the existence of stationary moments, see Eqs. (3).

The dependence of relaxation rates on the noise-controlling parameter is shown in Fig. 4. We see that in our model the (multiplicative) noise always slows down the process. What seems particularly interesting is that for (relatively) large $|a|$ (compared to λ) the presence of the small "negative" ($A/a < 0$) WSN results in drastic decreasing of the long-time relaxation rate. This case does not have a counterpart for GWN because in the limit (9) the condition $|a| > \lambda$ can no longer be satisfied (Sec. VI). [The relation $A = \sqrt{D/\lambda} > |a|/\lambda$, valid for sufficiently large λ, shows that case (a) (when the continuous spectrum does not appear) does not have a GWN counterpart as well.]

A few others quantities have been proposed to characterize globally the system's relaxation (not only the long-time regime). One of the most important is the so-called relaxation time T [5]

$$T = \sigma^{-1} \int_0^\infty d\tau K(\tau) \quad (A \neq 0). \tag{14}$$

If the "normalization" factor is $\sigma = 1$ or $\sigma = K(0)$ [5], T given by (14) becomes "by the definition" the relaxation time of the ACF or of the autocorrelation coefficient, respectively. Note that for the deterministic case ($A = 0$) Eq. (14) cannot be directly used, so the value of T should be, generally speaking, specified separately. The first quantity (with $\sigma = 1$) must be actually zero [23], whereas the second one should be taken $1/\mu a$ ("lifetime" of nonstationary states) [24] in the deterministic case. It might suggest to identify the relaxation time of the autocorrelation coefficient with the time of the system's relaxation. However, we will see on a particular example that just the first integral (over ACF) reflects better the influence of the noise on the system's relaxation, so it seems better to choose the σ as the noise-independent quantity, and interpret (14) as an additional noise-induced effect on the relaxation of the system.

Using Eqs. (A1) or (A3) and carrying out the integration in (14) we obtain immediately the series (or contour integral) representation of T. In the general case these expressions are rather complicated. They radically simplify for the true Verhulst model (see the last paragraph of Appendix A).

A. Relaxation time T for the true Verhulst model

We have examined numerically the dependence of T vs A ($A < 1$) for different values of the deterministic coefficients a, $b > 0$ obtaining (with excellent accuracy, see the right graph in Fig. 3)

$$T = \frac{\lambda A^2}{\sigma b^2}, \qquad \nu = 1 \, (\lambda = 1). \tag{15}$$

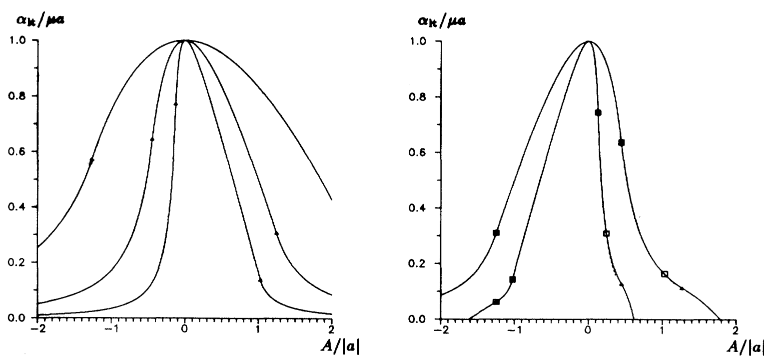


FIG. 4. (Normalized) long-time relaxation rate against the "noise ratio." The left and right graphs correspond to $\mu = 1$, $a = 0.2$ (upper), 1, 5 and $\mu = -0.2$, $a = -0.2$, -1, respectively. For the positive μ the relaxation of the mean value (to $\langle x \rangle_{st}$) and decay of the (stationary) autocorrelation function are governed by the same long-time rates: (12) if $A/|a|$ lies inside the interval determined by the positions of appropriate marks, or (13) when it is outside. Note that in the latter case the relaxation is not purely exponential. On the right graph the "triangles" and "squares" indicate the points where the "transitions" in the relaxation rates of $\langle x_t \rangle$ and of $K(\tau)$ appear, respectively. So, the "empty square" marks the position where relaxation of ACF becomes dominated by the term C_0 (see the text), and these two rates are no longer equal. The dashed line (in fact almost invisible on the graph), showing the relaxation of ACF for such a case, rapidly terminates on a certain value of $A/|a|$, starting from which ACF no longer exists.

The analytical proof of (15) is presented in Appendix B.

This very simple result shows that T given by Eq. (15) (with the noise-independent σ) is proportional to $D = \lambda A^2$ ("noise intensity" in the GWN limit). The "effective rate" $\alpha_{\rm eff} \sim T^{-1} \sim A^{-2}$ decreases with $|A|$, which remains in qualitative agreement with the previous analysis of long-time relaxation rates. Let us observe that after the integration in (14) the asymmetry with respect to the sign of A is lost.

As we have already mentioned the integral over the autocorrelation coefficient [$\sigma = K(0) = \mathrm{Var}(x)$], which is equal to $(1 - A)/a$, is not a good quantity to represent the system relaxation. It linearly decreases with A, whereas the relaxation time should increase with $|A|$. In the GWN limit (9) (cf. Marchesoni, Ref. [5]) it becomes independent on the noise intensity and equal to the "lifetime" $(1/a)$ of the deterministic nonstationary state, but this is just coincidence.

VI. CONCLUDING REMARKS

The relaxation and stationary properties of the exactly solvable Verhulst model driven by the linearly coupled white shot noise (with exponentially distributed weights) have been considered. The stationary probability density distribution, its shape, domain, and character of boundaries, have been examined versus the noise-controlling parameter A, Fig. 1. We have shown that in spite of occurrences of the so-called noise-induced transitions the averages (moments) change continuously with A, Fig. 2. The explicit expressions for the stationary autocorrelation function have been found, and the asymptotic behavior has been analyzed. It was shown that the long-time relaxation rate is continuous as a function of A, but given piecewise by different analytical formulas. The long-time relaxation rate monotonically decreases with $|A|$ (but it is not symmetric with respect to the sign of A), Fig. 4. It means that the noise slows down the relaxation. It was shown that this effect may be relatively large, even for the case of "small noise." The so-called relaxation time T (defined as the integral over the autocorrelation function) has been examined for the true Verhulst model, and we have found the simple relation $T \propto A^2$. The comparison of numerical and analytical results is provided in Fig. 3.

In Sec. V we have pointed out certain qualitative effects, which cannot appear for the case of GWN. Let us now consider the GWN limit in more details. First of all note that in order to carry out (9) we should remove the *implicit* assumption $\lambda = 1 (= D/A^2)$, replacing the dimensionless parameters a, b, and t by aA^2/D, bA^2/D, and Dt/A^2 in the formula of interest. The resulting expression, as a function of A, is usually *singular* at $A = 0$, so the GWN limit should be taken with care. In particular, limits $A \to 0^+$ and $A \to 0^-$ may be different, and then only one of them will lead to correct results. Using Eq. (4) we obtain that for the positive system parameters the first limit $A \to 0^+$ is appropriate, whereas for the negative a, b, μ the point $A = 0$ should be approached through negative values of A. In fact,

$$P_{\rm st}(x) \sim \begin{cases} x^{\beta_0^{(\pm)}} & \text{if } x \to 0 \\ x^{\beta_\infty^{(\pm)}} & \text{if } x \to \infty\,, \end{cases} \tag{16}$$

where

$$\beta_0^{(+)} = -1 + a/(D - aA) \underset{A\to 0}{\longrightarrow} -1 + a/D\,,$$

$$\beta_\infty^{(+)} = -1 - \mu - 1/A \underset{A\to 0^\pm}{\longrightarrow} \mp\infty\,,$$

$$\beta_0^{(-)} = \beta_\infty^{(+)}\,, \qquad \beta_\infty^{(-)} = \beta_0^{(+)}\,,$$

and where the sign $(+)$ or $(-)$ refers to the case of positive or negative system parameters, respectively. To be normalizable [on $(0,\infty)$] (16) requires just such a way of carrying out the limit, the way resulting in

$$P_{\rm st}(x) \propto x^{-1+a/D} \exp\Big(-\frac{bx^\mu}{D\mu}\Big)\,,$$

which is the well-known form of the stationary probability density for the GWN present. Therefore, only the WSN cases specified by the relations

$$0 < a < A \quad \text{or} \quad A < a < 0, \tag{17}$$

with dimensionless parameter $a[= O(A^2)]$ and with sufficiently small $|A|$, correspond to appropriate results for GWN. Thus, the autocovariance function for GWN (and, in particular, the behavior of the long-time relaxation rate, cf. Graham and Schenzle [4]; [19]) is given by a limiting form of Eq. (A3b) [case (b)]. Conditions (17) mean also that the areas $A/|a| < 0$ (on the left graph in Fig. 4) and $A/|a| > 0$ (on the right graph in Fig. 4) are not accessible in the case of GWN (especially, the difference between long-time relaxation rates of the mean value and of the autocorrelation is then not possible).

ACKNOWLEDGMENTS

I would like to thank Professor F. Marchesoni and Professor J. M. Sancho for bringing Refs. [5,17] to my attention. I would also like to thank Professor A. Fuliński and Dr. K. Rościszewski for helpful comments on the manuscript. This work was supported by the Polish KBN Grant No. 2.0387.91.01.

APPENDIX A: AUTOCOVARIANCE FUNCTION

Suppose that the second moment $\langle x^2\rangle_{\rm st}$ exists [see the conditions in Eqs. (3)]. Then, the stationary autocovariance function $C(\tau)$ is given by the following formulas:

(a) For $0 < \mu A < \mu a$

$$C(\tau) = Q^{-2\nu} \frac{\left[\left(\Gamma(1+\frac{\nu}{A}-\nu)\right)\right]^2 \Gamma(1+\vartheta_0)}{\frac{\nu}{A}\left[\Gamma\left(\frac{\nu}{A}\right)\right]^2 \Gamma(1-\nu+\vartheta_0)} \sum_{j=0}^{\infty} \frac{\Gamma(\nu+j)\Gamma(\vartheta_j)[j+\vartheta_j]\exp[\eta(j)\tau]}{j!\Gamma(\nu)\Gamma(1-\nu+\vartheta_j)[j+\nu/A]} \times {}_3F_2\Big(-j, \vartheta_j, 1+\frac{\nu}{A}-\nu; \frac{\nu}{A}, 1-\nu+\vartheta_0; 1\Big) \tag{A1}$$

where

$$\vartheta_z = \frac{\nu}{A} + \frac{\nu}{(a-A)(1+z\mu A)},$$

$$\eta(z) = -\mu(a-A)z - 1 + \frac{1}{1+z\mu A}.$$

This equation results from term-by-term integration in (10) [by the use of (11)], where $\langle x_\tau \rangle$ is given by Eq. (4.5) of Ref. [15].

(b) For $0 < \mu a < \mu A$ and (c) for $\mu A < 0 < \mu a$ the calculation of $C(\tau)$ is more complicated, because appropriate conditions required by (11) are not satisfied in general. Let us introduce the notation

$$a_z = \tfrac{\nu}{A} \mp z, \qquad \tilde{a}_z = \tfrac{\nu}{A} \mp \tfrac{R^2}{z},$$
$$b_z = \nu - \tfrac{\nu}{A} \pm z, \qquad \tilde{b}_z = \nu - \tfrac{\nu}{A} \pm \tfrac{R^2}{z},$$
$$c_z = \nu - \tfrac{\nu}{a-A} \mp z, \qquad \tilde{c}_z = \nu - \tfrac{\nu}{a-A} \mp \tfrac{R^2}{z},$$
$$d_z = 1 + z, \qquad \tilde{d}_z = 1 + \tfrac{R^2}{z}.$$

Here and hereafter the upper and lower signs correspond to cases (b) and (c), respectively, and $R = \sqrt{\frac{\nu^2}{(A-a)A}}$. Let

$$\Psi(z) = \frac{|Q|^{-2\nu}\Gamma(a_z)\Gamma(\tilde{a}_z)\Gamma(b_z)\Gamma(\tilde{b}_z)\Gamma(c_z)\Gamma(\tilde{c}_z)}{z\Gamma(\nu)\Gamma(\frac{\nu a}{A(A-a)})\Gamma(-z+R^2/z)\Gamma(z-R^2/z)\Gamma(a_z+\tilde{c}_z)} \exp[\psi(z)\tau]F_\mp(z)\,, \tag{A2a}$$

where

$$F_-(z) = \frac{\Gamma(1-\nu+\frac{\nu}{A})\Gamma(1-\frac{\nu}{a-A})}{\Gamma(1+\frac{\nu}{A})\Gamma(\nu-\frac{\nu}{a-A})}\,{}_3F_2\Big(\nu-1, c_z, \tilde{c}_z; a_z+\tilde{c}_z, \nu-\frac{\nu}{a-A};1\Big)\,, \tag{A2b}$$

$$F_+(z) = \frac{\Gamma(-\frac{\nu}{A})\Gamma(d_z)\Gamma(\tilde{d}_z)}{\Gamma(\nu-\frac{\nu}{A})\Gamma(\frac{\nu}{a-A})}\,{}_3F_2(a_z+\tilde{c}_z-\nu+1, c_z, \tilde{c}_z; a_z+\tilde{c}_z, d_z+\tilde{c}_z;1)\,, \tag{A2c}$$

respectively, and

$$\psi(z) = \frac{a}{A} - 2 \mp \mu(a-A)\Big(z + \frac{R^2}{z}\Big).$$

Then, the following representation of autocovariance function:

$$C(\tau) = \left[\int_{-R-i\infty}^{-R+i\infty} + \tfrac{1}{2}\oint\right] \frac{dz}{2\pi i}\,\Psi(z)\,, \tag{A3a}$$

where the second integral is taken along the counterclockwise oriented circle $C(0,R)$ of the radius R at the origin, is valid under the condition that all poles $\zeta = z_K(u)$ (where $u_\zeta = -K$, and $K = 0, 1, 2, \ldots$; $u = a, b, c$ [and d in case (c)]) lie outside the circle $C(0,R)$. All the remaining poles $z_K(\tilde{u})$ are then inside, as the images of the former poles by the inversion with respect to this circle.

The restrictions related to the location of $z_K(a)$, $z_K(b)$ [and $z_K(d)$] validate the proper integral representation of the transient mean value (Eq. (B1) of Ref. [15]), and the condition for $z_K(c)$ enables us to carry out the integration (over y) in Eq. (10) by the use of (11). Then, the final form (A2) of the integrand in (A3a) results from the identity (Luke [21])

$$ {}_3F_2(p,q,r;v,w;1) = \frac{\Gamma(v)\Gamma(w)\Gamma(x)}{\Gamma(p)\Gamma(x+q)\Gamma(x+r)} \times {}_3F_2(v-p, w-p, x; x+q, x+r;1),$$

where $x = v + w - p - q - r$.

However, all these restrictions are not essential for the convergence of the integrals in (A3a), and may be removed in the following way (compare [15]). Let $U_K = \mathrm{res}[\Psi(\zeta), \{\zeta{:}u_\zeta = -K\}]$, where $U = A$, B, C, or D, respectively, when $u = a$, b, c, or d.

The first integral in Eq. (A3a) may be evaluated as the sum of residues at the poles lying to the left of the line $\mathrm{Re}(z) = -R$, $\sum_K B_K$ in case (b) and $\sum_K(A_K + C_K + D_K)$ in case (c). The relations $U_K = -\tilde{U}_K$, which are easily verified by direct calculations of appropriate residues, show that the second integral contains implicitly the contribution $-\frac{1}{2}\sum_K(A_K+B_K+C_K)$ in case (b) and $-\frac{1}{2}\sum_K(A_K+B_K+C_K+D_K)$ in case (c); among that from the *essential singularity* (at $z = 0$).

It means that $C(\tau)$ consists of the continuous part $K_c(\tau)$ given by the integral over the circle in (A3a), and of the discrete part such that the total contribution from the successive pairs of poles $[z_K(u), z_K(\tilde{u})]$ is $B_K/2 - A_K/2 - C_K/2$ in case (b) and $A_K/2 + C_K/2 + D_K/2 - B_K/2$ in case (c), respectively. This formulation enables us to write the following generally valid expressions.

For case (b)

$$C(\tau) = K_c(\tau) - \sum_{K=0}^{K_{\max}} A_K - \sum_{L=L_{\min}}^{L_{\max}} C_L + \sum_{M=0}^{\infty}{}' B_M\,, \tag{A3b}$$

where $K_{\max} = [R - \frac{\nu}{A}]$, $L_{\min} = \max\{0, [-R-\nu+\frac{\nu}{a-A}]\}$, $L_{\max} = [R - \nu + \frac{\nu}{a-A}]$, and the prime above the summation symbol indicates that terms B_M with $M \in [\frac{\nu}{A} - \nu - R, \frac{\nu}{A} - \nu + R]$ should be removed.

For case (c)

$$C(\tau) = K_c(\tau) - \sum_{K=0}^{K_{\max}} B_K + \sum_{L=0}^{\infty}{}' A_L + \sum_{M=0}^{\infty}{}' C_M + \sum_{N=[1+R]}^{\infty} D_N\,, \tag{A3c}$$

where $K_{\max} = [R - \nu + \frac{\nu}{A}]$, and all the terms A_L with $L \in [-\frac{\nu}{A} - R, -\frac{\nu}{A} + R]$, and C_M with $M \in [-R - \nu + \frac{\nu}{a-A}, R - \nu + \frac{\nu}{a-A}]$ are excluded from the summation.

The middle sum of (A3b) and the first sum of (A3c) are empty for $\nu > 0$. For $\nu = 1$ the last series in Eq. (A3c) disappears (all D_N vanish). The term A_0 in Eqs. (A3b) and (A3c) does not depend on τ being equal to $\langle x\rangle^2_{\rm st}$. The integrand of $K_{\rm c}(\tau)$ written in the angular parametrization ($z = Re^{i\phi}$) becomes explicitly real.

In a number of special cases the Clausen function ${}_3F_2$ of the unit argument may be summed up (see, e.g., Luke [21]). For instance, for $\nu = 1$ it simply turns to be unity in Eq. (A2b) or becomes a Gauss hypergeometric function for Eqs. (A1) and (A2c), and may be expressed in terms of Γ functions,

$$ {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \qquad \mathrm{Re}(c-a-b) > 0. \quad \text{(A4)} $$

APPENDIX B: ANALYTICAL DERIVATION OF EQ. (15)

We want to show that for $\nu = 1, A < 1$

$$ T \equiv \mathcal{T}_+(\infty) \equiv \int_0^\infty d\tau K(\tau) = \frac{A^2}{b^2}, \quad \text{(B1)} $$

provided the integral converges.

Note that it is sufficient to prove (B1) in the certain area of parameters a, A only, because such a result may be extended by *analytical continuation* on the whole domain of the existence of the integral. Consider the case $0 < A < a$. Then $K(\tau)$ is given by the series (A1) (without the first term with $j = 0$), which takes the particular form $K(\tau) = \mathcal{T}_-(\infty)$

$$ \mathcal{T}_\mp(N) = \frac{a}{b^2}\sum_{j=1}^{N}\left[1 + \frac{r\alpha}{(\alpha+j)^2}\right]\frac{\{rj/(\alpha+j)\}_j}{\{\alpha+r\}_j}Q_\mp(j), \quad \text{(B2)} $$

where $\alpha = A^{-1}$, $r = (a-A)^{-1}$, $\{u\}_j = u(u+1)\dots(u+j-1)$, and

$$ Q_-(j) = \frac{1}{r}\exp\left[-\tau\frac{j(r+\alpha+j)}{r(\alpha+j)}\right]. $$

Thus, in the series representation of $T = \mathcal{T}_+$ the functions $Q_+(j) = \frac{\alpha+j}{j(r+\alpha+j)}$ are included.

Consider the contour integral

$$ T_R = \frac{a}{b^2}\oint\frac{d\xi}{2\pi i}\left[1 + \frac{r\alpha}{(\alpha+\xi)^2}\right]\frac{\alpha+\xi}{\xi(r+\alpha+\xi)} \times\sum_{j=0}^{[R]}\frac{\{r\xi/(\alpha+\xi)\}_j}{\{\alpha+r\}_j(\xi-j)}, \quad \text{(B3)} $$

where the path of integration consists of a large counterclockwise oriented circle of radius R at the origin, and of three sufficiently small (clockwise oriented) circles centered at $\xi = -\alpha - r$, $\xi = -\alpha$, and $\xi = 0$, respectively.

Evaluating the integral by the use of the Cauchy theorem one obtains

$$ T_R = \mathcal{T}_+([R]). \quad \text{(B4a)} $$

On the other hand,

$$ T_R = \oint_{C(0,R)} - \oint_{C(0,\epsilon)} - \oint_{C(-\alpha,\epsilon)} - \oint_{C(-\alpha-r,\epsilon)}, \qquad \epsilon \propto \frac{1}{R}. \quad \text{(B4b)} $$

The first integral of (B4b) vanishes like R^{-1}. In fact, this integral for large R is estimated by the value of the sum in (B3). Each summand with $j < R/2$ is estimated by $2R^{-1}\{r\}_j/\{\alpha+r\}_j$, and each term from the remaining ones, say, with $j = [R/2]+k$ by $(\frac{r}{\alpha+r})^{[R/2]+k} = (1 - A/a)^{[R/2]+k}$. Therefore, the whole sum does not exceed $2R^{-1}{}_2F_1(1,r;\alpha+r;1) + \frac{a}{A}(1-A/a)^{[R/2]}$. For $A < 1$ the last series is convergent in view of (A4). Thus, the conclusion follows.

The contribution to the second integral comes from the first term of the series in (B3) only. Calculating the residuum at the (second order) pole $\xi = 0$ one obtains A^2/b^2, which is just the right-hand side of Eq. (B1).

In order to complete the proof, we have to show that the third and fourth integrals of (B4b) vanish when $R \to \infty$. To this end note that on the paths of integration $\mathrm{Re}(\xi) < 0$. Therefore, we can replace $(\xi - j)^{-1}$ by $-\int_0^\infty dq\, e^{q(\xi-j)}$ in all terms of (B3). In the limit $R \to \infty$ the resulting series approach ${}_2F_1[1, r\xi/(\alpha+\xi); \alpha+r; e^{-q}]$. Assuming for a moment that $\mathrm{Re}(\xi) > -1$ we can carry out the integration over q, obtaining

$$ \frac{a}{b^2}\oint\frac{d\xi}{2\pi i}\left[1 + \frac{r\alpha}{(\alpha+\xi)^2}\right]\frac{\alpha+\xi}{\xi(r+\alpha+\xi)} \times\frac{\pi\Gamma(\alpha+r)\Gamma(1-\frac{r\xi}{\alpha+\xi})}{\sin\pi\xi\,\Gamma(\alpha+r+\xi)\Gamma(1-\xi-\frac{r\xi}{\alpha+\xi})}, \quad \text{(B5)} $$

where the integral is taken along $C(-\alpha,\epsilon)$ or $C(-\alpha-r,\epsilon)$, respectively. [The restriction $\mathrm{Re}(\xi) > -1$ may be removed by analytical continuation.] The latter is equal to zero because there is no singularity at $\xi = -\alpha - r$. The former, in view of Eq. (7), vanishes like $\epsilon^\alpha \propto R^{-1/A}$.

In conclusion, taking the limit $R \to \infty$ in Eqs. (B4), we get

$$ T_\infty = A^2/b^2 = \mathcal{T}_+(\infty) = T. $$

This finishes the proof.

[1] A. Schenzle and H. Brand, Phys. Rev. A **20**, 1628 (1979); R. Graham and A. Schenzle, *ibid.* **26**, 2676 (1982).

[2] J.M. Sancho and M. San Miguel, Z. Phys. B **36**, 357 (1980); **43**, 361 (1981).

[3] N.G. Van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1981); C.W. Gardiner, *Handbook of Stochastic Methods* (Springer, Berlin, 1983).

[4] M. Suzuki, K. Kaneko, and S. Takesue, Prog. Theor. Phys. **67**, 1676 (1982); M. Suzuki, S. Takesue, and F. Sasagawa, *ibid.* **68**, 98 (1982); R. Graham and A. Schenzle, Phys. Rev. A **25**, 1731 (1982); C.W. Gardiner and R. Graham, *ibid.* **25**, 1854 (1982); Y. Hamada and K. Muto, Prog. Theor. Phys. **69**, 451 (1983).

[5] J.M. Sancho, M. San Miguel, S.L. Katz, and J.D. Gunton, Phys. Rev. A **26**, 1589 (1982); S. Faetti, C. Festa, L. Fronzoni, P. Grigolini, F. Marchesoni, and V. Paleschi, Phys. Lett. **99A**, 25 (1983); F. Marchesoni, Z. Phys. B **62**, 505 (1986).

[6] C. Van Den Broeck, J. Stat. Phys. **31**, 467 (1983).

[7] P. Hänggi and P. Riseborough, Phys. Rev. A **27**, 3379 (1983); C. Van Den Broeck and P. Hänggi, *ibid.* **30**, 2730 (1984).

[8] W. Horsthemke and R. Lefever, *Noise-Induced Transitions* (Springer, Berlin, 1984); C.R. Doering and W. Horsthemke, J. Stat. Phys. **38**, 763 (1985); C.R. Doering, Phys. Rev. A **34**, 2564 (1986); J. Łuczka, M. Niemiec, and E. Piotrowski, Phys. Lett. A **167**, 475 (1992).

[9] P. Hänggi, Z. Phys. B **36**, 271 (1980); **43**, 269 (1981); G. Faraci and A.R. Pennisi, Phys. Rev. A **33**, 583 (1986); J. Masoliver, *ibid.* **35**, 3918 (1987); J. Masoliver and G.H. Weiss, Physica A **149**, 395 (1988); S.C. Lowen and M.C. Teich, Phys. Rev. A **43**, 4192 (1991); J. Łuczka and M. Niemiec, J. Phys. A **24**, L1021 (1991).

[10] H. Risken, *The Fokker-Planck Equation: Methods of Solution and Applications* (Springer, Berlin, 1983); J.J. Brey, J.M. Casado, and M. Morillo, Z. Phys. B **66**, 263 (1986); J.J. Brey, C. Aizpuru, and M. Morillo, Physica A **142**, 637 (1987); S. Succi and R. Iacono, Phys. Rev. A **36**, 5020 (1987); M.J. Englefield, J. Stat. Phys. **52**, 369 (1988); A. Debosscher, Phys. Rev. A **40**, 3354 (1989); **42**, 4485 (1990); **44**, 908 (1991); **44**, 7929 (1991); M. Kuś, E. Wajnryb, and K. Wódkiewicz, *ibid.* **43**, 4167 (1991); K.J. Phillips, M.R. Young, and S. Singh, *ibid.* **44**, 3239 (1991).

[11] A. Fuliński, Phys. Lett. A **126**, 84 (1988); Acta Phys. Pol. A **74**, 193 (1988); A. Fuliński and T. Telejko, Phys. Lett. A **125**, 11 (1991).

[12] P. Hänggi, F. Marchesoni, and P. Grigolini, Z. Phys. B **56**, 333 (1984); P. Hänggi, T. Mroczkowski, F. Moss, and P.V.E. McClintock, Phys. Rev. A **32**, 695 (1985); P. Jung and H. Risken, Z. Phys. B **61**, 367 (1985); P. Jung and P. Hänggi, Phys. Rev. A **35**, 4464 (1987); S. Faetti and P. Grigolini, *ibid.* **36**, 441 (1987); J. Łuczka, Physica A **153**, 619 (1988); Phys. Lett. A **139**, 29 (1989); N.G. van Kampen, J. Stat. Phys. **54**, 1289 (1989); P. Hänggi, P. Jung, and F. Marchesoni, J. Stat. Phys. **54**, 1367 (1989); A.E. Sitnitsky, Phys. Lett. A **162**, 155 (1992).

[13] See, e.g., Ref. [3], and A. Papoulis, *Probability, Random Variables and Stochastic Processes* (McGraw-Hill, New York, 1965); W. Feller, *Introduction to Probability and its Application* (Wiley, New York, 1966).

[14] See, e.g., Refs. [2,3,6,7] and P. Hänggi, Z. Phys. B **31**, 407 (1978); **36**, 271 (1980); **43**, 269 (1981).

[15] R. Zygadło, Phys. Rev. E **47**, 106 (1993).

[16] C. Festa, L. Fronzoni, P. Grigolini, and F. Marchesoni, Phys. Lett. **102A**, 95 (1984); F. de Pasquale, J.M. Sancho, M. San Miguel, and P. Tartaglia, Phys. Rev. A **33**, 4360 (1986); P. Jung and P. Hänggi, Phys. Rev. Lett. **61**, 11 (1988); M.R. Young and S. Singh, Phys. Rev. A **38**, 238 (1988); M.C. Torrent and M. San Miguel, *ibid.* **38**, 245 (1988); R. Mannella, C.J. Lambert, N.G. Stocks, and P.V.E. McClintock, *ibid.* **41**, 3016 (1990); G. Debnath, F. Moss, T. Leiber, H. Risken, and F. Marchesoni, *ibid.* **42**, 703 (1990); J.B. Swift, P.C. Hohenberg, and G. Ahlers, *ibid.* **43**, 6572 (1991); E. Fick, M. Fick, and G. Hausmann, *ibid.* **44**, 2469 (1991); T.C. Elston and R.E. Fox, *ibid.* **44**, 8403 (1991); S. Zhu, *ibid.* **45**, 3210 (1992).

[17] J.M. Sancho, M. San Miguel, L. Pesquera, and M.A. Rodriguez, Physica A **142**, 522 (1987).

[18] F. Sasagawa, Prog. Theor. Phys. **69**, 791 (1983).

[19] R. Zygadło, Acta Phys. Pol. A **78**, 277 (1990).

[20] Throughout this paper we distinguish, both in notation and in terminology, between three related quantities: $C(\tau)$—autocovariance, $K(\tau)$—autocorrelation, and $K(\tau)/K(0)$—autocorrelation coefficient (normalized autocorrelation).

[21] Y.L. Luke, *The Special Functions and Their Approximations* (Academic, New York 1969); I.M. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series and Products* (Academic, New York, 1965); H. Exton, *Handbook of Hypergeometric Integrals* (Horwood, Chichester, 1978).

[22] The data points of the left graph in Fig. 3 have been computed as follows. For each realization of the compound Poisson process (see Ref. [15]) the appropriate "stationary" averages along the (resulting) sample trajectory of x_t were calculated, as the arithmetical averages of the type $\langle x_{I+j}x_I\rangle_{\rm path} = n^{-1}\sum_{i=I}^{I+n} x_{i+j}x_i$. I was sufficiently large (corresponding to the dimensionless time equal 50 in our case) in order to eliminate the effect of the initial condition, and $n = 250$. The data points represent the arithmetical average over $N = 50\,000$ such values obtained for different sample realizations.

[23] The phase space available in the deterministic stationary state consists of the one point only. Therefore $K(\tau)$ $[|K(\tau)| \le K(0) = \mathrm{Var}(x) = 0]$ must be identically zero. In this sense the "relaxation" of the stationary ACF follows immediately, so the "relaxation time" (of ACF) is zero.

[24] Let us define in the deterministic case

$$T = \lim_{t\to\infty}\int_0^\infty d\tau \frac{x(t+\tau)x(t) - x_{\rm st}x(t)}{x^2(t) - x_{\rm st}x(t)}\,,$$

where $x_{\rm st} = \lim_{t\to\infty} x(t)$, and where $x(t)$ is the (deterministic) solution of Eq. (2) with $A = 0$. Using the *explicit* form of $x(t)$ (see, e.g., [19]) one obtains $T = 1/\mu a$. Note that the result is irrespective of the order, in which the limit and the integration are carried out.