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Spin-depolarization mechanisms due to overlapping spin resonances in synchrotrons

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We studied depolarization mechanisms of polarized-proton acceleration in high-energy accelerators with snakes and found that the perturbed spin tune due to the imperfection resonance plays an important role in beam depolarization at snake resonances. We also found that *even*-order snake resonances exist in the overlapping intrinsic and imperfection resonances. Due to the perturbed spin tune of imperfection resonances, each snake resonance splits into two. Thus the available betatron tune space becomes smaller. Some constraints on polarized-beam colliders were also examined.

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I. INTRODUCTION

The spin equation of motion for a spin particle, governed by the magnetic interaction between the magnetic dipole moment of the particle and the static magnetic field in a synchrotron, is given by the Thomas-BMT equation [1] (where BMT denotes Bargmann-Michel-Telegdi),

$$\frac{d\mathbf{S}}{dt} = \frac{e}{\gamma m}\mathbf{S} \times [(1+G\gamma)\mathbf{B}_\perp + (1+G)\mathbf{B}_\parallel], \tag{1.1}$$

where $\mathbf{B}_\perp$ and $\mathbf{B}_\parallel$ are the transverse and longitudinal components of the magnetic fields with respect to the velocity vector, $\boldsymbol{\beta}$. In a planar synchrotron, vertical magnetic fields are needed to guide the orbiting particle around a closed path. Thus the spin vector is precessing with respect to the vertical axis at a frequency $G\gamma f_0$, where f_0 is the revolution frequency, $G = \frac{g}{2} - 1$ is the anomalous magnetic g factor, and γ is the relativistic Lorentz factor. The quantity $G\gamma$, representing the number of spin precessions per revolution, is called the spin tune.

In a synchrotron, strong quadrupole fields are also needed to focus the beam to a small size. Those particles moving off-center vertically in quadrupoles will experience horizontal fields, which will kick the spin vector away from the vertical axis. Since quadrupole magnets and the particle closed orbits are periodic in a circular accelerator and the betatron and the synchrotron motions are quasiperiodic, perturbing kicks to the spin vector can be decomposed into harmonics, K, given by

$$K = n + m\nu_z + \ell\nu_x + k\nu_{\text{syn}}, \tag{1.2}$$

where ν_z, ν_x and ν_{syn} are, respectively, the vertical betatron, the horizontal betatron, and the synchrotron tunes, and k, ℓ, m, n are integers. The imperfection resonances, due to the vertical closed orbit errors, are located at integer harmonics, $K = n$. The intrinsic resonances, due to the vertical betatron motion, are located at $K = nP + \nu_z$, where P is the superperiodicity of the accelerator. Other depolarizing resonances arise from linear or nonlinear betatron coupling, vertical dispersion, synchro-beta coupling, and random field errors. When the spin-precession frequency is in phase with the harmonics of perturbing kicks in a synchrotron, i.e.,

$$G\gamma = K, \tag{1.3}$$

these spin perturbing kicks add up coherently every revolution around the ring. Therefore the beam can be depolarized.

To avoid a spin-resonance condition, Derbenev and Kondratenko [3] proposed to use a local spin rotator, which rotates the spin vector 180° about an axis in the horizontal plane. These spin rotators are called snakes. Using snakes in an accelerator, the spin tune of the particle can become $\frac{1}{2}$ and independent of energy. The resonance condition of Eq. (1.3) can therefore be avoided.

However, subsequent studies show that when the resonance strength is large, new spin-depolarizing resonances occur at some fractional betatron tunes. These resonances are called *snake resonances* [4]. Snake resonances, due to coherent higher-order spin-perturbing kicks, are located at

$$\nu_s + \ell K = \text{ integer}, \quad \ell = 1, 3, 5, 7, \ldots, \tag{1.4}$$

where ν_s is the spin tune and K is the spin-depolarizing resonant harmonic. For $\nu_s = \frac{1}{2}$, we expect that snake resonances occur at the following fractional betatron tunes:

$$\nu_z = \tfrac{1}{2}, \tfrac{1}{6}, \tfrac{5}{6}, \tfrac{1}{10}, \tfrac{3}{10}, \tfrac{7}{10}, \tfrac{9}{10}, \tfrac{1}{14}, \tfrac{3}{14}, \ldots. \tag{1.5}$$

Here the lowest-order snake resonance has been observed [5]. Other higher-order snake resonances have been identified in numerical simulation shown in Fig. 1, where the final vertical spin vector, after passing through an isolated intrinsic spin resonance in an accelerator with two snakes, is plotted as a function of the vertical betatron tune ν_z. Higher-order snake resonances become important when the resonance strength is larger than $\epsilon \geq 0.1$. At a larger resonance strength, e.g., $\epsilon \approx 0.4$, snake resonances up to the $\ell \leq 7$ are important and $\epsilon \approx 0.5$, $\ell \leq 11$ are also important.

From Fig. 1, it is interesting to note that the numeri-

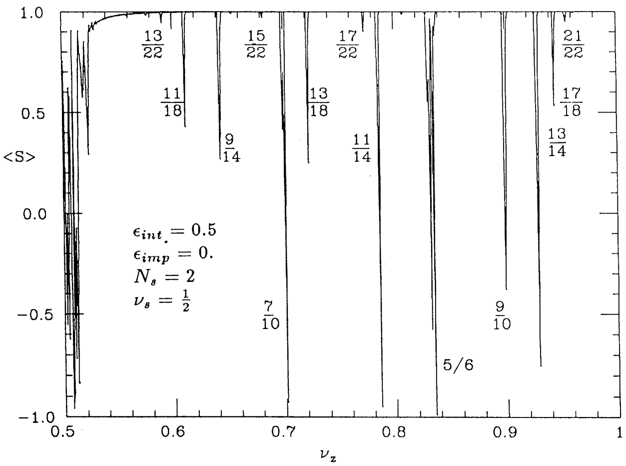

FIG. 1. The final vertical spin vector after passing through an intrinsic depolarization resonance with strength $\epsilon_{\rm int} = 0.5$ of an accelerator with two snakes is plotted as a function of the fractional part of the vertical betatron tune.

cal simulations show no apparent even-order snake resonances at

$$\nu_s + \ell K = \text{ integer}, \quad \ell = 2, 4, 6, 8, \ldots . \tag{1.6}$$

Several reasons for the nonexistence of even-order snake resonances were given in the past [4, 6]. However, the situation has never been tested in the case of overlapping resonances. With overlapping resonances, the cancellation of the depolarization perturbation is not guaranteed and the coherent kicks due to the imperfection resonance may induce strong perturbation to the spin vector. This may lead to beam depolarization at even-order snake resonances; therefore, careful studies are needed.

Overlapping resonances are important in high-energy accelerators. Important intrinsic spin-depolarizing resonances are located at harmonics nearest to [6]

$$K = mP \pm \nu_z \approx kPM \pm \nu_B, \tag{1.7}$$

where m and k are integers, P is the superperiodicity, M is the number of FODO cells per superperiod, and ν_B is the total accumulated betatron tune of those FODO cells which contain dipole magnets [for an explanation of FODO, see text after Eq. (3.1)]. Thus, important intrinsic resonances are well separated. On the other hand, an important imperfection resonance will occur at the integer nearest to an important intrinsic resonance. Therefore overlapping intrinsic and imperfection resonances constitute the most important problem in the spin dynamics during polarized-proton acceleration.

Previous studies [7] of overlapping resonances indicated that when the betatron tune is chosen *properly*, i.e., far away from low-order snake resonances, the *tolerable* or *critical* intrinsic resonance strength is given by

$$\epsilon_{\rm int,c} \le \tfrac{1}{5} N_s, \tag{1.8}$$

where N_s is number of snakes. However, these studies leave many open questions. Where is the proper tune? What is the depolarization mechanism for overlapping resonances? What are essential effects of imperfection resonances?

Without a detailed understanding of spin-depolarization mechanisms, it would be difficult to design hardware requirements for polarized colliders. This paper is intended to investigate spin-depolarization mechanisms of overlapping intrinsic and imperfection resonances. We organized the paper as follows. In Sec. II, we will study the spin tracking equation and investigate depolarization mechanisms. Effects of an imperfection resonance near an intrinsic resonance will also be studied. The dynamics of possible even-order snake resonances will be examined. Section III addresses the hardware requirements for a polarized collider. The conclusion is given in Sec. IV.

II. SPIN-DEPOLARIZATION MECHANISMS IN A SYNCHROTRON

In a synchrotron, the Thomas-BMT equation can be cast into the equation for the two-component spinor [8], Ψ, as

$$\frac{d\Psi}{d\theta} = -\frac{i}{2}\begin{pmatrix} G\gamma & -\xi \\ -\xi^* & -G\gamma \end{pmatrix}\Psi , \tag{2.1}$$

where components of the spin vector are given by $S_i = \langle\Psi|\sigma_i|\Psi\rangle$, θ is the orbital bending angle, and ξ arises from nonvertical magnetic fields in a synchrotron and is the main source for the beam depolarization. In a perfect circular accelerator, where $\xi = 0$, the spinor is transformed according to

$$\Psi(\theta_f) = e^{-\frac{i}{2}G\gamma(\theta_f-\theta_i)\sigma_3}\Psi(\theta_i). \tag{2.2}$$

Thus the spin closed orbit is in the vertical axis for a perfect planar circular accelerator. Any spin vector deviating from the vertical direction will precess about the vertical axis at a rate of $G\gamma$ turns per orbital revolution.

Due to periodic structure of a circular accelerator and the quasiperiodicity of the betatron and synchrotron motions, ξ can be expanded in Fourier harmonics as

$$\xi = \sum_K \epsilon_K e^{-iK\theta}. \tag{2.3}$$

The Fourier amplitude ϵ_K is called the resonance strength, and the corresponding frequency K of Eq. (1.2) is called the resonance tune.

In this section, we will study spin-depolarization mechanisms step by step by first deriving the spin transfer matrix for a single depolarization resonance without and with snakes and the requirements of snake configurations in accelerators. Using the evolution equation for the spin transfer matrix, we will discuss effects of the perturbed spin tune and snake resonances. The effects of overlapping intrinsic and imperfection resonances are then examined. Finally imperfections related to snakes will be discussed.

A. Spin transfer matrix of a single spin resonance

For a single resonance, i.e., $\xi(\theta) = \epsilon e^{-iK\theta}$, the spinor equation of motion can be solved analytically. Assuming a slow or zero acceleration rate, the equation for the spinor can be transformed to

$$\frac{d\Psi_K}{d\theta} = \frac{i}{2}\hat{n}_{\mathrm{CO}} \cdot \boldsymbol{\sigma}\Psi_K,$$

where the spinor wave function in the resonance precessing frame, Ψ_K, is given by

$$\Psi_K(\theta) = e^{\frac{i}{2}K\theta\sigma_3}\Psi(\theta). \tag{2.4}$$

Here $\hat{n}_{\mathrm{CO}}$ is the spin closed orbit in the resonance precessing frame given by

$$\begin{aligned} \hat{n}_{\mathrm{CO}} &= \frac{1}{\lambda}[\delta\hat{e}_3 + \epsilon_R\hat{e}_1 - \epsilon_I\hat{e}_2], \\ \lambda &= (\delta^2 + |\epsilon|^2)^{1/2}, \\ \delta &= K - G\gamma. \end{aligned} \tag{2.5}$$

with $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ as orthonormal bases corresponding to radially outward, longitudinal, and vertical axes. Thus the evolution of the spinor wave function is given by

$$\begin{aligned} \Psi(\theta_f) &= e^{-\frac{i}{2}K\theta_f\sigma_3}e^{\frac{i}{2}\lambda\hat{n}_{\mathrm{CO}}\cdot\sigma(\theta_f-\theta_i)}e^{\frac{i}{2}K\theta_i\sigma_3}\Psi(\theta_i) \\ &= t(\theta_f, \theta_i)\Psi(\theta_i), \end{aligned} \tag{2.6}$$

where for an intrinsic resonance, the spin closed-orbit vector, $\hat{n}_{\mathrm{CO}}$, of the resonance frame is precessing around the vertical axis, $\hat{e}_3$. For an imperfection resonance, the spin closed orbit is stationary at every azimuth position in the ring.

At $\delta = \pm|\epsilon|$, the spin closed-orbit vector is tilted 45° away from the vertical axis. The system has three eigenvalues, 0 and $\pm i\lambda$, which correspond to three eigensolutions describing the spin vector along the spin closed orbit and the spin vectors precessing right or left with respect to $\hat{n}_{\mathrm{CO}}$.

The matrix $t(\theta_f, \theta_i)$ of Eq. (2.6) is the spin transfer matrix, whose components are given by

$$t_{11}(\theta_f, \theta_i) = ae^{i[c-K(\theta_f-\theta_i)/2]}, \tag{2.7}$$

$$\begin{aligned} t_{12}(\theta_f, \theta_i) &= ibe^{-i[d+K(\theta_f+\theta_i)/2]}, \\ t_{21}(\theta_f, \theta_i) &= -t_{12}^*(\theta_f, \theta_i), t_{22}(\theta_f, \theta_i) = t_{11}^*(\theta_f, \theta_i), \end{aligned} \tag{2.8}$$

with

$$\begin{aligned} b &= \frac{|\epsilon|}{\lambda}\sin\left(\lambda\frac{\theta_f - \theta_i}{2}\right) = (1 - a^2)^{1/2}, \\ c &= \arctan\left[\frac{\delta}{\lambda}\tan\left(\lambda\frac{\theta_f - \theta_i}{2}\right)\right], \end{aligned} \tag{2.9}$$

and $d = \arg(\epsilon^*)$. The parameter b is the effective resonance strength with a maximum amplitude $\frac{|\epsilon|}{\lambda}$. The parameter δ is the distance between the spin tune and the resonance tune. The off-diagonal matrix elements, t_{12} and t_{21}, are depolarization driving terms.

The spin closed orbit $\hat{n}_{\mathrm{CO}}$ of Eq. (2.5) is precessing at a frequency K around the vertical axis in the laboratory frame. If the proton spin vector is injected along the vertical direction, the final spin vector will precess around the spin closed orbit $\hat{n}_{\mathrm{CO}}$ at a precession frequency λ. The net vertical spin vector is then given by

$$\begin{aligned} \langle S_3 \rangle &= |t_{11}|^2 - |t_{12}|^2 = 1 - 2b^2 \\ &= 1 - 2\frac{|\epsilon|^2}{\lambda^2}\sin^2\left[\frac{\lambda}{2}(\theta_f - \theta_i)\right]. \end{aligned} \tag{2.10}$$

If all particles in the bunch have identical spin tunes, $G\gamma$, then the polarization vector will precess around the spin closed orbit $\hat{n}_{\mathrm{CO}}$ without depolarization. On the other hand, if there are spin tunes spread in the bunch, the spin vectors of the bunch will decohere and the remaining polarization is the projection of the spin vectors of particles in the bunch onto the spin closed orbit $\hat{n}_{\mathrm{CO}}$. The measurable polarization in the accelerator will be the projection of the spin closed orbit onto the vertical axis.

Equation (2.10) indicates that the depolarization will occur at a spin-resonance condition $G\gamma = K$, where $\lambda = |\epsilon|$. To avoid the spin-resonance condition, the betatron tune can be changed suddenly during the acceleration. Note here that the $\hat{e}_3$ component of the spin closed-orbit vector, $\hat{n}_{\mathrm{CO}}$, changes sign in passing through a resonance. A sudden tune jump will cause a nonadiabatic transition of the spin closed-orbit vector. The polarization, which survives in the tune jump, is the projection of the initial spin closed-orbit vector onto the final spin closed-orbit vector, i.e.,

$$P_f = -\frac{\delta_1\delta_2 + |\epsilon|^2}{\lambda_1\lambda_2}P_i, \tag{2.11}$$

where P_f and P_i are the final and initial polarization, $\delta_1 = K_1 - G\gamma$ and $\delta_2 = K_2 - G\gamma$ with K_1, K_2 as the resonance harmonics before and after the jump and λ_i's are given by Eq. (2.5). For an optimal tune jump with $\delta_2 = -\delta_1 = \delta$, the polarization becomes [8]

$$P_f = \frac{\delta^2 - |\epsilon|^2}{\delta^2 + |\epsilon|^2}P_i. \tag{2.12}$$

To achieve a proper tune jump with 95% polarization survival, $\Delta\nu_z = 2\delta_{95\%} \approx 12|\epsilon|$ is needed. Since the betatron tune jump is limited to $\Delta\nu_z \leq 0.3$ by the betatron stop bands [2], the maximum intrinsic spin-depolarization resonance strength, which can be effectively overcome by the tune jump scheme, is $\epsilon \leq 0.025$. However, the actual polarization survival is the average of Eq. (2.12) over the bunch distribution. The applicability of the tune jump method may depend on the beam distribution.

On the other hand, when the beam is accelerated adiabatically through a spin resonance, the polarization vector follows the spin closed orbit and flips. Depending on the acceleration rate, the degree of spin flip (or degree of adiabaticity) is given by the Froissart-Stora formula [9],

$$\frac{P_f}{P_i} = 2e^{\frac{-\pi|\epsilon|^2}{2\alpha}} - 1, \quad \alpha = \frac{dG\gamma}{d\theta}, \tag{2.13}$$

where α is the acceleration rate. The tune jump method and the adiabatic spin flip method were used successfully in overcoming spin-depolarization resonances during the polarized-proton acceleration in low- to medium-energy synchrotrons. For high-energy synchrotrons, snakes are needed.

B. Effects of snakes on spin motion

When snakes are inserted into an accelerator, the spin-perturbation parameter, b in Eq. (2.9) becomes smaller due to a small orbital angle difference, $\theta_f - \theta_i$, between snakes. Snakes are local spin rotators, which rotate particle spin by π radians about a horizontal axis locally without perturbing particle orbits outside a snake region. A partial snake differs only in the amount of spin-rotation angle, e.g., a 10% snake rotates spin by 0.1π rad. Thus a snake is characterized by the amount of *spin-rotation angle*, ϕ, and the *snake axis angle*, ϕ_s, with respect to $\hat{e}_1$ (radially outward direction). The spin rotator which rotates the spin 180° about the $\hat{e}_1$ axis is usually called the type-II snake and the snake which rotates the spin about the $\hat{e}_2$ axis is called the type-I snake.

The spinor wave function at a snake will be transformed locally according to

$$\Psi(\theta^+) = e^{-i\frac{\phi}{2}\hat{n}_s\cdot\boldsymbol{\sigma}}\Psi(\theta^-), \tag{2.14}$$

where ϕ is spin-rotation angle and $\hat{n}_s = (\cos\phi_s, \sin\phi_s, 0)$ denotes the snake axis with respect to radially outward direction, $\hat{e}_1$. $\theta^\pm$ depict azimuthal orbit rotation angles just before and after the snake. More specifically, at $\phi = \pi$, or the 100% snake, the spinor wave function can be transformed as

$$\Psi(\theta^+) = e^{-i\frac{\pi}{2}\hat{n}_s\cdot\boldsymbol{\sigma}}\Psi(\theta^-) = T_s(\phi_s)\Psi(\theta^-), \tag{2.15}$$

where $T_s(\phi_s) = -i\hat{n}_s\cdot\boldsymbol{\sigma}$ is the spin transfer matrix for a 100% snake.

Let us consider a perfect circular accelerator with two snakes, $-i\sigma_1, -i\sigma_2$, separated by π orbital angle apart. The one-turn spin transfer matrix is given by

$$[-i\sigma_2]e^{-i\frac{G\gamma\pi}{2}\sigma_3}[-i\sigma_1]e^{-i\frac{G\gamma\pi}{2}\sigma_3} = i\sigma_3. \tag{2.16}$$

Thus the spin tune, obtained from the trace of the one-turn spin transfer matrix, is $\frac{1}{2}$ and the stable spin closed orbit is vertical. Now we introduce a small constant local spin angular precessing kick, χ, about an axis $\hat{n}_k$ in the horizontal plane, and the spin transfer matrix becomes

$$T_1 = e^{-i\frac{\chi}{2}\hat{n}_k\cdot\boldsymbol{\sigma}}i\sigma_3. \tag{2.17}$$

Because $\hat{n}_k$ is in the horizontal plane, the evolution of the spin transfer matrix at the nth revolution becomes

$$T^{(n)} = [T_1]^n = \begin{cases} [i\sigma_3]^n & \text{if } n = \text{even} \\ T_1[i\sigma_3]^{(n-1)} & \text{if } n = \text{odd}, \end{cases} \tag{2.18}$$

which means that the perturbed spin precessing kicks cancel each other every two turns around the accelerator. Thus the snake is very effective in correcting the imperfection resonances due to a localized constant spin-perturbing kick.

Extending the model a step further, we assume that the precessing kick is different in each turn, and the spin transfer matrix becomes

$$T^{(n)} = \prod_{m=1}^{n} T_m = e^{-i\frac{1}{2}[\sum_{m=1}^{n}(-1)^{n-m}\chi_m]\hat{n}_k\cdot\boldsymbol{\sigma}}[i\sigma_3]^n. \tag{2.19}$$

The vertical spin vector is given by

$$S_3^{(n)} = 1 - 2\sin^2\left(\frac{1}{2}\sum_{m=1}^{n}(-1)^{n-m}\chi_m\right). \tag{2.20}$$

Now if the spin perturbation kicks are due to a betatron motion, these kicks are correlated by

$$\chi_m = \chi_0\cos 2m\pi\nu_z, \tag{2.21}$$

where ν_z is the fractional part of the vertical betatron tune. When the vertical betatron tune is $\nu_z = \frac{1}{2}$, each kick adds up coherently. The spin vector will precess around the $\hat{n}_k$ axis at a precessing tune of $\frac{\chi_0}{2\pi}$, or in other words, it takes $\frac{2\pi}{\chi_0}$ orbital revolutions to complete one precessing turn around the $\hat{n}_k$ axis.

The spin perturbing kick of Eq. (2.21) arises mainly from the vertical betatron motion in a quadrupole, where the magnitude χ_0 depends on the betatron amplitude. Since a beam bunch is composed of particles with different betatron phases and amplitudes and the polarization of the beam is the ensemble average of Eq. (2.20) over the beam distribution, the polarization will therefore be lost at the $\nu_z = \frac{1}{2}$ resonance condition. On the other hand, the spin of an ideal beam bunch without spin-tune spread and phase spread will precess about the $\hat{n}_k$ axis at a precessing tune of $\frac{2\pi}{\chi_0}$ without depolarization at the resonance condition.

An interesting observation worth pointing out is that the depolarization occurs only at $\nu_z = \frac{1}{2}$ in the localized spin kick model discussed above, i.e., the localized spin kick does not explain higher-order snake resonances shown in Fig. 1.

C. Basic requirements of snake configurations in accelerators

Let us consider N_s snakes with snake axes $(\phi_1, \phi_2, ..., \phi_{N_s})$ distributed in an accelerator, and let $\theta_{i,i+1}$ be the azimuthal orbit rotation angle between the ith and $(i+1)$th snakes. The one-turn spin transfer matrix for a perfect circular accelerator is given by

$$e^{-i\frac{G\gamma}{2}[\theta_{0,1}-\theta]\sigma_3}e^{-i\frac{\pi}{2}\hat{n}_{N_s}\cdot\boldsymbol{\sigma}}\prod_{k=1}^{N_s-1}[e^{-i\frac{G\gamma}{2}\theta_{k,k+1}\sigma_3}e^{-i\frac{\pi}{2}\hat{n}_k\cdot\boldsymbol{\sigma}}] \times e^{-i\frac{G\gamma}{2}\theta\sigma_3} = e^{-i\pi\nu_s\hat{n}\cdot\boldsymbol{\sigma}}, \tag{2.22}$$

where the spin tune ν_s and the spin closed-orbit vector $\hat{n}_{CO}$ can be obtained by identifying the matrix elements of Eq. (2.22). To ensure that the spin tune is independent of the particle energy, the distribution of snakes should satisfy the following condition [6, 10]:

$$\theta_{\text{odd}} = \theta_{\text{even}} = \pi,$$
$$\theta_{\text{odd}} = \sum_{k=\text{odd}}^{N_s} \theta_{k,k+1}, \tag{2.23}$$
$$\theta_{\text{even}} = \sum_{k=\text{even}}^{N_s} \theta_{k,k+1},$$

where $\theta_{\text{odd}} + \theta_{\text{even}} = 2\pi$ is the total orbital angle for a circular path. If the odd (or even) orbital angle deviates from π, the spin tune becomes $\frac{1}{2} + G\gamma(1 - \theta_{\text{odd}}/\pi)$, i.e. the spin tune is *shifted* away from $\frac{1}{2}$ by an amount

$$\Delta\nu_s = G\gamma\left(1 - \frac{\theta_{\text{odd}}}{\pi}\right).$$

For high-energy storage rings, $G\gamma$ is a large number, e.g., $G\gamma = 450$ for the relativistic heavy ion collider (RHIC) and $G\gamma$=36 000 for the Superconducting Super Collider (SSC), accurate placement of snakes becomes important. This issue will be addressed in Sec. III. The spin tune can be obtained from the trace of the one-turn transfer map, i.e.,

$$\nu_s = \frac{1}{\pi}\sum_{k=1}^{N_s}(-1)^k\phi_k, \tag{2.24}$$

which can be used to set the spin tune to the most favorable number in avoiding spin-depolarizing resonances. For example, accelerators with two snakes, $N_s = 2$, should have those two snakes placed at the locations of the π orbital angle apart and the snake axes of these two snakes should be orthogonal to each other to obtain a spin tune of $\frac{1}{2}$. For accelerators with a large number of snakes, there are many ways to organize snakes to obtain proper snake superperiodicity and proper spin tune.

D. Spin tracking hierarchy equation

Let us consider an accelerator with N_s equally spaced snakes. The spin transfer matrix after passing through a pair of (ϕ_2, ϕ_1) snakes is given by

$$\tau\left(\theta_0 + \frac{4\pi}{N_s}, \theta_0\right) = T_s(\phi_2)t\left(\theta_0 + \frac{4\pi}{N_s}, \theta_0 + \frac{2\pi}{N_s}\right) \times T_s(\phi_1)t\left(\theta_0 + \frac{2\pi}{N_s}, \theta_0\right), \tag{2.25}$$

where $t(\theta_f, \theta_i)$ and $T_s(\phi_i)$ are given by Eqs. (2.7) and (2.14), respectively. The components of spin transfer matrix are given by

$$\tau_{11}\left(\theta_0 + \frac{4\pi}{N_s}, \theta_0\right) = -e^{-i(\phi_2-\phi_1)}(1 - 2b^2e^{i\Phi}\cos\Phi), \tag{2.26}$$

$$\tau_{12}\left(\theta_0 + \frac{4\pi}{N_s}, \theta_0\right) = -2iabe^{-i(c-2K\pi/N_s+\phi_2)}\cos\Phi, \tag{2.27}$$

with $\tau_{21} = -\tau_{12}^*$, $\tau_{22} = \tau_{11}^*$, where $\Phi = K\theta_0 + 2K\pi/N_s + d - \phi_1$ and parameters a, b, c, and d are defined in Eq. (2.9) with $\theta_f - \theta_i = 2\pi/N_s$ as

$$b = \frac{|\epsilon|}{\lambda}\sin\frac{\pi\lambda}{N_s} = (1 - a^2)^{1/2} \tag{2.28}$$

and

$$\lambda = (\delta^2 + |\epsilon|^2)^{1/2}, \quad \delta = K - G\gamma,$$
$$c = \arctan\left[\frac{\delta}{\lambda}\tan\frac{\pi\lambda}{N_s}\right], \quad d = \arg(\epsilon^*).$$

The spin motion in accelerator can then be obtained iteratively by using the spin tracking equation through pairs of snakes, i.e.,

$$T(\theta_{n+1}) = \tau(\theta_{n+1}, \theta_n)T(\theta_n), \tag{2.29}$$

where $\theta_{n+1} = \theta_n + 4\pi/N_s$. Here $\tau(\theta_{n+1}, \theta_n)$ is the kernel of the iterative equation, which can be solved iteratively using a power-series expansion in strength parameter b^2, i.e.,

$$T_{11} = T_{11}^{(0)} + T_{11}^{(1)} + T_{11}^{(2)} + \cdots,$$
$$T_{12} = T_{12}^{(1)} + T_{12}^{(2)} + T_{12}^{(3)} + \cdots, \tag{2.30}$$

where $T_{11}^{(i)} = O(b^{2i})$ and $T_{12}^{(i)} = O(ab^{2i-1})$. The final vertical spin vector is obtained from the expectation value of σ_3 in the spinor wave function, i.e.,

$$\langle S\rangle = |T_{11}|^2 - |T_{12}|^2 = 1 - 2|T_{12}|^2. \tag{2.31}$$

E. The perturbed spin tune

Without loss of generality, we will discuss an accelerator with two snakes ϕ_1, ϕ_2, located at an orbital angle of π from each other. The spin transfer matrix for passing through two snakes [or equivalently the *one-turn map* (OTM)] is given by

$$\tau_{11}(\theta_0 + 2\pi, \theta_0) = -e^{-i\pi\nu_s}(1 - 2b^2e^{i\Phi}\cos\Phi), \tag{2.32}$$
$$\tau_{12}(\theta_0 + 2\pi, \theta_0) = -2iabe^{-i(c-K\pi+\phi_2)}\cos\Phi, \tag{2.33}$$

where $\pi\nu_s = \phi_2 - \phi_1$ and $\Phi = K\theta_0 + K\pi + d - \phi_1$ is the characteristic betatron phase of the orbital motion.

The perturbed spin tune Q_s defined as the trace of OTM is given by

$$\cos\pi Q_s = -\cos\pi\nu_s + 2b^2\cos\Phi\cos(\Phi - \pi\nu_s) = b^2\sin(2\Phi). \tag{2.34}$$

Because of the betatron phase, the perturbed spin tune for an intrinsic resonance, Q_s, is oscillating around $\frac{1}{2}$ up to a maximum and minimum given by

$$Q_{s,\substack{\max\\ \min}} = \frac{1}{2} \pm \frac{1}{\pi}\arcsin\left[\sin^2\frac{\pi\epsilon}{N_s}\right]. \tag{2.35}$$

If the resonance strength of a spin resonance is $|\epsilon| \approx mN_s/2$, $m = 1, 3, \ldots$, the perturbed spin tune Q_s will cover a whole integer unit during the acceleration and cross the intrinsic resonance many times. The polarization may be lost. Figure 2 shows the final polarization after passing through a single resonance with two snakes

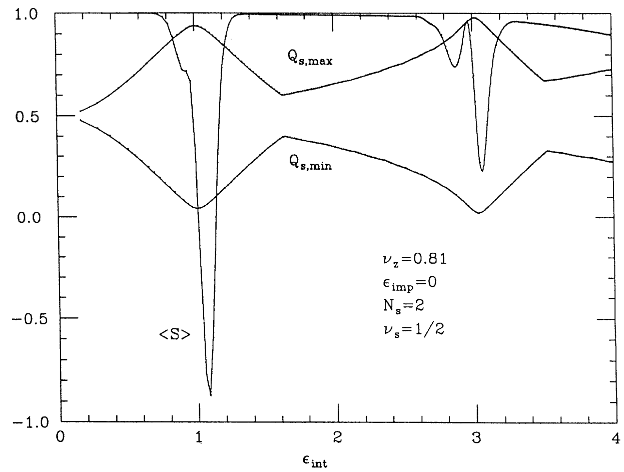

FIG. 2. The vertical spin vector and the perturbed spin tunes $Q_{s,\max(\min)}$ obtained from a numerical tracking calculation at intrinsic resonance $\nu_z = 0.81$ is plotted as a function of the resonance strength. Note that the perturbed spin tune covers the entire tune space at $\epsilon = 1, 3$.

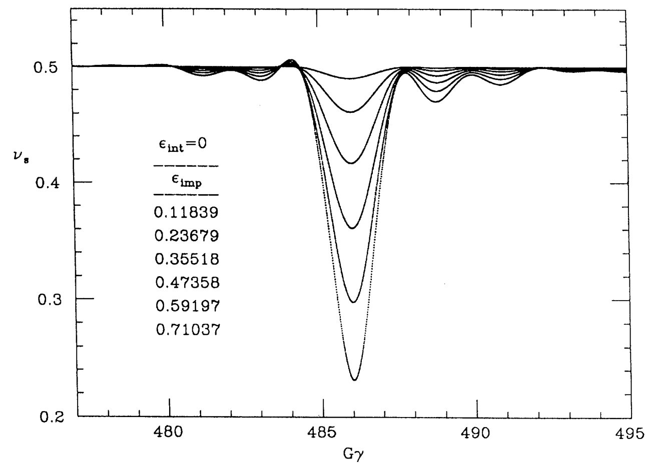

FIG. 3. The perturbed spin tune across imperfection resonances.

as a function of the intrinsic resonance strength at the betatron tune of $\nu_z = 0.81$. Note that the maximum and minimum perturbed spin tunes cover the entire tune spaces around $\epsilon = 1$ and 3, where beam depolarization occurs.

One might conclude that the critical spin-resonance strength $\langle\epsilon_c\rangle$ could be defined by requiring that the perturbed spin tune does not cross the spin resonance K during the acceleration, i.e.,

$$\langle\epsilon_c\rangle \le \frac{\arcsin(|\cos\pi K|^{1/2})}{\pi} N_s, \qquad (2.36)$$

where the resonance tune K is related to the betatron and/or synchrotron tunes of an accelerator by Eq. (1.2). Equation (2.36) indicates that the tolerable critical resonance strength will be larger when the betatron tune is nearer to an integer. This, however, is not a sufficient condition. At some special fractional betatron tunes, e.g., $1/3, 2/3, 1/4, 3/4, 1/8, 3/8$, etc., the spin motion is not affected by the spin tune. At these special betatron tunes, the maximum and minimum perturbed spin tunes during a resonance crossing depend also on the initial betatron phase. Results of tracking calculations show that the spin motion is not much affected by the perturbed spin tune. Similarly, at an odd-order snake resonance of Eq. (1.4), e.g., $\nu_z = 5/6$, depolarization occurs at a very small deviation of the perturbed spin tune from $\frac{1}{2}$ and the final spin vector after passing through the resonance *depends on the initial betatron phase.* However, if the betatron tunes are chosen to be far away from low-order rational numbers, the critical resonance strength obtained from numerical simulations agrees [6] with that of Eq. (2.36).

On the other hand, when $K =$ integer for an imperfection resonance, the phase Φ in Eq. (2.34) is constant (mod 2π). The perturbed spin tune of Eq. (2.34) varies smoothly across the resonance shown in Fig. 3, where the spin tune for various imperfection resonance strengths is shown for an accelerator with two snakes. The actual magnitude and sign of the spin tune shift for imperfection resonances depend on the vertical closed-orbit error.

One might argue that depolarization might occur when the perturbed spin tune Q_s satisfies

$$Q_s \pm \nu_z = \text{ integer}. \qquad (2.37)$$

However, this condition is irrelevant to the imperfection resonance shown in Fig. 4, where the betatron tune is chosen to satisfy the above condition. There is no effect on the spin motion across the resonance. This can be understood easily by the fact that there are no spin kicks coherent to the resonance condition of Eq. (2.37) due to the fact that the intrinsic resonance strength in this tracking demonstration is zero. When the intrinsic resonance strength is zero, the spin motion is not af-

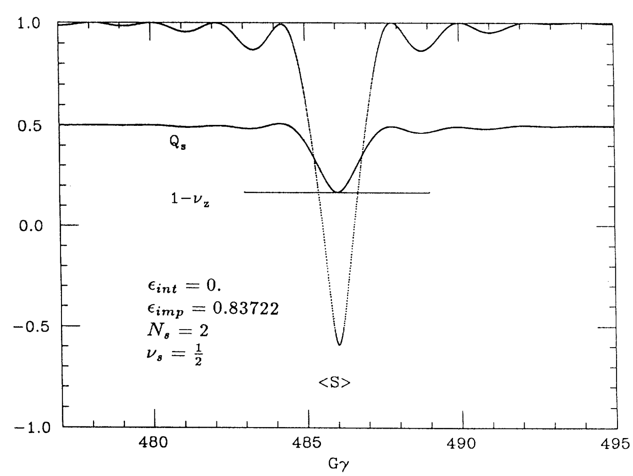

FIG. 4. The evolution of the vertical spin vector and the perturbed spin tune in the presence of an imperfection resonance at $\epsilon = 0.837\,22$. The betatron tune is chosen to satisfy Eq. (2.37) in order to demonstrate that the imperfection resonance does not contribute to the depolarization of a snake resonance.

fected by varying the betatron tune across the snake resonance condition of Eq. (2.37). However, we will show that Eq. (2.37) plays an important role when the intrinsic resonance strength is not zero, i.e., overlapping intrinsic and imperfection resonances.

F. Snake resonances

We have demonstrated that the spin depolarization mechanism is not determined solely by the criterion of spin tune alone. The simple model used in Sec. II B shows that snakes provide a mechanism for the self-cancellation of localized spin kicks every two revolutions in the accelerator, and a coherent enhancement of the perturbative kicks at the betatron tune of $\nu_z = \frac{1}{2}$. The Fourier harmonics of a localized spin kick is equivalent to the equally spaced spin resonances. In Sec. II B, we found that a localized spin kick can explain only the snake resonance at $\nu_z = \frac{1}{2}$. In this section, we will show that the cancellation is also valid for distributed spin-perturbative kicks, which corresponds to an isolated resonance. Solving the spin tracking equation (2.29) to the first order in parameter b, the spin transfer matrix is given by

$$T_{12}^{(1)}(\theta_{n+1}) = 2iab(-1)^n e^{-i(c-K\pi+\phi_2)} \times \tfrac{1}{2}\{e^{i(\Phi+nK\pi)}\zeta_{n+1}(K+\nu_s) + e^{-i(\Phi+nK\pi)}\zeta_{n+1}(K-\nu_s)\} \quad (2.38)$$

where $\zeta_n(q)$, called the enhancement function, is given by

$$\zeta_n(q) = \frac{\sin nq\pi}{\sin q\pi}. \quad (2.39)$$

At the first-order snake resonance condition,

$$\nu_s \pm K = \text{ integer}, \quad (2.40)$$

the off-diagonal kicks add up coherently each turn through snake pairs. The beam can be depolarized easily as shown in Fig. 1 at $\nu_z = \frac{1}{2}$. Since betatron tunes of an accelerator are not half integers, the first-order snake resonance condition can easily be avoided.

It is interesting to note that the enhancement factor ζ_n vanishes for an imperfection resonance when n is an even number. This means that the linear perturbing kicks to the spin vector cancel each other every two turns around an accelerator for an imperfection resonance discussed in Eq. (2.18). With imperfection resonances, there is no source of coherent spin kick at the condition of Eq. (2.40) and hence it does not cause depolarization shown in Fig. 4.

Avoiding snake resonances, the vertical spin vector across the resonance region will fall within the envelope of

$$\langle\langle S\rangle\rangle = 1 - 8a^2b^2, \quad b = \frac{|\epsilon|}{\lambda}\sin\frac{\pi\lambda}{2}, \quad (2.41)$$

which have been demonstrated in many numerical simulations [see, e.g., Fig. 1 of Ref. 4]. Note here that the envelope function has many nodal points at $\lambda = 2m$, corresponding to $b = 0$. At these nodal points, the depolarization driving component vanishes. Away from the central resonance location, the depolarization driving parameter b is usually small. Therefore if the spin vector is not restored to the vertical position at the first nodal location after passing through the resonance, the spin is depolarized. A few important observations can be drawn from Eqs (2.36)–(2.41).

(1) At an imperfection resonance, K = integer, $T_{12}^{(1)}(\theta_{n=\text{even}}) = 0$. This means that imperfection kicks cancel each other every two revolutions around accelerator. Thus snakes are most effective in overcoming imperfection resonances.

(2) At a fractional betatron tune, e.g., $K = \frac{q}{p} \neq \frac{1}{2}$, one obtains

$$T_{12}^{(1)}(\theta_m) = 0 \quad \begin{cases} m = p & \text{if } p \text{ is even} \\ m = 2p & \text{if } p \text{ is odd.} \end{cases}$$

Thus the linear depolarization driving terms tend to cancel at a betatron tune of a rational number. One might guess that the spin will be more stable against perturbation at a betatron tune which is a rational number with a small even-number denominator. Figure 5 shows the vertical spin vector, after passing through an intrinsic resonance, as a function of the intrinsic resonance strength for the betatron tune at $\nu_z = 3/4$. The vertical spin is characteristically different from that of Fig. 3 of intrinsic resonances, where the betatron tune is not a low-order rational number. This feature remains true to all low-order rational numbers, such as 1/3, 2/3, 1/4, 3/4, 1/5, 2/5, etc.

(3) The envelope function $\langle\langle S\rangle\rangle$ has many nodal points, where the depolarization driving term vanishes, i.e., $b = 0$ or 1. These nodal points correspond to the spin matching condition [10] where

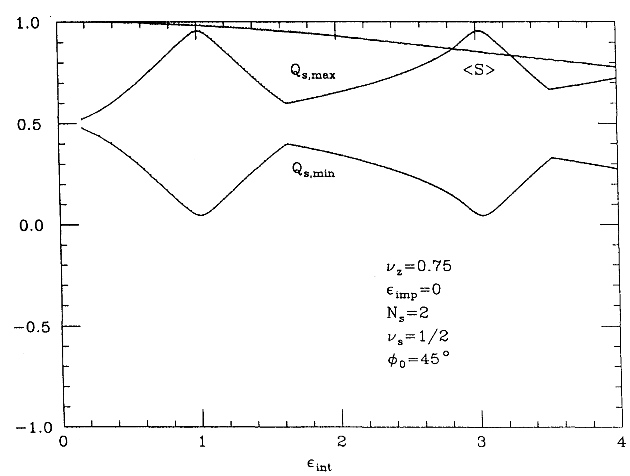

FIG. 5. The perturbed spin tune and the vertical spin component after passing through an intrinsic resonance at ν_z=3/4 are shown as a function of the intrinsic resonance strength for $\epsilon_{imp} = 0$.

$$\frac{\lambda}{N_s} = \text{integer} \quad \text{or} \quad G\gamma = K \pm \sqrt{(\text{integer} \times N_s)^2 - |\epsilon|^2}.$$

Thus these nodal locations are separated approximately by N_s units of $G\gamma$. These nodal points may play an essential role in spin restoration during the passage through a depolarization resonance.

(4) The width of envelope function is about $12|\epsilon|$ for 95% polarization. However, one can choose a nodal point to obtain 100% polarization.

(5) Based on the linear-response theory of Eq. (2.38), we expect that depolarization occurs at a betatron tune equal to a half integer shown clearly in Fig. 1. The $\nu_s = \frac{1}{2}$ snake resonance has been observed [5].

(6) Up to the linear order in the kick strength, the result of distributed spin-perturbative kick (a single isolated resonance) of Eq. (2.38) is equivalent to the localized spin kick (equal resonance strength at every harmonic) of Eq. (2.19).

From the discussions above, we might expect that the spin vector would be more stable at a betatron tune equal to a low-order rational number. However, Fig. 1 shows that there are many high-order depolarization resonances at a betatron tune of rational numbers, e.g., 1/6, 5/6, 1/10, 3/10, etc. Solving the spin tracking equation beyond linear order in b gives rise to snake resonance conditions given by Eq. (1.4) [4].

G. Overlapping resonances and even-order snake resonances

Basic accelerator theory [2] indicates that a closed-orbit distortion has largest amplitude at a harmonic nearest to the betatron tune. We thus expect a large imperfection resonance, located at an integer nearest to the important intrinsic resonance. The correlation remains important even after closed-orbit corrections, which minimizes error harmonics nearest to the betatron tunes.

For a single isolated intrinsic resonance at $\nu_z = 3/4$, we found that the spin is not much affected by the perturbative spin tune and is not depolarized at $\epsilon = 1, 3$ due to the cancellation of the linear spin kicks every four revolutions. Figure 6 shows the evolution of the vertical spin vector across the resonance region for $\nu_z = 3/4$. The upper part of the graph shows the vertical spin vector at $\epsilon_{\text{int}} = 0.23$ without imperfection resonance. Note that the deviation of the vertical spin from the value 1 is much smaller than the envelope function of Eq. (2.41). The lower part of the figure shows the *inverted* vertical spin vector (for clearly visible reasons) at the same intrinsic resonance strength with an overlapping imperfection resonance at the resonance strength $\epsilon_{\text{imp}} = 0.04$. When the imperfection resonance is introduced, the vertical spin vector is strongly perturbed so that the spin vector cannot retain full polarization at the first nodal point. The memory on the vertical spin vector is lost. A simple model to explain the coherent kicks at the even-order snake resonance will be discussed in the next section. Figure 7 shows the vertical spin vector after passing through overlapping resonances with a very small imperfection resonance strength $\epsilon_{\text{imp}} = 0.002$ as a function of

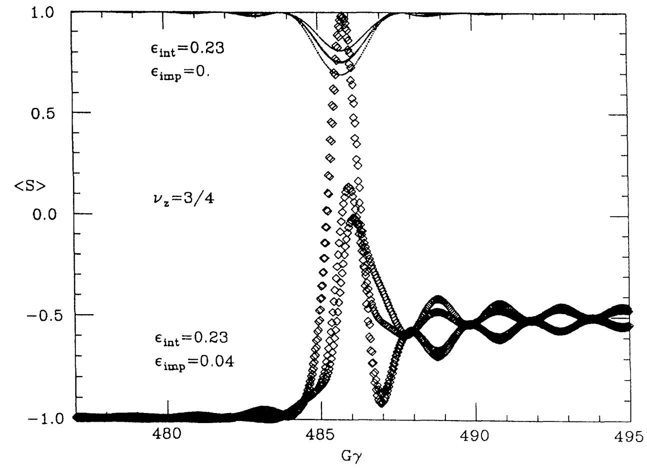

FIG. 6. Evolution of the spin vector at an even-order snake resonance for the case of an isolated intrinsic resonance (upper curve) and the case of overlapping intrinsic and imperfection resonances (lower curve).

the intrinsic resonance strength ϵ_{int} and $\nu_z = 3/4$. Note here that strong depolarization occurs at a small perturbed tune shift due to the even-order snake resonance condition.

To understand the effect of the imperfection on the spin motion, the imperfection spin-depolarizing kick is included in the spin tracking. Figure 8 shows the final vertical spin vector after passing through overlapping intrinsic and imperfection resonances, $\epsilon_{\text{int}} = 0.5, \epsilon_{\text{imp}} = 0.05$. We observe that when a small imperfection resonance strength $\epsilon_{\text{imp}} = 0.05$ is included, beam depolarization occurs at all even-order snake resonances shown in Fig. 8.

To understand further the effect of imperfection resonances on the spin motion, we reduce intrinsic resonance strength in our calculation to $\epsilon_{\text{int}} = 0.137$, where only

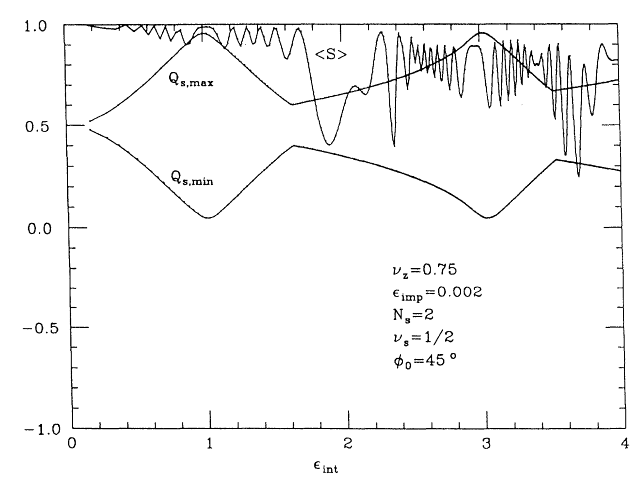

FIG. 7. The perturbed spin tune and the vertical spin component after passing through an intrinsic resonance at ν_z=3/4 are shown as a function of the intrinsic resonance strength for $\epsilon_{\text{imp}} = 0.002$.

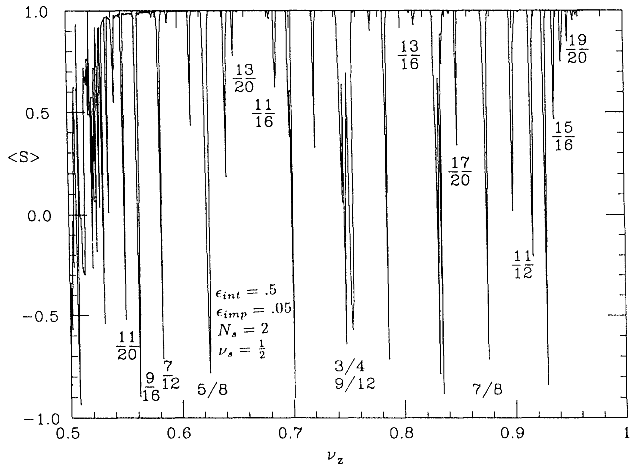

FIG. 8. Beam polarization after passage through overlapping intrinsic and imperfection spin resonance is shown as a function of the fractional part of spin-resonance tune. In comparison with that of Fig. 1, even-order snake resonances appear while the odd-order snake resonances are not much affected. At $\epsilon_{\rm imp} = 0.05$ for two snakes, the even-order snake resonances become more important than the odd-order snake resonances.

low-order snake resonances at $\nu_z = 1/2,\ 1/6,\ 5/6$ are important. When an imperfection resonance at $\epsilon_{\rm imp} = 0.13$ is included, we found that even-order snake resonances at $\nu_z = 3/4,\ 5/8,\ 7/8, \ldots$ appear. Furthermore, all snake resonances split into double peaks shown in Fig. 9. The distance of these two peaks increases with the strength of the imperfection resonance. There are two points worth mentioning. First, the even-order snake resonance becomes more important than the odd-order snake resonance, and the odd-order snake resonance is not much affected by the imperfection resonance. Second, double peaks occur for each snake resonance. The feature of double peaks can be understood easily knowing that the imperfection resonance generates a perturbed spin-tune shift. The snake resonance condition becomes

$$\tfrac{1}{2} + \Delta Q_s \pm \ell \nu_z = \text{ integer}, \quad \ell = \text{ integer}, \qquad (2.42)$$

where ΔQ_s is the perturbed spin-tune shift from the imperfection resonance given by

$$|\Delta Q_s| \leq \frac{1}{\pi} \arcsin\left(\sin^2 \frac{\pi \epsilon_{\rm imp}}{N_s}\right). \qquad (2.43)$$

The actual magnitude and sign of the spin-tune shift depend on the closed orbit of the circular accelerator. Because of spin tune shift, each snake resonance will split into two snake resonances separated by

$$\Delta \nu_z = \pm \frac{1}{\ell} \Delta Q_s. \qquad (2.44)$$

The distance of splitting becomes smaller at higher-order snake resonances as clearly seen in Fig. 9. The depolarization line shape of these double peaks shown in Fig. 9 reflects the important effect of perturbed spin-tune shift on the snake resonances at the maximum spin-tune shift shown in Figs. 3 and 4.

Thus the spin-depolarization mechanisms are due to the perturbed spin-tune shift of the imperfection resonance and the snake resonance conditions of Eqs. (1.4) and (1.6). Since betatron tunes of colliders, such as RHIC, Tevatron, and SSC, have to avoid similar low-order betatron resonances for orbital stability, snake resonances do not impose further constraints to the operation of colliders. One can generalize the discussion to multisnake accelerators, where the resonance condition of Eq. (1.4) will be modified by snake superperiodicity, P_s. At higher snake superperiodicity, there are fewer snake resonances, yet resonance width is also increased. Basic physics remains unchanged [6].

H. A simple model for even-order snake resonances

To understand the essential mechanism of the even-order snake resonances in the presence of overlapping spin

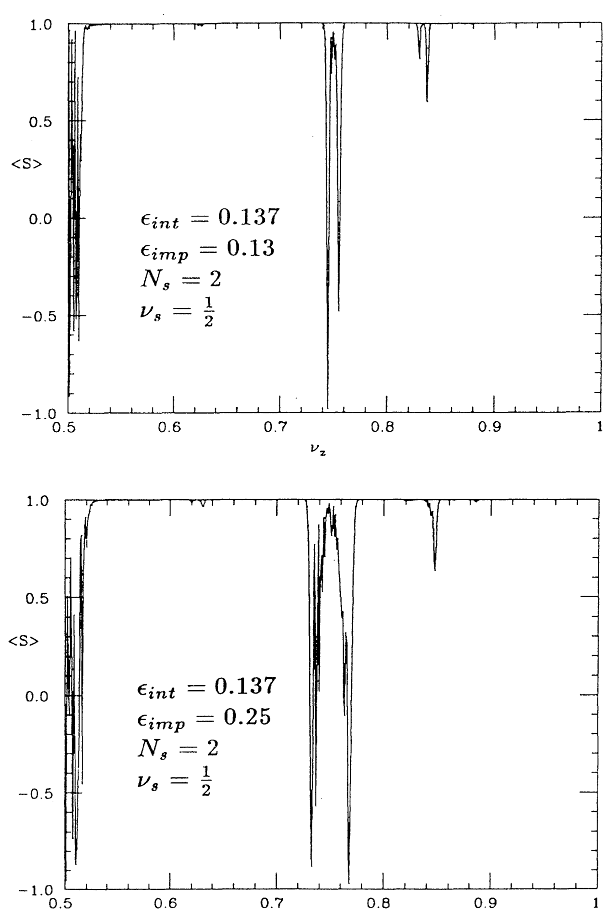

FIG. 9. The effect of the imperfection resonance on the snake resonances is shown for $\epsilon_{\rm imp} = 0.13$ (top) and $\epsilon_{\rm imp} = 0.25$ (bottom). Note that even-order snake resonances appear and each snake resonance splits into two resonance conditions due to the unperturbed spin tune of the imperfection resonance.

resonances, we consider the model of the spin transfer matrix shown in Eq. (2.25). The OTM of the overlapping intrinsic and imperfection resonances can be expressed as

$$\tilde{\tau} = e^{-i\frac{\chi}{2}\sigma_1}\tau(\theta_0 + 2\pi, \theta_0), \tag{2.45}$$

where $\tau(\theta_0 + 2\pi, \theta_0)$ is given in Eq. (2.25), and we have assumed a small local spin-precessing kick χ about the $\hat{e}_1$ axis. The resonance strength of the imperfection resonance is given by $\epsilon_{\rm imp} = \chi/2\pi$ at all integer harmonics. The matrix elements of the OTM are given by

$$\tilde{\tau}_{11} = -e^{-i\pi\nu_s}(1 - 2b^2e^{i\Phi}\cos\Phi)\cos\frac{\chi}{2} -2abe^{i(c-K\pi+\phi_2)}\cos\Phi\sin\frac{\chi}{2}, \tag{2.46}$$

$$\tilde{\tau}_{12} = -2iabe^{-i(c-K\pi+\phi_2)}\cos\Phi\cos\frac{\chi}{2} +ie^{i\pi\nu_s}(1 - 2b^2e^{-i\Phi}\cos\Phi)\sin\frac{\chi}{2}, \tag{2.47}$$

where parameters a, b, c, d, and Φ are given in Eq. (2.28) and $\tilde{\tau}_{21} = \tilde{\tau}_{12}^*, \tilde{\tau}_{22} = \tilde{\tau}_{11}^*$. Due to the imperfection resonance, the off-diagonal matrix elements now contain a term oscillating at two times the betatron frequency with an amplitude proportional to $b^2\sin\frac{\chi}{2}$. Following the same procedure in deriving Eq. (2.38), one obtains a snake resonance condition, $\nu_s \pm 2K$ = integer, which is the lowest even-order snake resonance condition of Eq. (1.6). By performing similar higher-order analysis, one can obtain all even-order snake resonances. Since the first term of $\tilde{\tau}_{12}$ in Eq. (2.47) depends on the imperfection resonance in $\cos\frac{\chi}{2}$, the odd-order snake resonance is not much affected by the overlapping imperfection resonance.

Since the real part of $\tilde{\tau}_{11}$ contains only terms oscillating with the betatron frequency Φ, the average spin tune shift is zero in this simple model with imperfection resonances. In reality, if the imperfection spin kicks are mixed with the intrinsic spin kicks, the spin tune of the one turn map will be shifted during the passage of a resonance region discussed in the last section. A slightly more complicated model given by

$$\tau\left(\theta_0 + \frac{4\pi}{N_s}, \theta_0\right) = e^{-i\frac{\chi_2}{2}\sigma_1}T_s(\phi_2)t\left(\theta_0 + \frac{4\pi}{N_s}, \theta_0 + \frac{2\pi}{N_s}\right) \times e^{-i\frac{\chi_1}{2}\sigma_1}T_s(\phi_1)t\left(\theta_0 + \frac{2\pi}{N_s}, \theta_0\right)$$

can generate a spin tune proportional to $\sin\frac{\chi_1}{2}\sin\frac{\chi_2}{2}$.

I. Snake imperfections

When the spin-rotation angle ϕ of Eq. (2.14) deviates from π by $\Delta\phi = \pi - \phi$, the spin transfer matrix of the snake becomes

$$T_s(\phi_s) = e^{i\frac{\Delta\phi}{4}\hat{n}_s\cdot\boldsymbol{\sigma}}e^{-i\frac{\pi}{2}\hat{n}_s\cdot\boldsymbol{\sigma}}e^{i\frac{\Delta\phi}{4}\hat{n}_s\cdot\boldsymbol{\sigma}}. \tag{2.48}$$

Comparing Eqs. (2.14) with (2.48), one obtains an equivalent imperfection spin resonance $\epsilon_{\rm imp}^{\rm eq}$ as

$$\epsilon_{\rm imp}^{\rm eq} = \frac{\Delta\phi}{\pi}. \tag{2.49}$$

Therefore the error in snake spin-rotation angle is equivalent to imperfection resonances at all integer harmonics. The result of the spin tracking calculation for two snakes at a spin rotation angle of 170° is shown in Fig. 10. The equivalent imperfection resonance strength is 0.055. The spin tune shift will cause each snake resonance to split into two resonances. The result is similar to the simple model of imperfection resonances discussed in the last section. When the spin-rotation angle deviates from 180°, the spin closed-orbit vector is also tilted away from the vertical axis.

Besides the effect of imperfection resonance, the spin tune becomes energy dependent. If we assume that identical snake pairs are installed in the accelerator, then the spin tune becomes

$$\cos\frac{2\pi Q_s}{N_s} = -\cos(\phi_2 - \phi_1) + \sin^2\left(\frac{\Delta\phi}{2}\right)[\cos(\phi_2 - \phi_1) + \cos(2G\gamma\pi/N_s)],$$

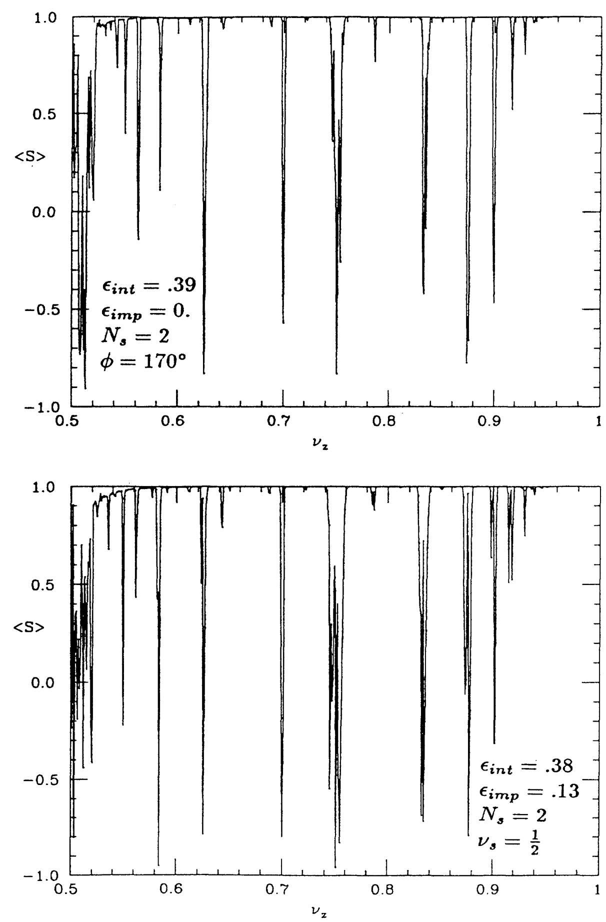

FIG. 10. The vertical spin vector after passing through an intrinsic resonance with $\epsilon = 0.39$ for two snakes with spin-rotation angle of 170° (top). The equivalent imperfection resonance strength is $\epsilon_{\rm imp}^{\rm eq} = 0.055$. On the bottom frame, the vertical spin vector after passing through an intrinsic resonance and an imperfection resonance at $\epsilon_{\rm imp} = 0.13$ is shown for comparison.

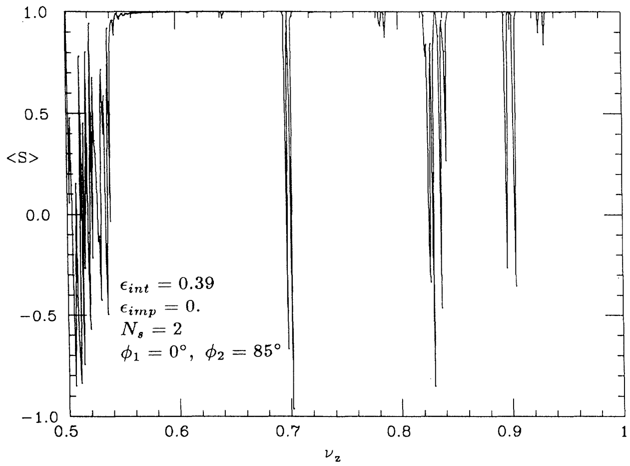

FIG. 11. The vertical spin vector after passing through an isolated intrinsic resonance for two snakes with spin axes angles with $\phi_1 = 0°$ and $\phi_2 = 85°$. The spin tune is shifted from $\frac{1}{2}$ by 0.028 without introducing imperfection resonances. Therefore each snake resonance is split into two without introducing even-order snake resonances.

where $\Delta\phi$ is the error in the spin-rotation angle of snakes. Here we expect that the spin-tune shift from $\frac{1}{2}$ will increase linearly with the number of snakes. In reality, the snake rotation angle may deviate from π randomly. The resulting spin-tune modulation will not increase linearly with the number of snakes. Because the perturbed spin tune is very important to snake resonances, the constraint on the spin-rotation angle of each snake is also important for an accelerator with a large number of snakes.

J. Snake axes imperfection

Besides the error in spin-rotation angles, ϕ, the snake axis angles ϕ_s may also deviate from the ideal situation. The resulting spin tune will deviate from $\frac{1}{2}$. The snake resonance condition of Eq. (2.42) indicates that each snake resonance will split into two. The corresponding tune space available will be smaller. However, the error in the spin-rotation angle does not generate imperfection resonance strength; therefore, the even-order snake resonances do not appear. Figure 11 shows the result of the vertical spin vector after passing through an isolated intrinsic resonance in an accelerator with an intrinsic resonance strength $\epsilon = 0.39$ and two snakes having $\phi_1 = 0, \phi_2 = 85°$, which corresponds to $\Delta Q_s = 0.028$. Note here that each snake resonance has split into two while the even-order snake resonances are missing.

III. CONSTRAINTS ON POLARIZED PROTON COLLIDERS

From our studies in Sec. II, we found that possible depolarization sources are (1) spin-tune modulation so that the spin tune overlaps with snake resonances, (2) betatron tune modulation so that snake resonances overlap with the spin tune, (3) the effects of beam-beam interactions, higher-order nonlinear resonances, and synchrotron depolarization resonances, (4) an uncompensated solenoid field at the interaction point (IP) of experimental detectors and the effect of spin rotators for helicity experiments, (5) effects of linear coupling, and (6) effects of rf noises. In this section, we will evaluate the importance of each issue. First, we make an estimation of the spin-resonance strengths for RHIC as a possible polarized proton collider.

Important intrinsic spin resonances are located at [6, 7]

$$K = nP \pm \nu_z \approx mPM \pm \nu_B, \tag{3.1}$$

where n and m are integers, P is the superperiodicity of the accelerator, M is the number of FODO cells per superperiod, and $2\pi\nu_B$ is the accumulated phase advance of all FODO cells, which contain bending dipoles. Here a FODO cell, composed in sequence of a focusing quadrupole, a dipole, a defocusing quadrupole, and a dipole, is the basic building block of synchrotrons. The corresponding resonance strengths are given by

$$\hat{\epsilon}_{K,\text{int}} \approx \frac{G}{4\pi}\sqrt{\frac{\gamma\epsilon_N}{\pi}}PM\frac{\sqrt{\beta_z(D)}}{f}\left(1+\sqrt{\frac{\beta_z(F)}{\beta_z(D)}}\right), \tag{3.2}$$

where $\beta_z(F)$ and $\beta_z(D)$ are vertical betatron amplitude functions at the location of focusing and defocusing quadrupoles, f is the focal length of quadrupoles, ϵ_N is the normalized emittance, and γ is the Lorentz factor. The maximum resonance strength for RHIC is about 0.45 at 250 GeV. Figure 12 shows intrinsic resonance strength for a RHIC lattice.

Important imperfection resonances are located at an interger nearest to an important intrinsic resonance of Eq. (3.1). The maximum imperfection resonance strengths are given by

$$\hat{\epsilon}_{K,\text{imp}} \approx \frac{G\gamma}{2\pi}PM\frac{\sigma_z}{f\sqrt{2\nu_z}}\left(1+\sqrt{\frac{\beta_z(F)}{\beta_z(D)}}\right), \tag{3.3}$$

where σ_z is the rms vertical closed orbit in the arc, and ν_z is the vertical betatron tune. We expect the imperfection resonance strength to be less than 0.05 for RHIC after a closed-orbit correction with $\sigma_z \approx 0.2$ mm.

A. Spin tune modulation

With snakes, the spin tune is independent of energy. Therefore the synchrotron motion does not give rise to spin tune spread. This has been verified indirectly in the snake experiment at the IUCF Cooler Ring, where one finds that there is no depolarization [5] at the synchrotron sideband for a 100% snake. Therefore the spin dispersion function, $\gamma\frac{\partial\hat{n}_{CO}}{\partial\gamma}$, and the spin chromaticity, $\gamma\frac{\partial\hat{Q}_{s,\text{imp}}}{\partial\gamma}$, are small for an accelerator with snakes [11]. However, the spin tune modulation may still arises from imperfect spin rotation in the snake, and imperfect orbital angle between snakes [Eq. (2.23)]. The errors in orbital angle may arise from survey error, closed-orbit error, and/or betatron motion.

First, let us set a basic constraint for the spin tune

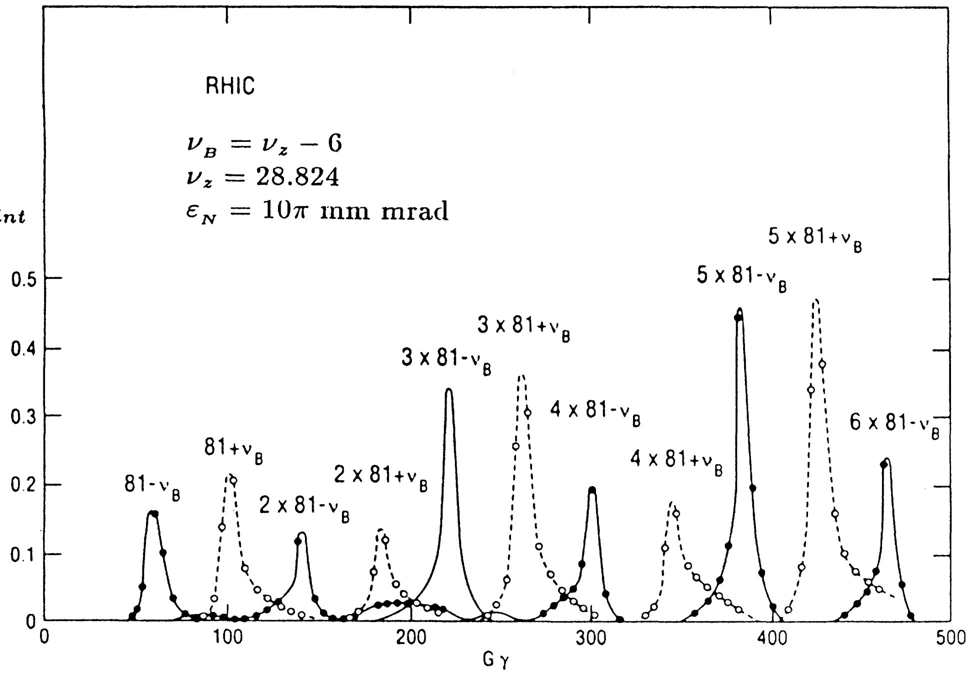

FIG. 12. The intrinsic resonance strengths calculated from a RHIC lattice. The expected resonance strength is about 0.45 at 250 GeV proton energy. The injection energy corresponds to $G\gamma \approx 48$, which happens to have a small intrinsic resonance strength.

spread. For an imperfection resonance strength at $\epsilon = 0.05$ for two snakes, the perturbed spin-tune shift is given by [Eq. (2.34)]

$$\Delta Q_s = \frac{\pi|\epsilon|^2}{4} = 0.002, \tag{3.4}$$

which will be used as a constraint for the tolerable spin tune spread for various sources.

We observe from Sec. II C that the deviation of snake spin-rotation angle from π is equivalent to intrinsic resonances at all integers. Assuming that the integrated field strength of each snake dipole is 10^{-3}, the error in the spin-rotation angle of a snake should be about $\sqrt{8} \times 180 \times 10^{-3} = 0.5°$, where we have assumed that a snake is constructed from eight dipoles. The effect of N_s snakes in the accelerator will give a resonance strength of the order

$$\epsilon_{\rm imp}^{\rm eq} \approx 0.004 \quad \text{for RHIC}, \tag{3.5}$$

where we have assumed two snakes for RHIC. The corresponding spin-tune modulation is negligible [See Sec. III].

The error in the orbital angle between odd and even snake pairs can give rise [Eq. (2.23)] to a spin-tune shift of $\Delta\nu_s = \frac{1}{\pi}G\gamma\Delta\theta_{\rm odd}$ with $\Delta\theta_{\rm odd} = (\pi - \theta_{\rm odd})$. If the spin-tune shift is less than 0.002, the tolerable survey error should be less than $\Delta\theta_{\rm odd} \leq 1.4\times10^{-5}$ rad at $G\gamma = 450$. Since the error in the orbital angle gives rise to spin-tune shift (not spin tune spread), one can compensate the effect by adjusting the spin-precessing axes of snakes. Thus the tolerance may be raised to about 0.01. The survey error can be about $\Delta\theta_{\rm odd} \approx 10^{-4}$ rad. At a larger survey error, active compensation by adjusting the snake spin axes is needed.

The closed orbit can also cause orbital angle error between snakes. Let us assume that the maximum closed orbit is about $\hat{a}\hat{\beta} \approx 6\sigma \approx 0.6$ mm. The angular deviation is of the order of

$$\Delta x'_{\rm CO} \approx \frac{\hat{a}}{\sqrt{\beta\hat{\beta}}}, \tag{3.6}$$

where $\hat{a}$ is the maximum orbit error and $\sqrt{\beta\hat{\beta}} \approx \frac{R}{\nu}$ is the average betatron amplitude function. The expected error is about $\Delta x'_{\rm CO} \approx 1 \times 10^{-5}$ for RHIC, which is smaller than the survey requirement.

Similarly the betatron oscillation can cause orbital angle modulations. The spin-tune modulation is given by

$$\Delta\nu_{s,\beta} \approx \frac{1}{\pi}G\gamma\sqrt{\frac{\epsilon}{\beta}} = \frac{1}{\pi}G\sqrt{\frac{\gamma\epsilon_N}{\beta}}. \tag{3.7}$$

The resulting spin-tune spread is about 0.005 for a beam with 10π-mm mrad normalized emittance at 250 GeV. Combining all the possible sources, we expect the total spin tune spread to be about 0.008 by taking quadrature of all sources.

B. Betatron tune spreads and modulations

For an accelerator with two snakes, we found that the ninth-order snake resonances are not important if the intrinsic resonance strength is kept below $\epsilon_{\rm int} \leq \frac{1}{5}N_s \approx 0.4$. The available tune space is about 0.03. The control of power supply ripples needed to avoid nonlinear betatron resonances for the orbit stability is sure enough to limit the snake resonance modulations.

At the injection energy, the space charge tune spread can be as large as 0.02 for RHIC. However, the corresponding spin-resonance strength at low energy is also about a factor of 3 smaller. Thus the spin is less susceptible to field errors at injection.

C. Beam-beam interaction, nonlinear depolarizing resonances

The linear component of the beam-beam interaction gives rise to tune spread in the beam. The spin perturbation due to beam-beam interaction is more important in the e^+e^- colliders, where the linear beam-beam tune shift is about 0.05. For a hadron collider, the typical linear beam-beam tune shift is 0.005, which is small relative to the available tune space of 0.03. Since the betatron tunes of high-energy colliders should be chosen to be free from high-order betatron resonances, the betatron tunes are also free from snake resonances. Higher-order snake resonances should not be important provided that the spin-tune spread is small.

The most important spin resonances, arising from sextupoles, are located at

$$K = nP \pm \nu_x \pm \nu_z \approx mPM \pm \nu_{Bx} \pm \nu_{Bz}.$$

The maximum resonance strength is given by

$$\epsilon_K \approx \frac{1+G\gamma}{8\pi}\sqrt{\frac{\epsilon_x\epsilon_z}{\pi^2}}\sqrt{\beta_x\beta_z}PM(|S_F|+|S_D|), \quad (3.8)$$

where S_F, S_D are respectively strengths of sextupoles located at the focusing and defocusing quadrupole locations. Because the emittance decreases with energy, the sextupole spin-resonance strength is energy independent in hadron storage rings. For RHIC, the resonance strength is about 1.5×10^{-4} at a normalized emittance of 10π mm mrad. At the lowest-order snake resonance, $\nu_s + \nu_x + \nu_z =$ integer the beam depolarization may occur, which has been observed in the IUCF Cooler Ring at a 100% snake [5]. Higher-order snake resonances are not important.

D. Uncorrected solenoid field at IP

High-energy particle detectors usually use solenoid magnets. The solenoid can contribute to the imperfection resonance strength,

$$\epsilon_{\text{imp,sol}} = \frac{1+G}{2\pi}\frac{\int B_\parallel d\ell}{B\rho}. \quad (3.9)$$

For a 5-T m integrated solenoid field strength, the resonance strength is about 0.02 at the injection energy and 0.003 at 250 GeV for RHIC.

To achieve helicity state collision, two spin rotators are needed. The spin transfer matrix at the IP can be expressed as

$$e^{i\frac{\pi}{4}\sigma_1}e^{i\frac{1}{2}\theta_s\sigma_2}e^{-i\frac{\pi}{4}\sigma_1} = e^{-i\frac{1}{2}\theta_s\sigma_3}. \quad (3.10)$$

Thus the combination of 90° spin rotation along the $\hat{e}_1$ direction and the solenoidal field in the $\hat{e}_2$ direction gives rise to a spin rotation about the vertical axis, i.e., the so called "type-3 snake" [5, 12]. The amount of spin precession is proportional to the solenoidal field precessing angle θ_s, which is about 0.017 rad at 250 GeV. The corresponding spin tune shift is about 0.003, which is tolerable at 250 GeV in RHIC. At lower energy, the solenoid field should be decreased accordingly. However, since this is a systematic effect, one can arrange the snake to be located between two detectors with identical spin precessions. Partial cancellation of the type-3 snake effect can be arranged. Another solution is to use a local solenoid compensation scheme. One can also correct this tune shift by a small adjustment in snake axes.

E. Linear coupling

Linear coupling, which arises from skew quadrupoles and solenoids, is important in high-energy colliders. Linear coupling limits the tune space available and can cause coupling snake resonances,

$$Q_s \pm \ell Q_x = \text{integer}, \quad \ell = \text{integer}. \quad (3.11)$$

Linear coupling correction in RHIC can minimize the coupling snake resonances. One can also choose the betatron tune to avoid these snake resonances.

F. Polarization lifetime

To prevent beam depolarization, the constraints listed above should be addressed. With careful manipulation of the operation condition, the polarization lifetime should be, at least, as long as the beam lifetime. Adiabatic modulation within the tolerable limit does not affect the beam polarization. The spin vector will follow the spin closed orbit adiabatically.

Nonadiabatic process, arising from rf noise at the spin-precession frequency, can indeed cause beam depolarization. Let us consider that a single dipole with strength θ_k is modulating at $\nu_s f_0$, which is about 39 kHz for RHIC. The corresponding induced spin-precessing kick is $G\gamma\theta_k$. The number of turns that the spin is perturbed to 80% of the original polarization is given by

$$N_p = \frac{\arccos[0.8]}{G\gamma\theta_k}. \quad (3.12)$$

Let us now consider the same angular kick to the orbital motion. If there is an rf source at $\nu_s f_0$, one expects to have a similar angular kick at the frequency qf_0, where q is the fractional part of the betatron tune. The orbital survival turn is given by

$$N_0 = \frac{A}{\langle\beta\rangle\theta_k}, \quad (3.13)$$

where $\langle\beta\rangle$ is the average betatron amplitude, and A is the dynamical aperture. Using $A = 0.01$ m, $\langle\beta\rangle = 20$ m for RHIC, we found that the orbital lifetime is not longer than the polarization lifetime.

Indeed, any rf sources at high frequencies around the synchrotron and betatron tunes are dangerous to the orbital stability of particles in accelerators. Similarly, any rf source at the spin tune can cause beam depolarization. These high-frequency rf sources should be addressed carefully in hardware design. Synchrotron radiation is known also to limit the polarization lifetime of electron beam. However, synchrotron radiation is not important in RHIC energy.

IV. CONCLUSIONS

In conclusion, we found that snake resonances, located at $\nu_s + \ell K =$ integer, are the major source of depolarization in synchrotrons with snakes, where the integer ℓ is called the order of snake resonance, K is the spin-resonance harmonic given by $K = mP \pm \nu_z$ for the intrinsic resonance, $K =$ integer for an imperfection resonance, $K = mP \pm \nu_x$ for a linearly coupling intrinsic resonances, etc. For a perfect accelerator with only intrinsic resonances, only odd-order, $\ell =$ odd integers, snake resonances exist. On the other hand, when imperfection resonances are overlapping with intrinsic resonances, even-order snake resonances appear. A simple model was used to explain the existence of even-order snake resonances. The perturbed spin tune, arising from imperfection resonances, is found to play an essential role in the depolarization mechanism; it causes each snake resonance to split into two resonances. Thus the available tune space becomes smaller.

For polarized colliders, the depolarization sources are the spin-tune modulation and/or the betatron tune modulation so that the betatron tune overlaps with an important snake resonance line. Spin-tune modulation may arise from the deviation of the orbital angle between snakes from π caused by survey and alignment error, the closed-orbit error, and the betatron motion. The most severe constraint might be the survey and alignment error of the orbital angle between snakes, where the magnitude of spin-tune shift is proportional to the $G\gamma$ value. An alignment angular error better than 10^{-4} rad is recommended. The expected spin-tune spread for the proton beam, arising from the imperfection resonance strength of the order 0.05, the rms closed-orbit error of 0.2 mm, and the betatron oscillation at 10π mm mrad normalized emittance, is about 0.008. The betatron tune modulation for the polarized proton collider mode should be kept under 0.005, which is required for maintaining transverse orbital dynamical aperture. We also found that the combination of the uncompensated solenoid and the helicity spin rotator contribute to a type-3 snake. Methods to compensate the effect were also discussed.

We define the critical snake resonance strength of order ℓ as the maximum resonance strength at the condition $\nu_s + \ell K =$ integer. We found that the critical snake resonance strength depends linearly on the order of snake resonance. At the same time, the critical resonance strength seems to depend on the acceleration rate in a power law. These studies are important to the depolarization lifetime for the storage ring. Details of these studies will be reported shortly.

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