Shock fluctuations in one-dimensional lattice fluids

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We present results of computer simulations of the motion of shock fronts in a variety of one-dimensional stochastic lattice models with parallel and serial dynamics, infinite and finite temperatures, and ferromagnetic and antiferromagnetic particle interactions. We find that fluctuations in the shock location, about an average determined by evolving an ensemble of systems with the same initial conditions, generically grow in time like $t^{1/3}$. We discuss the robustness of the $t^{1/3}$ growth and determine the density dependence of the coefficient in a simple case. We compare this with models where the dynamics are specially tuned so that the growth is reduced to $t^{1/4}$, and with the situation where the ensemble members have different initial conditions, in which case the growth is like $t^{1/2}$.

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I. INTRODUCTION

The familiar equations of hydrodynamics, such as Navier-Stokes and Euler, are "derived" by summing over the motion of a large number of molecules [1]. While motion at the molecular level appears random, the resulting hydrodynamic equations for the conserved quantities (momentum, energy, and mass density) are deterministic. These equations have been very successful at describing fluids on the macroscopic scale when the flow is well behaved [2, 3]. To see the molecular structure, however, one has to go beyond the macroscopic equations; this can be particularly important when the gradients in the hydrodynamic variables diverge and the hydrodynamic equations break down in which case these equations need no longer tell the whole story [1, 4]. This happens, for example, when discontinuities (such as shocks) in macroscopic variables (such as density) appear. Since these situations are not uncommon, we would like to gain a better understanding of them at both the macroscopic and microscopic levels.

The example we shall consider here is the one-dimensional analog of the Euler equations, the inviscid Burgers equation [5], written in standard form as

$$\frac{\partial \rho(x,t)}{\partial t} = -\rho(x,t)\frac{\partial \rho(x,t)}{\partial x}. \tag{1}$$

This equation, which can be derived from various microscopic models [6, 7] in the hydrodynamical scaling limit, i.e., when the ratio of microscopic to macroscopic scales formally goes to zero, is well known to permit the formation and propagation of shocks. To investigate the microscopic structure of these shocks one can use two approaches.

In the first approach one adds a conservative, stochastic noise term $\xi(x,t)$ plus a diffusive term $\partial^2\rho/\partial x^2$ to Eq. (1). This yields the noisy Burgers equation [8–10]

$$\frac{\partial \rho}{\partial t} = -\rho\frac{\partial \rho}{\partial x} + D\frac{\partial^2 \rho}{\partial x^2} + \frac{\partial \xi(x,t)}{\partial x}. \tag{2}$$

Both white noise, where the covariance is

$$\langle \xi(x,t)\xi(x',t')\rangle = 2D\delta(t-t')\delta(x-x'), \tag{3}$$

and noise terms with nontrivial correlations that are designed to more accurately model the microscopic dynamics, have been considered, [9, 11].

This approach has been used for a variety of fluid systems [8] including surface growth models [9, 12, 13]. Of interest are the statistical properties of the solutions to Eq. (2). One of the more successful techniques in the study of nonlinear stochastic partial differential equations like (2) is the dynamical renormalization-group approach [8, 9, 12–14], or, in a simpler version, the use of scaling concepts [15]. Using such methods, one can see, for example, how the resulting correlations $\langle \rho(x,t)\rho(x',t')\rangle$ behave for large $|x-x'|$ and $|t-t'|$, and how fluctuations in the fluid density profile spread [10].

The drawbacks of this type of approach are that the derivations are purely phenomenological and the assumptions made are impossible to justify rigorously. In particular the assumption of white noise, Eq. (3), may be too restrictive. It is generally expected, however, that other noise terms reduce to white noise on the time and length scales of relevance, unless one wants to consider long-range interactions between particles.

The alternate approach (and the one on which we focus here) circumvents these problems by defining the fluid

systems at the microscopic level. One simply prescribes the dynamical rules of evolution for the individual microscopic components and works from there. The advantage is that no assumptions concerning the macroscopic equation or noise term need to be made. The disadvantage, however, is that not just rigorous work, but any analytical work not reverting to the hydrodynamical level is extremely difficult, and so computer simulation is the only reliable tool for studying these systems at the microscopic level. For practical reasons this usually limits the types of interactions one can study to (often nonphysical) short-range interactions.

In this paper we study the asymptotic behavior in one-dimensional systems of shock spreading due to the randomness of the dynamical evolution, given initial conditions that are (microscopically) fixed. We find that the results of simulations confirm the predictions of exponents made by van Beijeren [10] on the basis of (2). By carrying out such simulations for a number of models we also check the class of particle dynamics for which (2) is applicable. We also address the problem of how long it takes models to get into the scaling regime, an important criterion if scaling is to have practical value.

II. MODELS

All models to be considered here are one-dimensional models with local stochastic dynamic rules, i.e., particles may make jumps between different sites with jump rates or jump probabilities determined by the occupation of sites in the direct neighborhood of the jumping particle. Configurations are denoted η; $\eta(x)$ is the number of particles at site x, $x \in \mathbb{Z}$. The parallel asymmetric simple exclusion process and the Boghosian-Levermore (BL) model are discrete time cellular automata (parallel dynamics); the other models we consider are continuous time stochastic processes (serial dynamics) for which analysis is often simpler. In continuous time it is not necessary to consider the jumps of more than one particle at a time, as the probability of this occurring is zero: each particle waits independently for an exponentially distributed time (with mean 1) and attempts to jump to another site. The location of the target site and the success or failure of the jump attempt depends on the specific model. We will provide the details below, but in all cases there will be an asymmetry with a preference for jumping to the right.

A. Shocks and shock fluctuations

For *all* of the models discussed here we begin with an ensemble of identical one-dimensional lattices with a preimposed shock, whose subsequent evolution we will track; i.e., we start initially with no particles to the left of the origin while for $x \geq 0$ there is a nonzero density of particles per site ρ. This convention is mainly for convenience. We could have instead begun with density of particles $\rho = \rho_-$ for $x < 0$ and $\rho = \rho_+$ for $x \geq 0$ with $\rho_- < \rho_+$, but tracking the shock would have been more complicated. With this simplification, however, the shock location is defined as the location of the leftmost particle [16]. The density profile as seen from this leftmost particle remains sharp (with asymmetric jump rates) in the sense that the average density profile in its moving frame approaches its asymptotic value exponentially fast (in space) [17].

Consider an initial configuration η chosen from some probability distribution μ with $\eta(x) = 0$ for $x < 0$ and a positive density for $x \geq 0$. Define the location of the shock (first particle) at time t to be x_t. If we average over all possible realizations of the dynamics resulting from this particular initial configuration, then we obtain the average location of the shock at time t for this particular initial configuration $E_\eta(x_t)$ where E_η is the expectation value taken under the fixed initial configuration η. The actual location of the shock in any one realization will deviate randomly from this average.

We are interested in how these deviations grow in time. Thus we study the growth of the variance of the shock position (with initial configuration η),

$$\sigma_\eta^2(t) \equiv E_\eta([x_t - E_\eta(x_t)]^2) \tag{4}$$

for many initial configurations η. We are also interested in average values of $\sigma_\eta^2(t)$ with respect to η—in particular, consider the product measure μ on the space of initial configurations such that $\mu\{\eta(x) = 1\} = 0$ for $x < 0$ and $\mu\{\eta(x) = 1\} = \rho_0$ for $x \geq 0$. Where computationally feasible, we study the evolution of $\sigma^2(t)$, defined by

$$\sigma^2(t) \equiv \int \mu(d\eta)\sigma_\eta^2(t) = E_\mu([x_t - E_\eta(x_t)]^2), \tag{5}$$

where E_μ is expectation with respect to the dynamics and the initial condition measure.

B. Asymmetric simple exclusion

In the asymmetric simple exclusion process (ASEP), there is at most one particle per site. A particle attempts to jump to the site immediately on its right with rate p, and to the site immediately on its left with rate $1 - p$, $1/2 < p \leq 1$. If the target site is unoccupied, then the jump succeeds; if not, then it fails. The asymmetry enhances jump attempts in one direction which induces a net particle current.

The ASEP is a good example to demonstrate the heuristics entering in the derivation of the Burgers equation. The time evolution of the average density is given by the continuity equation

$$\frac{\partial \rho(x,t)}{\partial t} = -\frac{\partial j}{\partial x}(x,t), \tag{6}$$

where $\partial/\partial x$ denotes a discrete derivative $\partial f(x)/\partial x = f(x + \frac{1}{2}) - f(x - \frac{1}{2})$. The average current between sites x and $x + 1$ is of the form

$$j(x+\tfrac{1}{2}) = \langle p\eta(x)[1-\eta(x+1)] - (1-p)\eta(x+1)[1-\eta(x)]\rangle$$
$$= (2p-1)\langle[\eta(x)+\eta(x+1)]/2 - \eta(x)\eta(x+1)\rangle - \langle\eta(x+1)-\eta(x)\rangle/2$$
$$= (2p-1)[\rho(x+1/2) - \rho^2(x+1/2)] - \frac{1}{2}\frac{\partial\rho(x+1/2)}{\partial x} + \cdots. \quad (7)$$

If we just keep the first term on the right-hand side of (7) we obtain the inviscid Burgers equation. Additional terms represent correlations (which vanish in the stationary state) and higher-order derivatives with respect to x which are neglected. The second term is a microscopic viscous correction.

An advantage of studying the ASEP is that a number of rigorous results exist [6, 17–24]. Of immediate relevance is that the rescaled particle density profiles in the ASEP are related to the solutions of the Burgers equation (1) and that the growth of the average variance has been proven to be sublinear [21]:

$$\lim_{t\to\infty} t^{-1}\sigma^2(t) = 0; \quad (8)$$

this gives an upper bound on the growth of dynamical fluctuations. A nonrigorous scaling analysis [10] indicates that the behavior of $\sigma^2(t)$ for the fluctuating Burgers equation (2) is asymptotically

$$\sigma^2(t) \sim t^{2/3}, \quad (9)$$

consistent with Eq. (8).

To test this result, we carried out simulations by a direct serial implementation of the asymmetric simple exclusion process described above. At time $t = 0$ we started a system of length $L = 1400$ by randomly depositing a particle with probability $\rho_0 = 0.5$ independently at each site x of the lattice $0 \le x < L$. Then we made 400 copies of this system and evolved them independently. Jumps were only to the right: $p = 1$. (Thus the system can be viewed as semi-infinite with no particles to the left of the origin.) A fixed, impenetrable wall at the right end of the lattice caused particles to pile up at that end. Two shocks result, one on the left sides between average densities 0 and $\frac{1}{2}$ and one on the right between average densities $\frac{1}{2}$ and 1. The dilute shock travels to the right with average speed $c = (2p-1)(1-\rho_0) = 0.5$ lattice units per Monte Carlo step (MCS) and the dense shock travels at the same speed to the left. The simulation is valid over a time such that the two shocks have not reached each other, and so we only looked at run times up to $t = 1000$ MCS's; since the time of the run is less than $L/(2c)$, they do not meet. At five-MCS intervals we calculated the average shock location for those initial conditions by averaging over location of the first particles (respectively the last holes) in each of the 400 systems. Then we determined the fluctuations of the first particles (last holes) about this average, yielding $\sigma_\eta^2(t)$ for the given initial condition. We repeated the process for a total of 110 different initial conditions (each of which was chosen independently from a product measure with average density $\rho_0 = 0.5$); this yields $\sigma^2(t)$.

In Fig. 1 we give the results of these simulations. The data indicate that by *first* averaging over the randomness in the dynamics with the initial condition fixed and *then* averaging over the initial conditions a power-law behavior for the fluctuations is induced with $\sigma(t) \sim t^{1/3}$. This result agrees with van Beijeren's scaling analysis of the fluctuating Burgers equation [10]. Note that this power law persists even to small times.

We also considered what happened when the final averaging over initial conditions was not carried out, i.e., if one just considered $\sigma_\eta(t)$ for an ensemble of systems with initial condition η. We discovered that the particular *short*-time growth can depend strongly on the initial conditions, while the long term behavior was universal. We saw this most clearly in our examination of periodic initial configurations.

For initial conditions with short periods, such as the "checkerboard" staggered particle-hole-particle-hole pattern ($\rho_0 = 0.5$), or a particle-hole-hole pattern ($\rho_0 = \frac{1}{3}$), the short- and long-time behaviors appear to correspond, so that $\sigma_\eta(t) \sim t^{1/3}$. Thus one can (to some extent) predict the average behavior of $\sigma(t)$ from such a single initial condition.

On the other hand, initial conditions with large-scale structure, such as a periodic configuration with a large period, will lead to deviations from $t^{1/3}$ behavior for short times, through a mechanism which is easily explained. (Here we consider shocks with density zero on the left; density 1 on the right is equivalent.) First consider the very low density limit $\rho_0 \ll 1$. Prior to a collision of the first particle with another, the first particle undergoes a random walk with a drift (although $p = 1$ the jump times are random). The fluctuations in its position are of order $t^{1/2}$. At longer times the particles interact, and the simple random-walk description breaks down; the interacting model then yields a crossover to $\sim t^{1/3}$.

Now consider the high-density limit $1 - \rho_0 \ll 1$. The same mechanism produces a different result: prior to the time when two *holes* have a significant probability of col-

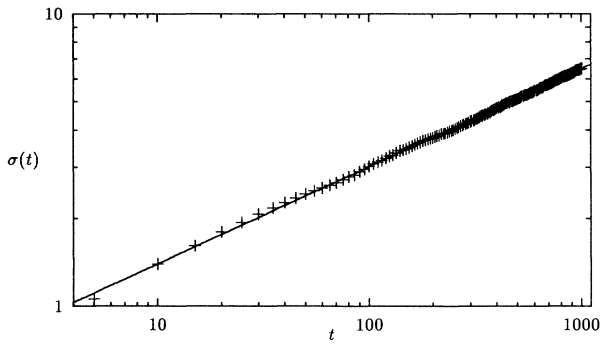

FIG. 1. Serial model, time dependence of fluctuations. The line has the form $0.65t^{1/3}$.

liding (near the shock) the *holes* undergo simple random walks with a drift. Thus provided $t^{1/2} < O((1-\rho_0)^{-1})$ the holes do not interact and the shock is hardly ever more than one site from its expected average position; the short-time behavior of $\sigma_\eta(t)$ therefore consists of maxima (with value $\frac{1}{2}$) at the expected arrival times of each hole and minima midway between expected arrival times, as seen in Fig. 2 for short times, prior to the crossover to $t^{1/3}$ behavior.

When neither the high- nor the low-density limit is applicable, the same basic mechanism is still responsible for the transient phenomena. Here larger-scale density fluctuations play the same role previously held by particles and/or holes.

C. Density dependence

If we vary the parameter D controlling the strength of the noise in Eq. (3), scaling analysis indicates that the fluctuations behave like $(D^2t)^{1/3}$ [15, 25]. In the limit where $1-\rho \ll 1$ (small hole density), $D \propto 1-\rho$. Thus we expect to see $\sigma(t) \sim (1-\rho)^{2/3}t^{1/3}$. To check this we took the density (on one side of the shock) very close to 1, and extracted the coefficient of the $t^{1/3}$ behavior. We reduced the computational load of simulating such systems by only keeping track of the holes and the distances between them, i.e., we do not keep track of attempted jumps to occupied sites; this allows one to study much larger systems in the dilute hole limit. This simulation method produces the same jumps (in the same order) as a direct simulation but at possibly different times; as long as the total number of holes is large this corresponds to an exponential waiting time distribution, like that which would be obtained by sampling from both particles and holes.

We performed an extensive series of simulations for low hole density. Typically we chose between 100 and 250 holes (with a periodic initial configuration) and observed the evolution of an ensemble of 500 to 1400 systems. Each system was allowed to evolve so that the $t^{1/3}$ behavior of $\sigma_\eta(t)$ was apparent (see Fig. 2); this required $T = 400\,000$ for the least dense system we examined $\rho_{\text{hole}} = 1/1600$. Such long times are possible for this specific set of simulations because the effective size of the system is greatly reduced by the hole tracking algorithm when the hole density is small. The coefficients of the growth are plotted in Fig. 3 and compare well with the predicted theoretical behavior.

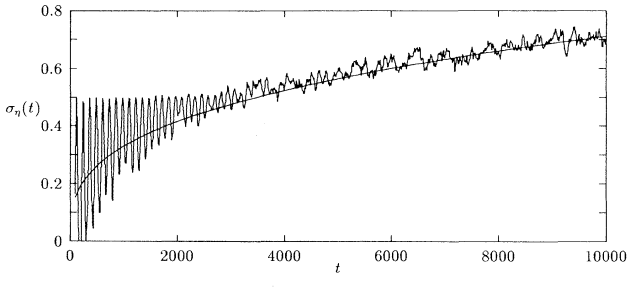

FIG. 2. Time dependence of fluctuations with density 0.992. Initial condition η has holes spaced uniformly, 125 sites apart. The solid line $0.033t^{1/3}$ gives the asymptotic behavior.

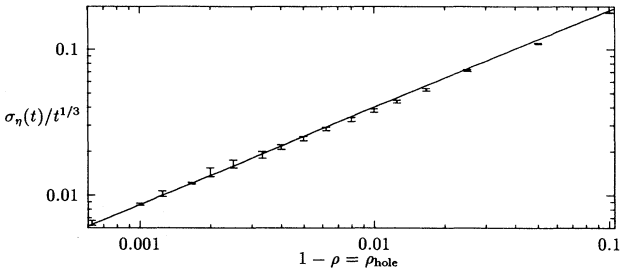

FIG. 3. Coefficient of $t^{1/3}$ growth of σ vs hole density for the ASEP. The line has the form $0.85\rho_{\text{hole}}^{2/3}$. Deviation for large ρ_{hole} is expected.

D. Parallel ASEP

We have also studied a parallel version of the dynamics, which was easily implemented on a Connection Machine. Here all particles *simultaneously* pick a direction in which to move. Each choice is made independently: to the right with probability p and to the left with probability $1-p$. As with serial dynamics, jumps are only made to unoccupied sites, but as a result of the simultaneous hopping, conflicts can arise when two particles try to occupy one previously unoccupied site at the same time. These conflicts are resolved in the following manner. After all particles have designated where they wish to jump, the number vying for each unoccupied site is determined. If no particles wish to occupy site x, then nothing is done. If exactly one particle wishes to occupy x, it succeeds. If, however, two particles attempt to occupy x (one jumping from the left and one jumping from the right), only one is chosen to succeed. This decision is random. Specifically, the particle moving to the right wins with probability p and the leftmoving one wins with probability $1-p$. Again we measured the fluctuations about the average over dynamics, given the same initial condition. Here, however, we restricted the simulations to 512 copies of a system with length $L = 2048$ and an initial staggered (periodic particle-hole) configuration at density $\rho_0 = 1/2$. We chose $p = 0.9$. Our findings were consistent with serial simulations in that the fluctuations also grew as $t^{1/3}$. We display the results in Fig. 4.

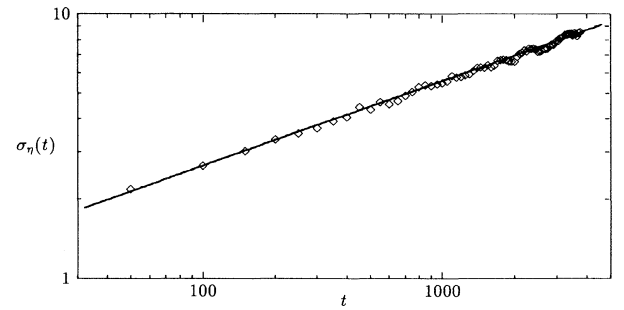

FIG. 4. $\sigma_\eta(t)$ vs time for parallel ASEP model. The line has the form $0.56t^{1/3}$.

E. Boghosian-Levermore model

The BL model [26] exists on a one-dimensional lattice whose sites can accommodate either 0, 1, or 2 particles. At each site there are two velocity modes—one for right-moving particles and one for left-moving particles. An exclusion prevents a single mode from being occupied by more than one particle at a time. The (discrete-time) evolution consists of two steps. First is advection. In this step each particle moves one lattice unit in the direction of its velocity. Second is the collision. When there are either zero or two particles at a site, nothing is done (the collision is trivial). If there is one particle at a site, then with probability $p > \frac{1}{2}$ it is placed in the right-moving mode, and with probability $1-p$ it is put in the left-moving mode. The asymmetry gives rise to a net current, as in the ASEP. Also like the ASEP, solutions of Burgers-like equations properly characterize the behavior of the BL model when viewed on a large scale [26, 27]. In the case of weak asymmetry, the Burgers equation can be proven to give the correct hydrodynamic limit [28], while in general, the appropriate equation on the Euler time scale is

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}\frac{1}{2}\left[1-(2p-1)^2\rho(2-\rho)\right]^{1/2} = -\frac{2p-1}{2\left[1-(2p-1)^2\rho(2-\rho)\right]^{1/2}}\frac{\partial}{\partial x}\rho(2-\rho), \tag{10}$$

where ρ is the spatially rescaled particle density.

Recently, by mapping the BL model onto the six-vertex model, Spohn and Gwa [29] proved that correlations in the BL model scale in the same way, $t^{-2/3}$, as the noisy Burgers equation (2), which suggests that shock fluctuations in the BL model should also scale as $t^{1/3}$. To check this, we performed a number of simulations on the BL model.

While we considered a wide range of parameters, we only performed extensive simulations on one system (preliminary analysis of the others indicated qualitatively identical behavior). We considered $p = 0.75$; the initial condition had all the left-moving states empty; the right-moving states to the left of the origin were empty and those to the right of the origin occupied.

The BL model possesses the subtlety that the odd and even sublattices decouple from each other [27]; we were thus concerned about using the leftmost particle as the shock identifier. To this end we studied four measures of the shock position: the leftmost particle, the leftmost particle on the even sublattice, the leftmost particle on the odd sublattice, and the average of the leftmost particle on the even *and* odd sublattices. As the data were similar for all four, we only present those for the leftmost particle. In these simulations we considered an ensemble of 1500 systems of length $L = 10\,000$. Every 30 MCS's we computed the shock positions and $\sigma_\eta^2(t)$. This is plotted in Fig. 5; notice that it takes quite a long time to establish the $t^{1/3}$ asymptotic behavior.

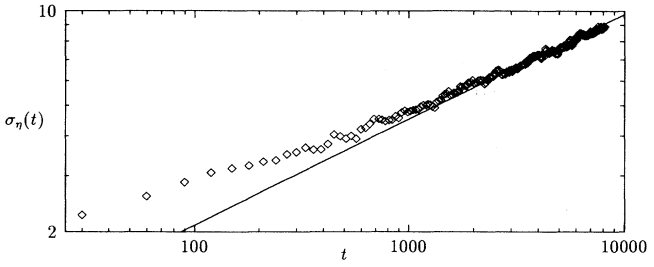

FIG. 5. $\sigma_\eta(t)$ vs time for the Boghosian-Levermore model. The line has the form $0.45t^{1/3}$.

F. ASEP with temperature

One can modify the ASEP to include a nearest neighbor interaction among particles [30, 31]. Let the configuration energy be given by

$$H(\eta) = -2J\sum_{i-j=1}\eta(i)\eta(j), \tag{11}$$

so that for $J > 0$ (ferromagnetic), the particles "attract" each other, while for $J < 0$ (antiferromagnetic), they repel. The asymmetry in the jump rates is now generated by an electric field with strength E which drives particles preferentially to the right. Starting with a configuration η, we select a pair of neighboring sites x, y and exchange their occupancy, resulting in a new configuration indicated by η^{xy}. The change in configuration energy and the work done by or against the field to perform this exchange is [30]

$$\Delta H = H(\eta^{xy}) - H(\eta) - E(x-y)\left[\eta(x)-\eta(y)\right]. \tag{12}$$

We imagine that the system is in contact with a heat bath at inverse temperature β, and accept or reject the exchange in a Metropolis-like manner. To determine whether the exchange is performed, we calculate $\phi = \exp(-\beta\Delta H)$ and select a uniformly distributed random number r, $0 \le r \le 1$. If $r \le \phi$, then the exchange is performed, if not, then it fails. [This model belongs to what is commonly referred to as the class of driven diffusive systems (DDS's), many of whose properties have been studied extensively [30–33].]

We performed simulations on the ferromagnetic model and found that they were consistent with $t^{1/3}$ behavior for $\sigma(t)$; the addition of the interaction did not seem to affect the qualitative behavior of the interface.

From a physical viewpoint a more interesting situation occurs when $J < 0$ [34, 35]. The ground state ($\beta = \infty$) of the equilibrium system $E = 0$ for $\rho = \frac{1}{2}$ is a staggered particle-hole configuration. At $\beta = \infty$ and $\rho = \frac{1}{2}$ there is no (steady-state) current for $|E| < 2J$, while for $\rho \neq \frac{1}{2}$ a small field does produce a current. One sees remnants of this behavior at low (but finite) temperature: at $\rho = \frac{1}{2}$ the only excitations are thermally activated and thus the current is exponentially small (in β); for $\rho \neq \frac{1}{2}$ there are free excitations and the current is much larger.

This difference in scales has a significant effect on the behavior of the shock fluctuations. For a given initial

configuration η with density $\rho = \frac{1}{2}$, the details of η will determine the shock fluctuations so long as the current due to excess "free" excitations is of the same order as the thermally activated excitations. For low temperatures, it may therefore be a very long time before the asymptotic $t^{1/3}$ behavior reveals itself: if $-\beta(2J + E)$ is large then thermally activated excitations are produced at a rate approximately given by

$$\exp[\beta(2J + E)] \tag{13}$$

while their rate of destruction is approximately given by the product of the excess particle and hole densities $\rho^2_{\text{excit}}/4$. This leads to the approximate equation

$$\frac{\partial \rho_{\text{excit}}}{\partial t} = \exp[\beta(2J + E)] - \rho^2_{\text{excit}}/4 \tag{14}$$

for the excitation density. For long times ρ_{excit} will approach its asymptotic density as

$$2\exp[\beta(J + E/2)] + \text{const} \times \exp[-e^{\beta(J+E/2)}t]. \tag{15}$$

This implies that the asymptotic behavior of $t^{1/3}$ should be valid in the region $\exp[-e^{\beta(J+E/2)}t] \lesssim 2\exp[\beta(J + E/2)]$. We examined a system where the parameters were $J = -4$, $E = 2$ and $\beta = 1$, so that the asymptotic result should be valid for $t \gtrsim 100$. This is the behavior we see in Fig. 6.

To reduce the problem of excess initial noise, we also considered the antiferromagnetic model where we chose the initial condition to contain the appropriate level of thermal noise. This was done by allowing a larger system (started with an η chosen from a product measure) to evolve for a long time. A relevant portion of this system was then used as the initial condition for the ensemble; we see in Fig. 6 that the $t^{1/3}$ behavior begins at a much earlier time than when we start with product measure.

G. Variable initial conditions

In all of the previous models, we have been considering the fluctuations about a fixed initial condition, i.e., fluctuations *due solely to the dynamics.* For completeness we wish to compare this with the result obtained when one

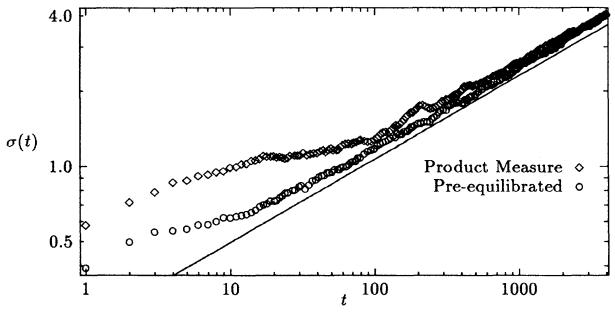

FIG. 6. Growth of shock fluctuations for model with antiferromagnetic interactions. In one case (◇) the initial condition is chosen from product measure (with density $\frac{1}{2}$), in the other (○) the system has been "preequilibrated." The solid line has the form const$\times t^{1/3}$.

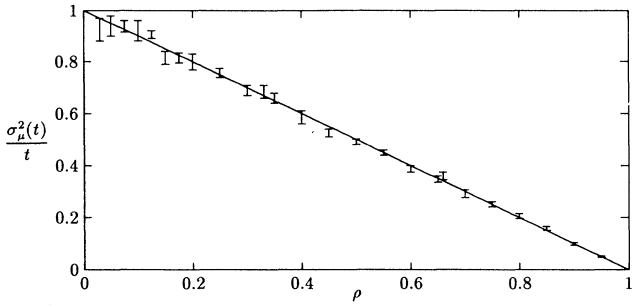

FIG. 7. Coefficient of linear time growth of σ^2_μ vs density for the ASEP. The line is $\sigma^2_\mu(t)/t = 1 - \rho$.

immediately averages over the initial conditions, obtaining the total variance:

$$\sigma^2_\mu(t) \equiv E_\mu([x_t - E_\mu(x_t)]^2) \ . \tag{16}$$

In the ASEP, one expects $\sigma^2_\mu(t) \sim t$: this has been proven for the case where the density is zero on one side of the shock in which case the limiting behavior of $x_t - E_\mu(x_t)$ is Brownian motion [16, 20] with diffusion constant $1 - \rho$.

There is a simple explanation for this behavior. Denote the density to the left of the shock by ρ_- and to the right by ρ_+. The variance of the shock position is the variance in the number of particles that it "sweeps through" [divided by $(\rho_+ - \rho_-)^2$ since one particle moves the shock a distance $1/(\rho_+ - \rho_-)$]. The shock moves with velocity $1 - \rho_+ - \rho_-$. The particles it is "running into" are moving with velocity $1 - \rho_+$ or $1 - \rho_-$, but those are irrelevant *phase* velocities. What is relevant is the *group* velocity $v_{\text{group}} = \partial j/\partial\rho = 1 - 2\rho$, since fluctuations on the basic density travel with this velocity.

The variance per "effective site" hitting the shock is $\rho_+(1 - \rho_+)$ on the right, $\rho_-(1 - \rho_-)$ on the left. The number of "effective sites" hitting the shock from either side after time t is $|v_{\text{group}} - v_{\text{shock}}|t = (\rho_+ - \rho_-)t$. Thus,

$$\begin{aligned}\sigma^2_\mu(t) &\approx (\rho_+ - \rho_-)^{-2}(\rho_+ - \rho_-)t[\rho_+(1 - \rho_+) \\ &\quad + \rho_-(1 - \rho_-)] \\ &= [\rho_+(1 - \rho_+) + \rho_-(1 - \rho_-)]\, t\big/(\rho_+ - \rho_-).\end{aligned} \tag{17}$$

In the case $\rho_- = 0$, $\rho_+ = \rho$ this reduces to

$$\sigma^2_\mu(t) \approx (1 - \rho)t. \tag{18}$$

In Fig. 7 we present our simulation results for this case which clearly agree with the theoretical results.

III. SYSTEMS WITH SPECIAL SYMMETRY

All the models discussed so far share one common feature: the current-density relationship is noncritical, i.e., $\partial^2 j/\partial\rho^2 \neq 0$. Since this is expected to have important repercussions [15], we also investigated a model tuned to make $\partial^2 j/\partial\rho^2 = 0$.

Consider an asymmetric exclusion process with the following jump probabilities:

(i) With probability $\frac{2}{3}$, attempt to jump to the nearest neighbor on the right. Succeed if the target site is empty.

(ii) With probability $\frac{1}{3}$, attempt to jump four sites to the right. Succeed if the target site is empty *and* the intervening three sites are either *all* occupied or *all* unoccupied.

In a product measure with density ρ (it is easily shown that product measure is stationary) the current is thus

$$j = \tfrac{2}{3}\rho(1-\rho) + 4\cdot\tfrac{1}{3}\rho\left[\rho^3 + (1-\rho)^3\right](1-\rho) = 2\rho(1-\rho)(1-2\rho+2\rho^2). \tag{19}$$

The derivatives of j are

$$\frac{\partial^2 j}{\partial \rho^2} = 12(1-2\rho)^2, \tag{20a}$$

$$\frac{\partial^3 j}{\partial \rho^3} = 48(2\rho-1), \tag{20b}$$

$$\frac{\partial^4 j}{\partial \rho^4} = 96. \tag{20c}$$

Thus at $\rho = \frac{1}{2}$, $\partial^2 j/\partial\rho^2 = \partial^3 j/\partial\rho^3 = 0$. The RG analysis is therefore different at this point than in the generic case, and may produce different scaling behavior for the fluctuations.

We performed a series of simulations for this model, considering both checkerboard and random initial conditions. For the random case, we chose 36 different initial conditions from a density $\frac{1}{2}$ product measure. For each initial condition η we considered an ensemble of 200 systems of length $L = 2240$; $\sigma_\eta^2(t)$ was computed every five MCS up to $t = 2000$. Then $\sigma^2(t)$ was determined by averaging the 36 different $\sigma_\eta^2(t)$. For the checkerboard initial condition we considered an ensemble of 800 systems of length 4400 which were allowed to run until $t = 4000$. This data is presented in Fig. 8. The asymptotic behavior appears consistent with $t^{1/4}$ behavior, *not* $t^{1/3}$; the single checkerboard simulation accurately depicts average behavior. The value of $\frac{1}{4}$ for the growth exponent is what one expects from an RG analysis given that the second and third derivatives of the current vanish.

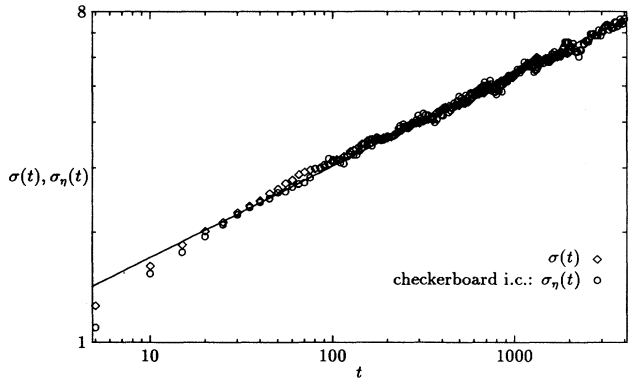

FIG. 8. Exclusion process with two types of jumps, time dependence of fluctuations. The line has the form $0.96t^{1/4}$.

IV. DISCUSSION

Our simulations give strong evidence to a universal behavior of corrections to hydrodynamics. Changes both in the interaction potential and in the details of the dynamics apparently do not affect the dominant $t^{1/3}$ fluctuations in the shock position at fixed initial configuration while changes in symmetry change the exponent from $\frac{1}{3}$ to $\frac{1}{4}$. These results are in agreement with the results obtained (or obtainable) from the fluctuating Burgers equation. The question thus arises of why is the fluctuating Burgers equation as robust as it apparently is? The first thing to remark is that the "derivation" described in Eqs. (6) and (7) is easily generalized—one combines a continuity equation with a constitutive equation expressing the average current density j in terms of $\rho(x,t)$ and its spatial derivatives. The constitutive equation can be obtained by using $j(\rho)$, the average current in a stationary state of average uniform density ρ, and supplementing it with a diffusive term of the form $-D\partial\rho/\partial x$. If the constitutive equation contains a term linear in ρ, this can be removed by an appropriate Galilean transformation. A quadratic term gives rise to the first term on the right-hand side of (2). Notice that the coefficient can always be adjusted by scaling ρ.

If the current depends on density in a local way, and a quadratic term is present, it will be the dominant nonlinearity: simple scaling arguments show that all higher-order local nonlinearities or local linear terms with second or higher derivatives with respect to x become irrelevant in the long-time large-scale limit. Generically a quadratic term will be present and it takes special care to suppress it, although this can be done, as we showed in Sec. III.

The other ingredient in the fluctuating Burgers equation is the assumption of the local nature of the noise. This is based on the assumption that the local structure is determined completely by the local density and relaxes rapidly to the characteristic structure imposed by the actual form of $\rho(x,t)$. While the precise form of the noise is not expected to be important the local character of the fluctuating current plays a crucial role in the derivation of the scaling laws. Zhang [11] recently showed that if one changes this character of the noise in the equivalent Kardar-Parisi-Zhang–type equation [12], the long-time behavior of density-density correlation functions (and as a consequence also that of shock fluctuations) may change drastically. In models with simple local dynamics, in which the only independent slowly decaying field is the density, it is hard to envision any mechanism by which the deviations of the instantaneous local currents from their expected values at a given density field could be correlated to the random currents at previous times. In none of the models considered here do we think that there are any additional independent slow variables. While it is true that in the models considered here the full current-current correlation function exhibits a $t^{-2/3}$ long-time tail, which will manifest itself in the form

$$\langle j(x,t)j(x',t')\rangle \sim t^{-4/3} f(|x-x'|/|t-t'|^{2/3}), \tag{21}$$

this basically is a correlation between the random current at the initial time and the resulting systematic current at the final time.

Of course the inclusion of sufficiently long-ranged interparticle interactions could also change the long-time behavior of shock fluctuations, but this falls outside the scope of the present paper.

In two and higher dimensions the fluctuating Burgers equation is also expected to be widely applicable. The computational demands of multidimensional systems almost certainly rules out a comprehensive numerical investigation, however.

Will any of the fluctuation behavior observed in stochastic lattice gases carry over into models with deterministic dynamics such as real physical systems? In one sense, this is the true test of the universality of the fluctuating Burgers equation. Clearly we can not export our results directly as the concept of "first averaging over the dynamics and then over the initial conditions" is trivial in the deterministic case, but this concept is easily generalized so that it can be applied. Additionally, under certain conditions, we can expect other aspects, such as current correlations, to behave similarly [15].

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