Weakly nonlocal hydrodynamics

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Anomalous viscoelastic, diffusion, and heat-transfer phenomena observed in spatially inhomogeneous simple and complex fluids are analyzed in this paper in the setting of weakly nonlocal hydrodynamics. Governing equations of this generalized hydrodynamics involve higher-order derivatives with respect to the position coordinate. The governing equations are obtained on the basis of the following consideration. The time evolution in both local and weakly nonlocal hydrodynamics is generated by a thermodynamic potential in a state space equipped with a Poisson structure and a dissipative potential. The Poisson structure is an expression of kinematics in the chosen state space. In weakly nonlocal hydrodynamics the Poisson structure remains the same as in local hydrodynamics but the potentials are generalized. The potentials are allowed to depend on higher-order derivatives of the hydrodynamic fields that are chosen as state variables. This way of introducing the governing equations guarantees that the equations are intrinsically compatible and that their solutions agree with certain fundamental macroscopic observations (e.g., the observations constituting the basis of equilibrium thermodynamics).

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I. INTRODUCTION

The governing equations of classical hydrodynamics can be introduced and justified on the basis of three types of arguments. Following arguments of the first type, called here microscopic arguments, the analysis begins with a microscopic theory (e.g., quantum mechanics of all molecules composing the fluids under consideration). The governing equations of hydrodynamics are introduced as follows. First, the governing equations of the chosen microscopic theory are solved for all admissible initial and boundary conditions. Then the pattern that is registered in the hydrodynamic measurements is extracted from the solutions. The equations that govern the time evolution of the pattern are the hydrodynamic equations. Following arguments of the second type, called here mesoscopic arguments, we try to understand and express dynamics of fluids without leaving the level of description introduced in hydrodynamics. In this way we introduce, for example, local conservation laws, hypothesis of local equilibrium, and the Lagrangian description in which fluid particles move according to Newton's law. Following arguments of the third type, called here macroscopic arguments, we look towards more macroscopic (less detailed) theories (e.g., equilibrium thermodynamics). We require that solutions to the hydrodynamic equations agree with some well-established macroscopic experience. For example, solutions of the hydrodynamic equations are required to obey global conservation of mass, momentum, energy, and the dissipation inequality that guarantees agreement with the experience constituting the basis of equilibrium thermodynamics.

None of these three types of arguments is sufficient by itself to provide a solid foundation of classical hydrodynamics. The difficulties with the microscopic approach lie in the knowledge of the starting microscopic theory

and, of course, in the task of finding all solutions and extracting from them the pertinent pattern. The mesoscopic arguments remain always limited in their implications. They limit the freedom of choice of the admissible governing equations but do not specify the equations completely. It is the combination and the mutual interrelationship of all three types of arguments that constitutes the foundation of classical hydrodynamics.

The setting of classical hydrodynamics becomes inadequate if, for example, the fluids under consideration become complex fluids (e.g., polymeric fluids) and/or the fluids involve strong inhomogeneities (e.g., interfaces). Hydrodynamics that is suitable for this type of fluids has to be formulated in an extended state space (that includes also quantities describing the internal structure of the fluids) and/or has to involve nonlocal (integral) operators in the position coordinate. We shall use the following terminology. Hydrodynamics will be called extended if the set of state variables is enlarged. Hydrodynamics will be called nonlocal (weakly nonlocal) if its governing equations involve nonlocal integral operators in the position coordinate (higher-order derivatives with respect to the position coordinate). The question arises of how the microscopic, mesoscopic, and macroscopic arguments developed in the context of classical hydrodynamics extend to extended and nonlocal hydrodynamics.

The main objective of this paper is to show how the mesoscopic arguments based on the Lagrangian viewpoint lead to the governing equations of weakly nonlocal classical hydrodynamics (Sec. II) and weakly nonlocal extended hydrodynamics of one- and two-component fluids (Sec. III). We recover the well-known and experimentally well-established effects (e.g., appearance of new stresses in spatially inhomogeneous complex fluids [1–3], non-Fickean diffusion in two-component complex fluids [4–6], and migration across streamlines in inhomogene-

ous flows of two-component complex fluids [7–11]) and show that these effects are necessarily accompanied by the appearance of other new stresses and reversible heat fluxes. The mesoscopic and the macroscopic arguments developed in Secs. II and III are then combined with microscopic arguments in Sec. IV.

II. WEAKLY NONLOCAL CLASSICAL HYDRODYNAMICS

In this section we recall first (Sec. II A) the mesoscopic and macroscopic analysis of classical hydrodynamics that is based on the insight provided by the Lagrangian view of fluid motion. This type of analysis is then applied in Sec. II B to weakly nonlocal classical hydrodynamics and in Sec. III to weakly nonlocal extended hydrodynamics.

A. Classical hydrodynamics

The fluids under consideration in this subsection are classical simple fluids (e.g., water). Following the experience collected in classical hydrodynamics, we choose $\rho(\mathbf{r})$, $s(\mathbf{r})$, $\mathbf{u}(\mathbf{r})$ denoting the fields of mass, entropy, and momentum, to characterize states of the fluids. One set of mesoscopic and macroscopic arguments that lead to the governing equations of classical hydrodynamics consists of the requirement of local conservation, hypothesis of local equilibrium, and the dissipation inequality. The first two arguments do not, however, extend to more general settings. An alternative set of mesoscopic and macroscopic arguments that lead to the same governing equations of classical hydrodynamics is based on the Lagrangian viewpoint of fluid motion and the dissipation inequality. The essence of these arguments is the realization that the nondissipative time evolution is Hamiltonian. The arguments become particularly well adaptable to different settings if the canonical Hamiltonian structure that arises in the Lagrangian description is transformed into a more general Poisson structure in the Eulerian description [12–15]. If the nondissipative Hamiltonian time evolution is combined with the dissipative time evolution (the requirement of the dissipation inequality is used) [16–21] then the complete time evolution can be regarded as a nonlinear extension of the Onsager-Casimir reciprocity relations [22,23] and also as an extension of the Cahn-Hilliard equation [24] to a dynamical equation that takes into account also the nondissipative time evolution. We now recall how the classical hydrodynamic equations arise from an analysis based on the Lagrangian description. The sequence of steps that will be established in this subsection will be then followed in the rest of the paper.

Following [16–21] we introduce the governing equations of classical hydrodynamics as time evolution equations generated by the thermodynamic potential Φ. Gradient of this potential is transformed into a vector field by the Poisson structure (characterized by a Poisson bracket $\{A,B\}$), expressing kinematics in the state space of classical hydrodynamic fields, and by the structure provided by another potential Ψ (called the dissipation potential), expressing the dissipative mechanism in the time evolution. We shall now introduce Φ, $\{A,B\}$, and Ψ.

First, we turn our attention to the thermodynamic potential Φ. Let the state space of classical hydrodynamics be denoted by the symbol M_{hyd} [i.e., $(\rho,s,\mathbf{u})\in M_{\text{hyd}}$]. The state space of equilibrium thermodynamics will be denoted by M_{ET}. Elements of M_{ET} are S, M, denoting entropy and mass, respectively. We need now to know how the thermodynamic variables are expressed in terms of the hydrodynamic state variables. From the mathematical point of view, we introduce into M_{hyd} a bundle structure: $M_{\text{hyd}} = (M_{\text{hyd}}, M_{\text{ET}}, \Pi_{\text{ET}}^{\text{hyd}})$, where M_{hyd} is the total space, M_{ET} is the base space, and $\Pi_{\text{ET}}^{\text{hyd}}$: $M_{\text{hyd}} \to M_{\text{ET}}$ is the bundle projection. The physical interpretation of the hydrodynamic fields suggests

$$S = S(\rho,s,\mathbf{u}) = \int d^3\mathbf{r}\, s(\mathbf{r}) \ ,$$
$$M = M(\rho,s,\mathbf{u}) = \int d^3\mathbf{r}\, \rho(\mathbf{r}) \ , \tag{1}$$

where the integration is over the space region in which the fluid under consideration is confined. The volume of this region will be denoted by the symbol V. The fundamental thermodynamic relation in M_{ET} is a relation,

$$E = E(S,M) \ , \tag{2}$$

where E is the energy. In M_{hyd}, the fundamental thermodynamic relation is a relation,

$$E = E(\rho,s,\mathbf{u}) = \int d^3\mathbf{r}\, e(\rho,s,\mathbf{u};\mathbf{r}) \ , \tag{3}$$

where $e(\rho,s,\mathbf{u};\mathbf{r})$ is the total-energy field. In order to relate (3) and (2) we introduce the thermodynamic potential

$$\Phi(\rho,s,\mathbf{u};T,\mu) = E(\rho,s,\mathbf{u}) - TS(\rho,s,\mathbf{u})$$
$$- \frac{\mu}{M_0} M(\rho,s,\mathbf{u}) \ , \tag{4}$$

where T is the equilibrium temperature, μ is the chemical potential, and M_0 is the molar mass. Next we look for solutions of

$$\frac{\delta\Phi}{\delta\rho(\mathbf{r})} = 0, \quad \frac{\delta\Phi}{\delta s(\mathbf{r})} = 0, \quad \frac{\delta\Phi}{\delta\mathbf{u}(\mathbf{r})} = 0 \ , \tag{5}$$

which we shall call thermodynamic equilibrium states [we denote them by $(\rho,s,\mathbf{u})_{\text{ET}}$]. The symbols $\delta/\delta\rho(\mathbf{r})$, etc., denote the Volterra functional derivatives. By evaluating Φ at $(\rho,s,\mathbf{u})_{\text{ET}}$ and equating the result with $-PV$, where P is the thermodynamic pressure, we obtain the dual (Gibbs-Duhem) form

$$P = P(\mu,T) \tag{6}$$

of (2). We shall leave the specification of the fundamental thermodynamic relation in M_{hyd} [see (3)] open. In the mesoscopic approach that we now follow, this relation enters the analysis as part of the input. We shall return to this point in Sec. IV, where we shall follow the microscopic arguments.

The second building block of dynamics is the Poisson structure characterized by the Poisson bracket $\{A,B\}$, where $A,B,\ldots$ are sufficiently regular functionals of $\rho,s,\mathbf{u}$. The canonical Hamiltonian structure revealed in the Lagrangian description transforms in the Eulerian

description into the Poisson structure characterized by the Poisson bracket [12–15]

$$\{A,B\} = \int d^3\mathbf{r}[u_\gamma(\partial_\alpha(A_{u_\gamma})B_{u_\alpha}-\partial_\alpha(B_{u_\gamma})A_{u_\alpha})$$

$$+\rho(\partial_\alpha(A_\rho)B_{u_\alpha}-\partial_\alpha(B_\rho)A_{u_\alpha})$$

$$+s(\partial_\alpha(A_s)B_{u_\alpha}-\partial_\alpha(B_s)A_{u_\alpha})] , \qquad (7)$$

where $A_\rho \equiv \delta A/\delta\rho(\mathbf{r})$ and $\partial_\alpha \equiv \partial/\partial r_\alpha$. It can be directly verified that the bracket (7) is antisymmetric ($\{A,B\}=-\{B,A\}$) and satisfies the Jacobi identity ($\{\{A,B\},C\}+\{\{B,C\},A\}+\{\{C,A\},B\}=0$). We also note that the base space M_{ET} is composed of distinguished functionals. A functional C is called distinguished if $\{A,C\}=0$ for all functionals A. The equations governing the nondissipative time evolution of $\rho,s,\mathbf{u}$ are obtained as follows. We require that

$$\partial_t F = \{F,\Phi\} \qquad (8)$$

holds for all functionals F. The left-hand side of (8) can be written as

$$\int d^3\mathbf{r}(F_{u_\alpha}\partial_t u_\alpha+F_\rho\partial_t\rho+F_s\partial_t s) \qquad (9)$$

and the right-hand side as

$$\int d^3\mathbf{r}(F_{u_\gamma}[-\partial_\alpha(u_\gamma\Phi_{u_\alpha})-u_\alpha\partial_\gamma(\Phi_{u_\alpha})$$

$$-\rho\partial_\alpha(\Phi_\rho)-s\partial_\alpha(\Phi_s)]$$

$$+F_\rho[-\partial_\alpha(\rho\Phi_{u_\alpha})]+F_s[-\partial_\alpha(s\Phi_{u_\alpha})]) , \qquad (10)$$

provided we use the boundary conditions that make all integrals over the surface arising in integration by parts equal to zero. Since (8) is required to hold for all F, we have from (9) and (10)

$$\partial_t\rho = -\partial_\gamma(\rho\Phi_{u_\gamma}) ,$$

$$\partial_t u_\alpha = -\partial_\gamma(u_\alpha\Phi_{u_\gamma})-\rho\partial_\alpha\Phi_\rho-s\partial_\alpha\Phi_s-u_\gamma\partial_\alpha\Phi_{u_\gamma} , \qquad (11)$$

$$\partial_t s = -\partial_\gamma(s\Phi_{u_\gamma}) .$$

If we now use the requirement of conservation of the total momentum $\mathbf{U}(\rho,s,\mathbf{u})=\int d^3\mathbf{r}\,\mathbf{u}(\mathbf{r})$, (i.e., a macroscopic argument) we see that there has to exist a quantity, which we shall call p, such that

$$-\partial_\alpha p = -\rho\partial_\alpha\Phi_\rho-s\partial_\alpha\Phi_s-u_\gamma\partial_\alpha\Phi_{u_\gamma} \qquad (12)$$

and thus

$$p = -e+\rho e_\rho+s e_s+u_\gamma e_{u_\gamma} . \qquad (13)$$

Equations (11)–(13) become the governing equations of classical (Euler) nondissipative hydrodynamics if $\Phi_\mathbf{u}$ is a fluid velocity (this is indeed the case if $e=\frac{1}{2}\mathbf{u}^2/\rho+$ an internal energy that is independent of $\mathbf{u}$) and p is interpreted as a local pressure. The relation (13) is the relation that is obtained in the standard mesoscopic approach (that begins with local conservation laws) as a consequence of the hypothesis of local equilibrium. Ac-

cording to this hypothesis $p(\mathbf{r})$ is related to $\rho(\mathbf{r}),e(\mathbf{r}),s(\mathbf{r})$, pointwise for all $\mathbf{r}$, as thermodynamic pressure is related to thermodynamic mass density, energy, and entropy. We see that this hypothesis is not needed if the mesoscopic argument based on the Lagrangian description is followed. The relation (13) arises as a result. This represents a considerable advantage over the standard approach since extensions of the hypothesis of local equilibrium needed in settings generalizing classical hydrodynamics are far from obvious. We shall appreciate this advantage in Secs. II B and III. The equation governing the nondissipative time evolution of the energy field $e(\rho,s,\mathbf{u};\mathbf{r})$ is obtained as follows:

$$\partial_t e = e_\rho\partial_t\rho+e_s\partial_t s+e_{u_\gamma}\partial_t u_\gamma . \qquad (14)$$

By using (11)–(13) we recast (14) into

$$\partial_t e = -\partial_\gamma(e\Phi_{u_\gamma}+p\Phi_{u_\gamma}) . \qquad (15)$$

This means that in classical hydrodynamics the heat flux is absent in the nondissipative time evolution of classical hydrodynamics.

In order to introduce the dissipative part of the time evolution, we shall follow [25–27,16] and introduce a real-valued function of

$$D_{\alpha\beta}=\frac{1}{2}(\partial_\alpha\Phi_{u_\beta}+\partial_\beta\Phi_{u_\alpha})$$

and $Q_\alpha=\partial_\alpha\Phi_s$, called a dissipation potential Ψ. A potential $\Psi(x)$ is called a dissipation potential if it satisfies the following properties:

$$\Psi(0)=0 ,$$

Ψ reaches its minimum at 0 , $\qquad (16)$

Ψ is convex in a neighborhood of 0 .

The dissipative time evolution is governed by

$$\partial_t\rho=0 ,$$

$$\partial_t u_\alpha=\partial_\gamma\Psi_{D_{\alpha\gamma}} , \qquad (17)$$

$$\partial_t s=\partial_\gamma\Psi_{Q_\gamma}+\sigma ,$$

where

$$\sigma=\frac{1}{e_s}(D_{\alpha\gamma}\Psi_{D_{\alpha\gamma}}+Q_\alpha\Psi_{Q_\alpha})\geq 0 . \qquad (18)$$

The inequality (18) arises as a consequence of the properties (16) and $e_s>0$. The time evolution equations (17) imply then

$$\partial_t e=\partial_\gamma(\Phi_{u_\alpha}\Psi_{D_{\alpha\gamma}})+\partial_\gamma(e_s\Psi_{Q_\gamma}) . \qquad (19)$$

If we now put together (11)–(13) and (17)–(19) we obtain the complete set of the time evolution equations of classical hydrodynamics.

We note that solutions of these equations satisfy the following two properties.

(i) The global conservation laws

$$\partial_t M=0, \quad \partial_t\mathbf{U}=0, \quad \partial_t E=0 , \qquad (20)$$

expressing conservation of the total mass, total momentum, and the total energy. Equations (20) follow directly from the degeneracy of the Poisson bracket (7) and from (12).

(ii) The global dissipation law

$$\partial_t \Phi \leq 0 \qquad (21)$$

that allows one to interpret the thermodynamic potential Φ as a Lyapunov function associated with the approach to thermodynamic equilibrium states. The inequality (21) follows directly from (8) (note that $\partial_t \Phi = \{\Phi, \Phi\} = 0$) and from (18). The equality sign in (21) [i.e., the case when (21) represents an additional conservation law] holds only if the dissipation potential $\Psi \equiv 0$. The time evolution equations for which the equality sign in (21) holds are called nondissipative (Euler hydrodynamics equation in the context of hydrodynamics).

An additional insight into the nature of the time evolution equations (11)–(13) and (17)–(19) can be gained by linearizing them about an equilibrium state [i.e., about a solution of (5)]. Let $\partial_t \varphi = P^- \varphi + P^+ \varphi$ be the equation governing the linearized time evolution in a small neighborhood U_{ET} of the thermodynamic equilibrium state, $\varphi \in U_{ET}$. By P^- we denote the linear operator governing the nondissipative time evolution [i.e., the time evolution with $\Psi \equiv 0$ and thus with the equality sign in (21)] and P^+ the linear operator governing the dissipative part of the time evolution. Let F denote the Hessian of the thermodynamic potential Φ evaluated at the equilibrium state. The properties of the Poisson bracket and the properties (16) of the dissipative potential imply that the linear operator FP^- is formally skew adjoint with respect to the L_2 inner product and the linear operator FP^+ is formally self-adjoint with respect to the L_2 inner product and nonpositive (we have to insist on formal self-adjointness and skew-adjointness since we do not specify details of the domains of the linear operators involved). These properties of FP^- and FP^+ constitute the statement of the Onsager-Casimir reciprocity relations [22,23]. We can thus regard the nonlinear time evolution formulated in terms of Poisson brackets and dissipative potentials as a nonlinear extension of the Onsager-Casimir reciprocity relations. We note that P^- and P^+ (and similarly also the nondissipative and dissipative parts of the nonlinear time evolution equations) can be distinguished also by introducing the so-called parity operator $\hat{I}$;

$$(\rho, s, \mathbf{u}) \xrightarrow[\hat{I}]{} (\rho, s, -\mathbf{u}) \ .$$

Note that $\hat{I} \circ \hat{I}$ = identity. The state variables ρ and s are said to have positive parity and $\mathbf{u}$ negative parity (a magnetic field is another example of a state variable having negative parity). If the dissipation potential $\Psi \equiv 0$ (i.e., the time evolution is nondissipative) then the parity operator $\hat{I}$ compensates the reversion of time. We see easily that $\hat{I} FP^- \hat{I} = -FP^- = (FP^-)^\dagger$ [$( )^\dagger$ stands for "formally adjoint"]. The time evolution is called in this case time reversible. The presence of the operator P^+ makes the time evolution time irreversible. We easily verify that $\hat{I} FP^+ \hat{I} = FP^+ = (FP^+)^\dagger$. The linearized time

evolution arising in fully microscopic (particle) description of a macroscopic system is known to be time reversible with P^- satisfying $\hat{I} FP^- \hat{I} = -FP^- = (FP^-)^\dagger$ and $\Psi \equiv 0$. The mesoscopic linear time evolution equations thus retain this feature of the fully microscopic time evolution in the nondissipative part of the time evolution. Similarly, the nonlinear fully microscopic time evolution equations are known to be Hamiltonian. The mesoscopic nonlinear time evolution equations thus retain this feature of the microscopic time evolution in the nondissipative part involving the Poisson bracket. The rest of the time evolution (i.e., the part involving the dissipation potential) is time irreversible. Onsager [22] has introduced an argument relating the microscopic time reversibility to the Onsager-Casimir relations on mesoscopic levels of description. Whether and how the Onsager argument extends to the nonlinear time evolution and can thus be used to relate the microscopic time reversibility to the generalized Hamiltonian structure of the nonlinear time evolution equations remains an open question.

B. Weakly nonlocal extension of classical hydrodynamics

The fluids under consideration in this subsection and in Sec. III are fluids that are strongly inhomogeneous (e.g., fluids that involve interfaces) and/or fluids with long-range interactions and long-range correlations (e.g., electrically charged fluids). The setting of classical hydrodynamics becomes inadequate to describe dynamics of these fluids. To obtain a suitable setting, three modifications of classical hydrodynamics have been suggested. First, the regions of strong inhomogeneity (e.g., interfaces [28,29]) or a substance, real or fictitious, that mediates the long-range interactions (e.g., electromagnetic field in electrically charged fluids or a network of entangled macromolecules in polymeric fluids [30–32]) is added as a new component. The state space is thus extended by the fields characterizing the new component. In the second modification, the time evolution equations are allowed to include nonlocal operators (integral operators in the position coordinate) [33]. In the third modification, higher-order derivatives with respect to position coordinates are allowed to appear in the governing equations. The third modification is thus a particular case of the second modification. We recall that, according to the terminology introduced in the Introduction, the hydrodynamics arising in the third modification is called weakly nonlocal. This hydrodynamics has been discussed by using various mesoscopic and macroscopic arguments in [34–39,11] and microscopic arguments (higher-order approximations in the Chapman-Enskog solution of the Boltzmann kinetic equation) in [40–42]. We now proceed to discuss the weakly nonlocal hydrodynamics by using the approach introduced in Sec. II A.

In this subsection, we shall still take the classical hydrodynamic fields $\rho(\mathbf{r}), s(\mathbf{r}), \mathbf{u}(\mathbf{r})$ as the only state variables. We shall, however, assume that the thermodynamic potential Φ can be a function of $\rho(\mathbf{r}), s(\mathbf{r}), \mathbf{u}(\mathbf{n})$ and also of their derivatives with respect to the position coordinate. The dissipation potential Ψ is also allowed to depend on first and higher derivatives of $\rho(\mathbf{r}), s(\mathbf{r}), \mathbf{u}(\mathbf{r})$ with

respect to $\mathbf{r}$. For the sake of simplicity, we shall assume in this subsection that Φ depends only on $\rho(\mathbf{r}), s(\mathbf{r}), \mathbf{u}(\mathbf{r})$ and their first derivatives with respect to $\mathbf{r}$. Specification of the functional dependence of Φ on $\rho, s, \mathbf{u}$ and their derivatives is left open. A microscopic analysis leading to such specification is introduced in Sec. IV. Thermodynamic potentials depending on derivatives of the chosen state variables with respect to $\mathbf{r}$ or on quantities that become later in the analysis associated with such derivatives have also been introduced, for example, in [43,44,24,1–3]. We shall focus our attention on the nondissipative time evolution. Some of the methods to include higher-order derivatives in the dissipative part of the time evolution have already been explored in [37,38].

We recall that the Poisson bracket expresses in mathematical terms kinematics in the chosen state space. Since we keep the same state variables as in the previous subsection, we shall also keep the same Poisson bracket (7). The derivatives $\delta/\delta\rho(\mathbf{r})$, etc., are the variational derivatives. If Φ depends on $\rho(\mathbf{r})$ and also on $\partial_\alpha\rho(\mathbf{r})$, we have

$$\frac{\delta\Phi}{\delta\rho(\mathbf{r})} = \Phi_{\rho(\mathbf{r})} - \partial_\alpha\Phi_{\partial_\alpha\rho(\mathbf{r})} \tag{22}$$

and similarly for other fields. If we now replace in (7) Φ_ρ, etc., by (22) and follow exactly the same steps as in Sec. II A, we obtain, after straightforward calculations, the following result:

$$\partial_t\rho = -\partial_\gamma(\rho(\Phi_{u_\gamma} - \partial_\beta\Phi_{\partial_\beta u_\gamma})),$$

$$\partial_t u_\alpha = -\partial_\gamma(u_\alpha(\Phi_{u_\gamma} - \partial_\beta\Phi_{\partial_\beta u_\gamma})) - \partial_\alpha p - \partial_\gamma\tau_{\alpha\gamma},$$

$$\partial_t s = -\partial_\gamma(s(\Phi_{u_\gamma} - \partial_\beta\Phi_{\partial_\beta u_\gamma})),$$

$$\partial_t e = -\partial_\gamma((e+p)(\Phi_{u_\gamma} - \partial_\beta\Phi_{\partial_\beta u_\gamma})$$

$$+\tau_{\gamma\alpha}(\Phi_{u_\alpha} - \partial_\beta\Phi_{\partial_\beta u_\alpha}) + q_\gamma),$$
$$\tag{23}$$

where

$$p = -\Phi + \rho\Phi_\rho + s\Phi_s + u_\gamma\Phi_{u_\gamma}$$

$$+\partial_\alpha\rho\Phi_{\partial_\alpha\rho} + \partial_\alpha s\Phi_{\partial_\alpha s} + \partial_\alpha u_\gamma\Phi_{\partial_\alpha u_\gamma}, \tag{24}$$

$$\tau_{\alpha\beta} = -\rho\partial_\alpha\Phi_{\partial_\beta\rho} - s\partial_\alpha\Phi_{\partial_\beta s} - u_\gamma\partial_\alpha\Phi_{\partial_\beta u_\gamma}, \tag{25}$$

$$q_\alpha = -\Phi_{\partial_\alpha u_\gamma}(\partial_\gamma p + \partial_\beta\tau_{\gamma\beta}) - \rho\Phi_{\partial_\gamma\rho}\partial_\gamma(\Phi_{u_\alpha} - \partial_\beta\Phi_{\partial_\beta u_\alpha})$$

$$-s\Phi_{\partial_\gamma s}\partial_\gamma(\Phi_{u_\alpha} - \partial_\beta\Phi_{\partial_\beta u_\alpha})$$

$$-u_\gamma\Phi_{\partial_\epsilon u_\gamma}\partial_\epsilon(\Phi_{u_\alpha} - \partial_\beta\Phi_{\partial_\beta u_\alpha}). \tag{26}$$

Equations (23)–(26) are the equations governing the nondissipative time evolution in weakly nonlocal hydrodynamics generated by the thermodynamic potential depending on $\rho(\mathbf{r}), s(\mathbf{r}), \mathbf{u}(\mathbf{r})$ and $\partial_\alpha\rho(\mathbf{r})$, $\partial_\alpha s(\mathbf{r})$, $\partial_\alpha u_\beta(\mathbf{r})$. We stress that Eqs. (23)–(26) have arisen as a direct consequence of the Poisson bracket (7) and the requirement of conservation of total momentum. There are no other physical considerations and assumptions involved in the derivation of (23)–(26).

A few comments concerning the physical interpretation of (23)–(26) are now in order. We note that if the thermodynamic potential Φ is independent of $\partial_\alpha\rho, \partial_\alpha s, \partial_\alpha u_\beta$ then (23)–(26) reduce to (11)–(15) governing the time evolution in classical hydrodynamics. If Φ depends on $\rho, s, \mathbf{u}$ as well as on $\partial_\alpha\rho, \partial_\alpha s, \partial_\alpha u_\beta$ then the conjugate state variables [i.e., $\delta\Phi/\delta\rho(\mathbf{r})$, $\delta\Phi/\delta s(\mathbf{r})$, $\delta\Phi/\delta u(\mathbf{r})$] appear as having the form of a sum of two terms [see (22)]. The first term can be interpreted physically as the conjugate state variable corresponding to the homogeneous component and the second as corresponding to another component that we can call an inhomogeneous component. To see more explicitly this two-component view of a one-component inhomogeneous fluid we introduce a particular form

$$\Phi(\rho, s, \mathbf{u}, \partial_\alpha\rho, \partial_\alpha s, \partial_\alpha u_\beta jT, \mu) = \Phi_{\text{cl. hyd}}(\rho, s, \mathbf{u}, T, \mu)$$

$$+a(\rho, s, \mathbf{u})\partial_\alpha\rho\partial_\alpha\rho$$

$$+b(\rho, s, \mathbf{u})\partial_\alpha s\partial_\alpha s$$

$$+c(\rho, s, \mathbf{u})\partial_\alpha u_\beta\partial_\alpha u_\beta \tag{27}$$

of the thermodynamic potential. By $\Phi_{\text{cl. hyd}}$ we denote the thermodynamic potential introduced in classical hydrodynamics (i.e., in Sec. II A) and a, b, c are functions of $\rho, s, \mathbf{u}$. Thermodynamic potentials that bear some similarity to (27) arise, for example, in extended irreversible thermodynamics [44] [see also (74)]. From (27), we see that the velocity field [conjugate to the momentum field $\mathbf{u}(\mathbf{r})$] equals $(\Phi_{\text{cl. hyd}})_{u_\alpha}$, which can be interpreted as the velocity of the homogeneous component, minus $\partial_\gamma(c(\rho, s, \mathbf{u})\partial_\gamma u_\alpha)$, which can be interpreted as the velocity of the inhomogeneous component. The quantity c is related to the mass density of the inhomogeneous component. This two-component view is then consistent also with the generalization (24) of the hypothesis of local equilibrium. The terms in (24) that are new if compared with (13) can be interpreted as arising due to the appearance of a new component. The new component then also makes the fluid behave as a viscoelastic fluid [due to the appearance of the nondissipative extra stress tensor (25)] and a "heat-elastic" fluid [due to the appearance of the nondissipative heat flux (26)]. It is also important to note that if Φ depends on the fields as well as on their derivatives then we cannot pass freely from the energy representation to the entropy representation [in which the state variables are the fields $\rho(\mathbf{r}), e(\mathbf{r}), \mathbf{u}(\mathbf{r})$]. Also, the relation between state variables and their conjugates is not one-to-one.

Weakly nonlocal extension of linear hydrodynamic equations (linearized about a thermodynamic equilibrium state) has been discussed in [36]. Let

$$\partial_t\varphi = P^-\varphi + P^+\varphi \tag{28}$$

be the governing equation of the linearized classical hydrodynamics. Equation (28) thus possesses the Onsager-Casimir symmetry, (i.e., FP^+ is a formally self-adjoint linear operator, FP^- is a formally skew-adjoint operator, and F is the Hessian of the classical thermodynamic potential that depends on $\rho, s, \mathbf{u}$ but not on their derivatives). Now, linear operators $\tilde{P}^+$ and $\tilde{P}^-$ are introduced

by adding to P^+ and P^- terms involving higher-order derivatives. One looks then for $\bar{F}$ so that $\bar{F}P^+$ is a formally self-adjoint linear operator and $\bar{F}P^-$ is a formally skew-adjoint linear operator. The $\bar{F}$ found in this way is then interpreted as the Hessian of an extended thermodynamic potential that depends on $\rho(\mathbf{r})s(\mathbf{r}),\mathbf{u}(\mathbf{r})$ and also on $\partial_\alpha\rho(\mathbf{r})$, $\partial_\alpha s(\mathbf{r})$, $\partial_\alpha u_\beta(\mathbf{r})$. The analysis presented in this section can be thus seen as a nonlinear extension of the analysis of linear weakly nonlocal hydrodynamics presented in [36].

As for the dissipative time evolution, we shall be content with (17). Since $\Phi_\mathbf{u}$ and Φ_e are replaced now by variational derivatives (22), the dissipative time evolution equation involves fourth-order derivatives with respect to $\mathbf{r}$. Dissipation potentials involving higher-order derivatives can be introduced, for example, by casting the analysis in [37,38] into the form involving a dissipation potential.

III. WEAKLY NONLOCAL EXTENDED HYDRODYNAMICS

The fluids under consideration in this section are complex fluids such as, for example, polymeric fluids, suspensions, and liquid crystals. Since the internal structure in these fluids evolves on the time scale that is comparable with the time scale on which the classical hydrodynamic fields evolve, the time evolution of the internal structure cannot be separated from the time evolution of the hydrodynamic fields. Consequently, the state space has to be enlarged by adopting new state variables characterizing the internal structure. Following Hand [45] (see also [21]) we shall choose a symmetric tensor field $C(\mathbf{r})$, called a conformation tensor field, to characterize the internal structure. We could also choose, for example, a configuration space distribution function $\psi(\mathbf{r},\mathbf{R})$, where

$\mathbf{R}$ is the end-to-end vector of a macromolecule or a suspended particle [the conformation tensor C can be expressed in terms of ψ as $c_{\alpha\beta}(\mathbf{r})=\int \alpha^3\mathbf{R}\,R_\alpha R_\beta\psi(\mathbf{r},\mathbf{R})$] or a vector field $\mathbf{n}(\mathbf{r})$ characterizing the end-to-end vector field of macromolecules or suspended particles (used, for example, in the Leslie-Ericksen theory of liquid crystals [1–3]).

In Sec. III A we consider one-component weakly nonlocal complex fluids. Kinematics (i.e., the Poisson bracket) of the state variables $\rho,s,\mathbf{u},C$ has been analyzed in [16,46,47]. The route to the time evolution equations established in Sec. II can be thus followed in a straightforward manner. In Sec. III B we consider two-component weakly nonlocal complex fluids. Kinematics and the corresponding Poisson bracket for two-component complex fluids is introduced in Sec. III B. Having the Poisson bracket, we can then arrive at the time evolution equations (describing, among other effects, the nonlinear non-Fickean diffusion and migration across streamlines) by following the same route as in Sec. II.

A. One-component weakly nonlocal complex fluids

Kinematics in the classical hydrodynamic state space is the kinematics deduced from the Lagrangian description [this part of the kinematics is expressed in the first term of (7)] and from viewing the scalar fields $\rho(\mathbf{r})$ and $s(\mathbf{r})$ as being simply advected by the material flow [this part of the kinematics is expressed in the second and the third term of (7)]. If we now adopt the conformation tensor field $C(\mathbf{r})$ as an extra state variable, we view its kinematics also as a simple advection by the material flow [16,46,47]. This then implies that

$$\{\,A,B\,\}=\{\,A,B\,\}^{(\text{cl.hyd})}+\{\,A,B\,\}^{(c)}\,, \tag{29}$$

where $\{\,A,B\,\}^{(\text{cl.hyd})}$ is the Poisson bracket (7) and

$$\{\,A,B\,\}^{(c)}=\int d^3\mathbf{r}\, c_{\gamma\beta}\left[\partial_\alpha\left[\frac{\delta A}{\delta c_{\gamma\beta}}\right]\frac{\delta B}{\delta u_\alpha}-\partial_\alpha\left[\frac{\delta B}{\delta c_{\gamma\beta}}\right]\frac{\delta A}{\delta u_\alpha}\right]$$
$$+c_{\gamma\beta}\left[\frac{\delta A}{\delta c_{\alpha\beta}}\partial_\gamma\left[\frac{\delta B}{\delta u_\alpha}\right]+\frac{\delta A}{\delta c_{\gamma\alpha}}\partial_\beta\left[\frac{\delta B}{\delta u_\alpha}\right]-\frac{\delta B}{\delta c_{\alpha\beta}}\partial_\gamma\left[\frac{\delta A}{\delta u_\alpha}\right]-\frac{\delta B}{\delta c_{\gamma\alpha}}\partial_\beta\left[\frac{\delta A}{\delta u_\alpha}\right]\right]\right]\,. \tag{30}$$

If we now assume that the thermodynamic potential Φ depends on $\rho(\mathbf{r}),s(\mathbf{r}),\mathbf{u}(\mathbf{r}),C(\mathbf{r})$ and proceed as in Sec. II, we arrive at equations governing the time evolution of $\rho(\mathbf{r})$, $s(\mathbf{r})$, $\mathbf{u}(\mathbf{r})$, and $C(\mathbf{r})$. These are the equations that have been used extensively to model polymeric fluids [47,48,21] and liquid crystals [48,21].

Similarly as in the preceding section, we can now consider the thermodynamic potential as depending also on derivatives with respect to $\mathbf{r}$ of the fields $\rho,s,\mathbf{u},C$ and derive the time evolution equations of weakly nonlocal extended hydrodynamics. We shall work out here only the case where Φ depends on $\rho(\mathbf{r}),s(\mathbf{r}),\mathbf{u}(\mathbf{r}),C(\mathbf{r})$, and $\partial_\alpha c_{\beta\gamma}$. Having the bracket, derivation of the time evolution equations is completely algorithmic [see (8)–(11)].

We can therefore present only the resulting time evolution equations:

$$\partial_t\rho=-\partial_\gamma(\rho\Phi_{u_\gamma})\,,$$

$$\partial_t s=-\partial_\gamma(s\Phi_{u_\gamma})\,,$$

$$\partial_t u_\alpha=-\partial_\gamma(u_\alpha\Phi_{u_\gamma})-\partial_\alpha p-\partial_\gamma\tau_{\alpha\gamma}\,, \tag{31}$$

$$\partial_t c_{\alpha\beta}=-\partial_\gamma(c_{\alpha\beta}\Phi_{u_\gamma})+c_{\epsilon\beta}\partial_\epsilon\Phi_{u_\alpha}+c_{\alpha\epsilon}\partial_\epsilon\Phi_{u_\beta}\,,$$

$$\partial_t e=-\partial_\gamma((e+p)\Phi_{u_\gamma}+\tau_{\alpha\gamma}\Phi_{u_\alpha}+q_\gamma)\,,$$

where

$$p = -\Phi + \rho\Phi_\rho + s\Phi_s + u_\gamma\Phi_{u_\gamma} + c_{\alpha\beta}\Phi_{c_{\alpha\beta}} + \partial_\gamma c_{\alpha\beta}\Phi_{\partial_\gamma c_{\alpha\beta}} , \tag{32}$$

$$\tau_{\alpha\gamma} = -2c_{\gamma\beta}\Phi_{c_{\alpha\beta}} - c_{\epsilon\beta}\partial_\alpha\Phi_{\partial_\gamma c_{\epsilon\beta}} + 2c_{\gamma\beta}\partial_\epsilon\Phi_{\partial_\epsilon c_{\alpha\beta}} , \tag{33}$$

$$q_\alpha = -c_{\beta\gamma}\partial_\epsilon(\Phi_{u_\alpha})\Phi_{\partial_\epsilon c_{\beta\gamma}} + 2c_{\beta\gamma}\partial_\beta(\Phi_{u_\epsilon})\Phi_{\partial_\alpha c_{\epsilon\gamma}} . \tag{34}$$

As in Sec. II B, the new terms arising in (31)–(34) due to the dependence of Φ on $\partial_\alpha c_{\beta\gamma}$ can again be physically interpreted as corresponding to a new component (called the inhomogeneous component) that modifies [see the last term in (32)] the hypothesis of local equilibrium [that is, however, already modified by adopting C(r) as the state variable—see the fifth term in (32)], contributes to the viscoelastic behavior [see the last two terms in (33)], and brings about the nondissipative heat flux (34).

Let us now choose the vector field n(r) instead of the tensor field C(r) as the state variable characterizing the internal structure. In terms of C(r), the vector n(r) can be regarded as the principal eigenvector of C(r). In the particular case of isothermal and incompressible fluids, the equations governing the time evolution of n(r) have been worked out, in the setting of Poisson brackets, in [48]. The additional terms that appear in the expression for the extra stress tensor due to the dependence of Φ on $\partial_\alpha n_\beta$ are the same as those introduced originally on the basis of other considerations in [1–3]. These additional stresses are called Frank-Ericksen stresses. We shall use the same terminology also for the additional stresses [the last two terms in (33)] that arise in the setting in which the tensor field C(r) rather than the vector field n(r) serves as the extra state variable. The time evolution equations with the configuration distribution function $\psi(\mathbf{r},\mathbf{R})$ serving as the extra state variable have been introduced in [49].

Now we turn our attention to the dissipative time evolution. By using the experience collected in theoretical and experimental investigations of polymeric fluids [21,47–50] we let the dissipation potential Ψ depend only on Φ_C. The dissipative time evolution is thus governed by

$$\partial_t\rho = 0 ,$$

$$\partial_t u_\alpha = 0 ,$$

$$\partial_t s = 0 , \tag{35}$$

$$\partial_t c_{\alpha\beta} = -\Psi_{\Phi_{c_{\alpha\beta}}} ,$$

where the entropy production

$$\sigma = \frac{1}{T}\Phi_{c_{\alpha\beta}}\Psi_{\Phi_{c_{\alpha\beta}}} \geq 0 . \tag{36}$$

The inequality in (36) holds due to the requirement that Ψ satisfies (16). We note that the complete time evolution equations (31)–(36) imply the dissipation inequality (21) as well as the conservation laws (20). The thermodynamic potential Φ and the dissipation potential Ψ have been specified for many polymeric fluids and solutions of (31)–(36) have been compared favorably with results of rheological measurements [21,50–52].

B. Two-component weakly nonlocal complex fluids

The fluids under consideration are now mixtures of two complex fluids. One possible set of state variables that we can suggest to describe states of the mixture is $\rho_1,s_1,\mathbf{u}_1,\mathbf{C}_1,\rho_2,s_2,\mathbf{u}_2,\mathbf{C}_2$, where the quantities with the subscript 1 (2) describe states of the first (second) fluid. Kinematics of the mixture is characterized by the Poisson bracket $\{A,B\} = \{A,B\}^{(1)} + \{A,B\}^{(2)}$, where $\{A,B\}^{(1)}$ ($\{A,B\}^{(2)}$) is the Poisson bracket (29) with the state variables $\rho_1,s_1,\mathbf{u}_1,\mathbf{C}_1$ ($\rho_2,s_2,\mathbf{u}_2,\mathbf{C}_2$). Now let the components of the mixture be simple fluids. In this case the conformation tensors $\mathbf{C}_1$ and $\mathbf{C}_2$ (describing the internal structure of the components) can be omitted and thus the set of the state variables that we suggest to describe states of mixtures of two simple fluids is $\rho_1,s_1,\mathbf{u}_1,\rho_2,s_2,\mathbf{u}_2$. Experience with mixtures of simple fluids indicates, however, that a smaller set of state variables consisting of $\rho_1,\rho_2,s,\mathbf{u}$, where s and $\mathbf{u}$ correspond now to the mixture as a whole, is often sufficient to describe states of the mixture. We proceed now to introduce a similar smaller state space and its kinematics in the case of mixtures of complex fluids.

We shall consider only a special case of the two-component mixture: one component is a simple fluid, the second component is a complex fluid. We can think, for example, about a polymer solution in a solvent that behaves as a simple fluid. As in Sec. III A, we begin with the classical hydrodynamic description of the fluid as a whole. The state variables are thus $\rho,s,\mathbf{v}$, where the symbol $\mathbf{v}$ is now used to denote the momentum field, and the corresponding Poisson bracket is the bracket (7). The fact that the fluid is a two-component mixture is expressed through the internal structure. The state variables with which we choose to describe the internal structure are $\mathbf{m}(\mathbf{r}),\mathbf{w}(\mathbf{r}),\mathbf{C}(\mathbf{r})$ representing mass density, momentum, and the conformation tensor of the complex component. Contrary to the analysis in Sec. III A, where we allowed C to be simply advected by the material flow, we now allow the internal structure (characterized by $\rho,\mathbf{w},\mathbf{C}$) to evolve as a complex fluid. This means that the corresponding Poisson bracket is

$$\{A,B\} = \{A,B\}^{(\text{cl.hyd})} + \{A,B\}' , \tag{37}$$

where $\{A,B\}^{(\text{cl.hyd})}$ is the bracket (7) (we recall that the symbol $\mathbf{v}$ now denotes the momentum field) and

$$\{A,B\}' = \int d^3\mathbf{r} \left\{ w_\gamma \left[\partial_\alpha \left[\frac{\delta A}{\delta w_\gamma} \right] \frac{\delta B}{\delta w_\alpha} - \partial_\alpha \left[\frac{\delta B}{\delta w_\gamma} \right] \frac{\delta A}{\delta w_\alpha} \right] + m \left[\partial_\alpha \left[\frac{\delta A}{\delta m} \right] \frac{\delta B}{\delta w_\alpha} - \partial_\alpha \left[\frac{\delta B}{\delta \mu} \right] \frac{\delta A}{\delta w_\alpha} \right] \right.$$

$$+ c_{\gamma\beta} \left[\partial_\alpha \left[\frac{\delta A}{\delta c_{\gamma\beta}} \right] \frac{\delta B}{\delta w_\alpha} - \partial_\alpha \left[\frac{\delta B}{\delta c_{\gamma\beta}} \right] \frac{\delta A}{\delta w_\alpha} \right]$$

$$\left. + c_{\gamma\beta} \left[\frac{\delta A}{\delta c_{\alpha\beta}} \partial_\gamma \left[\frac{\delta B}{\delta w_\alpha} \right] + \frac{\delta A}{\delta c_{\gamma\alpha}} \partial_\beta \left[\frac{\delta B}{\delta w_\alpha} \right] - \frac{\delta B}{\delta c_{\alpha\beta}} \partial_\gamma \left[\frac{\delta A}{\delta w_\alpha} \right] - \frac{\delta B}{\delta c_{\gamma\alpha}} \partial_\beta \left[\frac{\delta A}{\delta w_\alpha} \right] \right] \right\} . \tag{38}$$

Since $\{A,B\}^{(\text{cl.hyd})}$ as well as $\{A,B\}'$ are Poisson brackets, both involving different sets of state variables ($\{A,B\}^{(\text{cl.hyd})}$ involves $\rho, s, \mathbf{v}$ and $\{A,B\}'$ involves $m, \mathbf{w}, \mathbf{C}$), the bracket (37) is also a Poisson bracket. So far, the kinematics of the fluid as a whole and the kinematics of the internal structure remain separated. We shall couple them in the same way as in Sec. III A. We shall regard the internal structure as being advected with the material flow. We thus introduce the transformation from the set of state variables $\rho, \mathbf{v}, s, m, \mathbf{w}, \mathbf{C}$ to another set of state variables $\rho, \mathbf{u}, s, m, \mathbf{w}, \mathbf{C}$ by

$$\rho = \rho ,$$
$$\mathbf{u} = \mathbf{v} + \mathbf{w} ,$$
$$\mathbf{w} = \mathbf{w} ,$$
$$\tag{39}$$

$$s = s ,$$
$$m = m ,$$
$$\mathbf{C} = \mathbf{C} .$$

This one-to-one transformation implies that the velocity with which the complex component moves is the sum of the total fluid velocity $\Phi_\mathbf{u}$ (i.e., conjugate of the momentum $\mathbf{u}$) and the velocity $\Phi_\mathbf{w}$ (i.e., conjugate of the momentum $\mathbf{w}$ that constitutes a part of the state variables chosen to characterize the internal structure). Since the transformation (39) is one-to-one the Poisson bracket (37) transforms under this transformation into another Poisson bracket,

$$\{A,B\} = \int d^3\mathbf{r} \left\{ \rho \left[\partial_\alpha \left[\frac{\delta A}{\delta \rho} \right] \frac{\delta B}{\delta u_\alpha} - \partial_\alpha \left[\frac{\delta B}{\delta \rho} \right] \frac{\delta A}{\delta u_\alpha} \right] + u_\gamma \left[\partial_\alpha \left[\frac{\delta A}{\delta u_\gamma} \right] \frac{\delta B}{\delta u_\alpha} - \partial_\alpha \left[\frac{\delta B}{\delta u_\gamma} \right] \frac{\delta A}{\delta u_\alpha} \right] \right.$$

$$+ s \left[\partial_\alpha \left[\frac{\delta A}{\delta s} \right] \frac{\delta B}{\delta u_\alpha} - \partial_\alpha \left[\frac{\delta B}{\delta s} \right] \frac{\delta A}{\delta u_\alpha} \right] + m \left[\partial_\alpha \left[\frac{\delta A}{\delta m} \right] \frac{\delta B}{\delta u_\alpha} - \partial_\alpha \left[\frac{\delta B}{\delta m} \right] \frac{\delta A}{\delta u_\alpha} \right]$$

$$+ w_\gamma \left[\partial_\alpha \left[\frac{\delta A}{\delta w_\gamma} \right] \frac{\delta B}{\delta u_\alpha} - \partial_\alpha \left[\frac{\delta B}{\delta w_\gamma} \right] \frac{\delta A}{\delta u_\alpha} \right] + w_\gamma \left[\partial_\alpha \left[\frac{\delta A}{\delta w_\gamma} \right] \frac{\delta B}{\delta w_\alpha} - \partial_\alpha \left[\frac{\delta B}{\delta w_\gamma} \right] \frac{\delta A}{\delta w_\alpha} \right]$$

$$+ w_\gamma \left[\partial_\alpha \left[\frac{\delta A}{\delta u_\gamma} \right] \frac{\delta B}{\delta w_\alpha} - \partial_\alpha \left[\frac{\delta B}{\delta u_\gamma} \right] \frac{\delta A}{\delta w_\alpha} \right] + c_{\gamma\beta} \left[\partial_\alpha \left[\frac{\delta A}{\delta c_{\gamma\beta}} \right] \frac{\delta B}{\delta u_\alpha} - \partial_\alpha \left[\frac{\delta B}{\delta c_{\gamma\beta}} \right] \frac{\delta A}{\delta u_\alpha} \right]$$

$$+ c_{\gamma\beta} \left[\frac{\delta A}{\delta c_{\alpha\beta}} \partial_\gamma \left[\frac{\delta B}{\delta u_\alpha} \right] + \frac{\delta A}{\delta c_{\gamma\alpha}} \partial_\beta \left[\frac{\delta B}{\delta u_\alpha} \right] - \frac{\delta B}{\delta c_{\alpha\beta}} \partial_\gamma \left[\frac{\delta A}{\delta u_\alpha} \right] - \frac{\delta B}{\delta c_{\gamma\alpha}} \partial_\beta \left[\frac{\delta A}{\delta u_\alpha} \right] \right]$$

$$+ c_{\gamma\beta} \left[\partial_\alpha \left[\frac{\delta A}{\delta c_{\gamma\beta}} \right] \frac{\delta B}{\delta w_\alpha} - \partial_\alpha \left[\frac{\delta B}{\delta c_{\gamma\beta}} \right] \frac{\delta A}{\delta w_\alpha} \right]$$

$$\left. + c_{\gamma\beta} \left[\frac{\delta A}{\delta c_{\alpha\beta}} \partial_\gamma \left[\frac{\delta B}{\delta w_\alpha} \right] + \frac{\delta A}{\delta c_{\gamma\alpha}} \partial_\beta \left[\frac{\delta B}{\delta w_\alpha} \right] - \frac{\delta B}{\delta c_{\alpha\beta}} \partial_\gamma \left[\frac{\delta A}{\delta w_\alpha} \right] - \frac{\delta B}{\delta c_{\gamma\alpha}} \partial_\beta \left[\frac{\delta A}{\delta w_\alpha} \right] \right] \right\} . \tag{40}$$

We are now in position to write the equations governing the nondissipative time evolution of $\rho, s, \mathbf{u}, m, \mathbf{w}, \mathbf{C}$. We shall first consider only the thermodynamic potentials Φ that depend on $\rho, s, \mathbf{u}, m, \mathbf{w}, \mathbf{C}$ and are independent of derivatives with respect to $\mathbf{r}$ of these fields. Following the standard calculations [see (8)–(10)] we obtain

$$\partial_t \rho = -\partial_\gamma (\rho \Phi_{u_\gamma}) ,$$

$$\partial_t u_\alpha = -\partial_\gamma (u_\alpha \Phi_{u_\gamma}) - \partial_\alpha p - \partial_\gamma \tau_{\alpha\gamma} ,$$

$$\partial_t s = -\partial_\gamma (s \Phi_{u_\gamma}) ,$$

$$\partial_t m = -\partial_\gamma (m \Phi_{u_\gamma}) - \partial_\gamma (m \Phi_{w_\gamma}) , \tag{41}$$

$$\partial_t c_{\alpha\beta} = -\partial_\gamma (c_{\alpha\beta} \Phi_{u_\gamma}) + c_{\gamma\beta} \partial_\gamma \Phi_{u_\alpha} + c_{\gamma\alpha} \partial_\gamma \phi_{u_\beta}$$
$$\qquad -\partial_\gamma (c_{\alpha\beta} \Phi_{w_\gamma}) + c_{\gamma\beta} \partial_\gamma \Phi_{w_\alpha} + c_{\gamma\alpha} \partial_\gamma \Phi_{w_\beta} ,$$

$$\partial_t w_\alpha = -\partial_\gamma (w_\alpha \Phi_{w_\gamma}) - m \partial_\alpha \Phi_m - w_\gamma \partial_\alpha \Phi_{w_\gamma}$$
$$\qquad - c_{\gamma\beta} \partial_\alpha \Phi_{c_{\gamma\beta}} - \partial_\gamma (w_\alpha \Phi_{w_\gamma}) - w_\gamma \partial_\alpha \Phi_{u_\gamma}$$
$$\qquad + 2 \partial_\gamma (c_{\alpha\beta} \Phi_{c_{\beta\gamma}}) ,$$

$$\partial_t e = -\partial_\gamma ((e + p) \Phi_{u_\gamma} + \tau_{\gamma\alpha} \Phi_{u_\alpha} + q_\gamma) ,$$

where

$$p = -\Phi + \rho \Phi_\rho + s \Phi_s + u_\gamma \Phi_{u_\gamma}$$
$$\qquad + m \Phi_m + w_\gamma \Phi_{w_\gamma} + c_{\alpha\beta} \Phi_{c_{\alpha\beta}} , \tag{42}$$

$$\tau_{\alpha\beta} = w_\alpha \Phi_{w_\beta} - 2 c_{\alpha\gamma} \Phi_{c_{\gamma\beta}} , \tag{43}$$

$$q_\alpha = (m \Phi_m + w_\alpha \Phi_{w_\alpha} + c_{\alpha\beta} \Phi_{c_{\alpha\beta}}) \Phi_{w_\alpha} - 2 c_{\alpha\beta} \Phi_{c_{\beta\gamma}} \Phi_{w_\gamma} . \tag{44}$$

If needed, the weakly nonlocal extension of these equations can be worked out in a straightforward manner by considering thermodynamics potentials depending also on derivatives of the fields $\rho, s, \mathbf{u}, m, \mathbf{C}, \mathbf{w}$ with respect to $\mathbf{r}$. The Volterra derivatives Φ_ρ, etc., are then replaced by the variational derivatives (22).

We now turn our attention to equations governing the dissipative time evolution. Following Sec. III A, we shall first introduce dissipation potentials [see (16)] depending only on Φ_m, Φ_C, Φ_w. The dissipative time evolution is again required to satisfy the dissipation inequality (21) and the conservation laws (20). The equations governing the dissipative time evolution are thus

$$\partial_t \rho = 0 ,$$

$$\partial_t u_\alpha = 0 ,$$

$$\partial_t s = \sigma ,$$

$$\partial_t m = -\Psi_{\Phi_m} , \tag{45}$$

$$\partial_t c_{\alpha\beta} = -\Psi_{\Phi_{c_{\alpha\beta}}} ,$$

$$\partial_t w_\alpha = -\Psi_{\Phi_{w_\alpha}} ,$$

where

$$\sigma = \frac{1}{T} (\Phi_{w_\alpha} \Psi_{\Phi_{w_\alpha}} + \Phi_{c_{\alpha\beta}} \Psi_{\Phi_{c_{\alpha\beta}}} + \Phi_m \Psi_{\Phi_m}) . \tag{46}$$

The inequality (46) holds due to the requirement that Ψ satisfies (16). We shall end this subsection with three remarks.

Remark 1. Equations (41)–(46) represent a complete set of nonlinear equations governing the time evolution of a mixture of a simple fluid and a complex fluid. Equations of this type have been introduced before (for example, in the context of extended irreversible thermodynamics [6]) but not as fully nonlinear equations. For this reason we see in (41)–(46) new terms as, for example, the diffusion contribution to the extra stress tensor [the first term on the right-hand side of (43)] and the nondissipative heat flux (44).

Remark 2. Equations (41)–(46) describe non-Fickean diffusion since the diffusion flux $\mathbf{w}$ enters dynamics as an independent state variable. We can show easily that under some simplifying assumptions the non-Fickean diffusion reduces to the usual Fickean diffusion. We assume that $\mathbf{w}$ evolves in time much faster than the rest of the state variables so that, after some time, we can set $\partial_t \mathbf{w} = 0$ in the sixth equation of (41) and the sixth equation of (45). We shall assume moreover that Φ is a quadratic function of $\mathbf{w}$ so that $\Phi_{\mathbf{w}} \sim \mathbf{w}$. If we then neglect in the sixth equation of (41) and the sixth equation of (45) the nonlinear terms, we obtain $w_\alpha \sim \partial_\alpha \Phi_m$, that is, the Fickean diffusion law.

Remark 3. Observations of polymeric solutions in strongly inhomogeneous flows indicate that diffusion couples in an interesting way with higher-order gradients of the flow. In particular, we have in mind observations of the migration across streamlines [7–11]. In order to see these effects in the setting introduced in this subsection, we have to choose a dissipation potential that is different from the one introduced in (45). In this remark we shall limit ourselves only to the simplest dissipation potential allowing display of the migration across streamlines. We introduce the notation

$$w_\alpha^* = \Phi_{w_\alpha} ,$$
$$\tag{47}$$
$$z_\alpha^* = c_{\beta\gamma} (\partial_\alpha \partial_\beta \Phi_{u_\gamma}) ,$$

and choose

$$\Psi = \tfrac{1}{2} \int d^3 r (\mathbf{w}^*, \mathbf{z}^*) \Lambda \begin{bmatrix} \mathbf{w}^* \\ \mathbf{z}^* \end{bmatrix} , \tag{48}$$

where

$$\Lambda = \begin{bmatrix} \Lambda_1 \underline{\delta} & \Lambda_2 \underline{\delta} \\ \Lambda_2 \underline{\delta} & \Lambda_3 \underline{\delta} \end{bmatrix} \tag{49}$$

is a symmetric positive-definite matrix, $\underline{\delta}$ denotes the 3×3 unit matrix, and $\Lambda_1, \Lambda_2, \Lambda_3$ are constants. With the dissipation potential (48) the equations governing the dissipative time evolution are

$$\partial_t \rho = 0 \ ,$$

$$\partial_t u_\alpha = -\partial_\gamma \partial_\beta \Psi_{\partial_\gamma \partial_\beta \Phi_{u_\alpha}} \ ,$$

$$\partial_t s = \sigma \ ,$$

$$\partial_t m = 0 \ , \tag{50}$$

$$\partial_t c_{\alpha\beta} = 0 \ ,$$

$$\partial_t w_\alpha = -\Psi_{w_\alpha^*} \ ,$$

where

$$\sigma = \frac{1}{T}(z_\alpha^* \Psi_{z_\alpha^*} + w_\alpha^* \Psi_{w_\alpha^*}) \geq 0 \ . \tag{51}$$

We now assume, as in Remark 1, that w evolves in time much faster than the rest of state variables to that, after some time, we can set $\partial_t w = 0$ in the sixth equation of (50). Since

$$\Psi_{w_\alpha^*} = \Lambda_1 w_\alpha^* + \Lambda_2 z_\alpha^* \ ,$$

the sixth equation of (50) implies $w_\alpha^* \sim z_\alpha^*$, which in turn implies

$$w \sim z^* \tag{52}$$

if Φ is a quadratic function of w. The diffusion flux z^* is the flux across streamlines. Indeed, let us consider a flow in the x direction between two horizontal parallel plates. This means that only the x component of the velocity and only its z-direction gradient are different from zero. Consequently, only the z component of z^* [see (47)], that equals $c_{31}\partial_3^2 \Phi_{u_1}$ is different from zero. By looking at the second equation of (50) we see that in addition to migration across streamlines we also predict the appearance of a new viscosity term (proportional to the third-order derivative of the material velocity) that has to always accompany the migration across streamlines.

IV. WEAKLY NONLOCAL THERMODYNAMIC POTENTIAL

In the context of the mesoscopic and the macroscopic approaches to the derivation of hydrodynamic equations that we followed in previous sections, the thermodynamic potential Φ and the dissipation potential Ψ remain unspecified. We recall that it is in these potentials that the individual features of fluids are expressed. If we remain only with the mesoscopic and macroscopic arguments we can specify Φ and Ψ only by making a comparison with experimental results. The comparison consists of first finding solutions of the governing equations for all Φ and Ψ and, second, choosing one particular pair of these potentials for which the predicted and the measured results show the best fit. All other ways to specify (Φ, Ψ) require the use of a more microscopic point of view. We can either introduce an insight into the microscopic nature of the macromolecules composing the fluids and use for example Gibbs's equilibrium statistical mechanics to specify Φ (for examples of this approach see [48,21,52]) or extract (Φ, Ψ) from solutions of the govern-

ing equations formulated on a more microscopic level of description. We shall follow here the latter route. The process of extraction can be regarded as a pattern recognition process in the set of solutions of the microscopic governing equations found for all admissible initial and boundary conditions. The more microscopic theory that we choose in this section is kinetic theory.

States of fluids are characterized in kinetic theory by the one-particle distribution function $f(r, v)$, where (r, v) denotes the position and the velocity of one particle. The state space composed of one-particle distribution functions will be denoted by the symbol M_{kt} (i.e., $f(r, v) \in M_{kt}$). We shall consider M_{kt} as a linear vector space equipped with the L_2 inner product. Contrary to the analysis in the previous sections, we shall narrow our attention to a specific fluid, namely, an ideal gas whose time evolution is governed by the Boltzmann kinetic equation (see Appendix). To simplify the notation we shall set the mass of one particle equal to 1. We are interested in extracting the thermodynamic potential used in weakly nonlocal hydrodynamics from solutions of the Boltzmann kinetic equation.

Let us assume that the Boltzmann equation has been solved for all initial conditions and we are looking at the resulting set of trajectories in M_{kt}. The first feature that we note is the two-dimensional submanifold of M_{kt}, denoted by the symbol $\mathcal{M}_{ET}$, $\mathcal{M}_{ET} \subset M_{kt}$, formed by the Maxwell distribution functions

$$f_{ET}(v) = e^{(\mu/T)-1} e^{-(1/2T)v^2} \ , \tag{53}$$

where T and μ are real numbers that will be interpreted below as thermodynamic temperature and chemical potential. The submanifold $\mathcal{M}_{ET}$ is noted since f_{ET} are fixed points of the time evolution. To prove it, we note that f_{ET} is a solution of

$$\frac{\delta \Phi_{ET}^{kt}}{\delta f(r, v)} \ , \tag{54}$$

where [compare with (4)]

$$\Phi_{ET}^{kt}(f; T, \mu) = E(f) - TS(f) - \mu N(f) \ , \tag{55}$$

$$N(f) = \int d^3 r \int d^3 v \, f(r, v) \ ,$$

$$S(f) = -\int d^3 r \int d^3 v \, f(r, v) \ln f(r, v) \ , \tag{56}$$

and

$$E(f) = \int d^3 r \int d^3 r \, \tfrac{1}{2} v^2 f(r, v) \ . \tag{57}$$

Since the potential Φ_{ET}^{kt} introduced in the Boltzmann kinetic equations (A2) and (A7) is the potential (55), we see that the distribution functions (53) are indeed the time-independent solutions of the Boltzmann equation. The functional $N(f)$ has the physical meaning of the total number of moles (or the total mass, since we put the mass of one particle equal to 1), $S(f)$ is the Boltzmann entropy, and $E(f)$ is the total energy if there are no interactions among the particles. From the mathematical point of view, it is useful to interpret (56) as an introduction of a bundle structure into M_{kt}, i.e., $M_{kt} \equiv (M_{kt}, \mathcal{M}_{ET}, \Pi_{ET}^{kt})$,

where M_{ET} is the base space and Π_{ET}^{kt}: $M_{kt} \rightarrow M_{\text{ET}}$, is the bundle projection specified by (56) [compare with (1)]. The relation (57) is interpreted as the fundamental thermodynamic relation in M_{kt} that characterizes in M_{kt} an ideal gas. The potential, Φ_{ET}^{kt}, introduced in (56), is the potential that arises in Legendre transformations corresponding to the fibration (56) and the fundamental thermodynamic relation (57). If we now evaluate Φ_{ET}^{kt} at f_{ET} and equate the result with $-PV$, where P is the thermodynamic pressure and V is the volume, we obtain the fundamental thermodynamic relation $P = P(\mu, T)$ [compare with (6)].

$$\Phi_{\text{ET}}^{kt}(f_{\text{ET}}; T, \mu) = -T \int d^3\mathbf{r} \int d^3\mathbf{v}\, f_{\text{ET}}$$

$$= -TVe^{(\mu/T)-1}(2\pi T)^{3/2} = -PV \; , \qquad (58)$$

characterizing, now in the thermodynamic state space M_{ET}, an ideal gas.

Another submanifold that is noted in M_{kt} filled with solutions of the Boltzmann equation is the submanifold (denoted by the symbol $\mathcal{M}_{\text{hyd}}^{(0)}, \mathcal{M}_{\text{hyd}}^{(0)} \subset M_{kt}$) composed of local Maxwellian distribution functions

$$f_{\text{hyd}}^{(0)}(\mathbf{r}, \mathbf{v}) = \exp\left[\frac{\mu(\mathbf{r})}{T(\mathbf{r})} + \frac{[\mathbf{u}^*(\mathbf{r})]^2}{2T(\mathbf{r})} - 1\right]$$

$$\times \exp\left[-\frac{[\mathbf{v} - \mathbf{u}^*(\mathbf{r})]^2}{2T(\mathbf{r})}\right] \; . \qquad (59)$$

The quantities $\mu(\mathbf{r})$, $\mathbf{u}^*(\mathbf{r})$, and $T(\mathbf{r})$ will be interpreted below as thermodynamic duals of the hydrodynamic fields. We note that the distribution functions $f_{\text{hyd}}^{(0)}$ are solutions of $\mathcal{A} = 0$ [see (A6)] and thus the time-independent solutions of the dissipative part (A7) of the

Boltzmann kinetic equation. If collisions play the dominant role in the time evolution then the submanifold $\mathcal{M}_{\text{hyd}}^{(0)}$ will be close to being an invariant submanifold. This is the reason why the submanifold $\mathcal{M}_{\text{hyd}}^{(0)}$ is noted. Similarly as in the preceding paragraph, we note that $f_{\text{hyd}}^{(0)}$ are solutions of

$$\frac{\delta \Phi_{\text{hyd}}^{kt0}}{\delta f(\mathbf{r}, \mathbf{v})} \; , \qquad (60)$$

where

$$\Phi_{\text{hyd}}^{kt0} = E(f) - \int d^3\mathbf{r}[T(\mathbf{r})s(f; \mathbf{r}) + \mu(\mathbf{r})\rho(f; \mathbf{r})$$

$$+ u_\alpha^*(\mathbf{r})\mathbf{u}(f; \mathbf{r})] \qquad (61)$$

and

$$\rho(f; \mathbf{r}) = \int d^3\mathbf{v}\, f(\mathbf{r}, \mathbf{v}) \; ,$$

$$s(f; \mathbf{r}) = -\int d^3\mathbf{v}\, f(\mathbf{r}, \mathbf{v}) \ln f(\mathbf{r}, \mathbf{v}) \; , \qquad (62)$$

$$\mathbf{u}(f; \mathbf{r}) = \int d^3\mathbf{v}\, \mathbf{v} f(\mathbf{r}, \mathbf{v}) \; .$$

We interpret (62) again as an introduction of a bundle structure into M_{kt}, i.e., $M_{kt} \equiv (M_{kt}, M_{\text{hyd}}, \Pi_{\text{hyd}}^{kt0})$, where M_{hyd} is the hydrodynamic state space serving now as the base space and Π_{hyd}^{kt0}: $M_{kt} \rightarrow M_{\text{hyd}}$ is the bundle projection specified in (62). The potential Φ_{hyd}^{kt0} introduced in (61) can be seen again as arising in Legendre transformations corresponding to the fibration (62) and the fundamental thermodynamic relation (57). If we now evaluate Φ_{hyd}^{kt0} at $f_{\text{hyd}}^{(0)}$ and equate the result with $-\int d^3\mathbf{r}p(\mathbf{r})$, where $p(\mathbf{r})$ is the local pressure, we obtain

$$\Phi_{\text{hyd}}^{kt0}(f_{\text{hyd}}^{(0)}; T(\mathbf{r}), \mu(\mathbf{r}), \mathbf{u}^*(\mathbf{r})) \equiv \Phi_{\text{hyd}}^{(0)} = -\int d^3\mathbf{r} \int d^3\mathbf{v}\, T(\mathbf{r})f_{\text{hyd}}^{(0)}(\mathbf{r}, \mathbf{v})$$

$$= -\int d^3\mathbf{r}\, T(\mathbf{r}) \exp\left[\frac{\mu(\mathbf{r})}{T(\mathbf{r})} + \frac{[u^*(\mathbf{r})]^2}{2T(\mathbf{r})} - 1\right][2\pi T(\mathbf{r})]^{3/2}$$

$$= -\int d^3\mathbf{r}\, p(\mathbf{r}) \; , \qquad (63)$$

that is, the fundamental thermodynamic relation in M_{hyd}. The thermodynamic potential $\Phi(\rho, e, \mathbf{u}; T, \mu)$ used in Sec. II is obtained from (63) by Legendre-transforming (63) into a potential depending on the hydrodynamic fields that are dual fields to the fields $\mu(\mathbf{r}), T(\mathbf{r}), \mathbf{u}^*(\mathbf{r})$. The thermodynamic potential $\Phi(\rho, e, \mathbf{u}; T, \mu)$ obtained in this way is then a specific example of the thermodynamic potential (4) that in hydrodynamics characterizes an ideal gas.

Finally, we introduce a third submanifold in M_{kt} that distinguishes itself in M_{kt} filled with solutions of the Boltzmann equation. We shall denote it by the symbol $\mathcal{M}_{\text{hyd}}^{(1)}$, and its elements by $f_{\text{hyd}}^{(1)}$. In order to introduce $\mathcal{M}_{\text{hyd}}^{(1)}$ we shall return to the submanifold $\mathcal{M}_{\text{hyd}}^{(0)}$. As we have noted in the previous paragraph, $\mathcal{M}_{\text{hyd}}^{(0)}$ is close to be-

ing an invariant submanifold if collisions play a dominant role in the time evolution. We shall now examine more closely the invariance of $\mathcal{M}_{\text{hyd}}^{(0)}$. The Boltzmann vector field attached to $\mathcal{M}_{\text{hyd}}^{(0)}$ is

$$-v_\alpha \frac{\partial}{\partial r_\alpha} f_{\text{hyd}}^{(0)} \; , \qquad (64)$$

while the vector field that is tangent to $\mathcal{M}_{\text{hyd}}^{(0)}$ is

$$\frac{\partial f_{\text{hyd}}^{(0)}}{\partial T(\mathbf{r})}\partial_t T(\mathbf{r}) + \frac{\partial f_{\text{hyd}}^{(0)}}{\partial \mu(\mathbf{r})}\partial_t \mu(\mathbf{r}) + \frac{\partial f_{\text{hyd}}^{(0)}}{\partial u_\alpha^*(\mathbf{r})}\partial_t u_\alpha^*(\mathbf{r}) \; , \qquad (65)$$

where the time evolution of $T(\mathbf{r}), \mu(\mathbf{r}), \mathbf{u}^*(\mathbf{r})$ is governed by the Euler hydrodynamic equation (11) with Φ being the dual of $\Phi_{\text{hyd}}^{(0)}$ introduced in (63) and $\rho(\mathbf{r}), e(\mathbf{r}), \mathbf{u}(\mathbf{r})$ be-

ing dual [again with respect to the potential (63)] of the fields $\mu(\mathbf{r}), T(\mathbf{r}), \mathbf{u}^*(\mathbf{r})$. If the two vector fields (64) and (65) were identical then the submanifold $\mathcal{M}_{hyd}^{(0)}$ would be invariant. It is easy to calculate that the difference between (64) and (65) is

$$\frac{1}{T^2(\mathbf{r})}\xi_\alpha(\mathbf{v}-\mathbf{u}^*)\frac{\partial T}{\partial r_\alpha}+\frac{1}{T(\mathbf{r})}\zeta_{\alpha\beta}(\mathbf{v}-\mathbf{u}^*)D_{\alpha\beta} , \quad (66)$$

where

$$D_{\alpha\beta}=\frac{1}{2}\left[\frac{\partial u_\alpha^*}{\partial r_\beta}+\frac{\partial u_\beta^*}{\partial r_\alpha}\right] ,$$

$$\xi_\alpha(\mathbf{v})=(\tfrac{1}{2}v^2-\tfrac{5}{2}T)v_\alpha , \quad (67)$$

$$\zeta_{\alpha\beta}(\mathbf{v})=(v_\alpha v_\beta-\tfrac{1}{3}v^2\delta_{\alpha\beta}) .$$

Since (66) is not identically equal to zero, the submanifold $\mathcal{M}_{hyd}^{(0)}$ is not an invariant submanifold. Now we shall try to deform $\mathcal{M}_{hyd}^{(0)}$ to arrive at a submanifold that will be closer to being an invariant submanifold. Following Chapman and Enskog [53,54] we deform $\mathcal{M}_{hyd}^{(0)}$ into $\mathcal{M}_{hyd}^{(1)}$ by requiring that the vector field (66), which is sticking out of $\mathcal{M}_{hyd}^{(0)}$, is exactly compensated by the dissipative vector field (A5) evaluated at $\mathcal{M}_{hyd}^{(1)}$ In this way we shall not guarantee that $\mathcal{M}_{hyd}^{(1)}$ is an invariant submanifold. The complete Boltzmann vector field attached to $\mathcal{M}_{hyd}^{(1)}$ will still not be tangent to $\mathcal{M}_{hyd}^{(1)}$. However, if collisions play a dominant role in the time evolution then the part of the complete Boltzmann vector field that is sticking out of $\mathcal{M}_{hyd}^{(1)}$ will be smaller than (66), which is sticking out of $\mathcal{M}_{hyd}^{(0)}$. After some calculations (see, e.g., [41]) we find that (up to and including the terms linear in φ) $\mathcal{M}_{hyd}^{(1)}\ni f_{hyd}^{(1)}$,

$$f_{hyd}^{(1)}(\mathbf{r},\mathbf{v})=f_{hyd}^{(0)}(\mathbf{r},\mathbf{v})e^{\varphi(\mathbf{r},\mathbf{v})} , \quad (68)$$

where

$$\varphi=\frac{1}{T^2}\hat{\xi}_\alpha\frac{\partial T}{\partial r_\alpha}+\frac{1}{T}\hat{\zeta}_{\alpha\beta}D_{\alpha\beta} , \quad (69)$$

$$\hat{\xi}_\alpha=B^{-1}\xi_\alpha; \quad \hat{\zeta}_{\alpha\beta}=B^{-1}\zeta_{\alpha\beta} , \quad (70)$$

and B is the Boltzmann collision operator [see the right-hand side of (A5)] linearized about $f_{hyd}^{(0)}$. As in the two preceding paragraphs, we note that (68) are solutions of

$$\frac{\delta\Phi_{hyd}^{kt1}}{\delta f(\mathbf{r},\mathbf{v})}=0 , \quad (71)$$

where

$$\Phi_{hyd}^{kt1}=E(f)-\int d^3\mathbf{r}[T(\mathbf{r})s^{(1)}(f;\mathbf{r})+\mu(\mathbf{r})\rho(f;\mathbf{r})$$
$$+u_\alpha^*(\mathbf{r})\mathbf{u}(f;\mathbf{r})] ; \quad (72)$$

$\rho(f;\mathbf{r}), s(f;\mathbf{r})$, and $\mathbf{u}(f;\mathbf{r})$ are the same as in (62), and

$$s^{(1)}(f;\mathbf{r})=s(f;\mathbf{r})+\int d^3\mathbf{v}\,\varphi(\mathbf{r},\mathbf{v})f(\mathbf{r},\mathbf{v}) . \quad (73)$$

Again, we interpret the introduction of $\rho(f;\mathbf{r}), \mathbf{u}(f;\mathbf{r})$, and $s^{(1)}(f;\mathbf{r})$ in (62) and (73) as an introduction of a bundle structure in M_{kt} and Φ_{hyd}^{kt1} as the thermodynamic potential that arises in Legendre transformations corresponding to the fibration (62), (73), and the fundamental thermodynamic relation (57). If we now evaluate Φ_{hyd}^{kt1} at $f_{hyd}^{(1)}$ and equate the result with $-\int d^3\mathbf{r}\,p(\mathbf{r})$, we obtain an example of the thermodynamic potential used in weakly nonlocal hydrodynamics. If we limit ourselves only to the terms proportional to φ^k, $k\leq2$, we obtain

$$\Phi_{hyd}^{kt1}(f_{hyd}^{(1)};T(\mathbf{r}),\mu(\mathbf{r}),\mathbf{u}^*(\mathbf{r}))\equiv\Phi_{hyd}^{(1)}$$

$$=-\int d^3\mathbf{r}\int d^3\mathbf{v}\,T(\mathbf{r})f_{hyd}^{(1)}(\mathbf{r},\mathbf{v})$$

$$=-\int d^3\mathbf{r}\,T(\mathbf{r})\exp\left[\frac{\mu(\mathbf{r})}{T(\mathbf{r})}+\frac{[u^*(\mathbf{r})]^2}{2T(\mathbf{r})}-1\right][2\pi T(\mathbf{r})]^{3/2}$$

$$-\int d^3\mathbf{r}\left[\frac{1}{2T^2(\mathbf{r})}\langle\hat{\xi}_\alpha,\hat{\xi}_\alpha\rangle\left[\frac{\partial T}{\partial\mathbf{r}}\right]^2+\frac{1}{2T(\mathbf{r})}\langle\hat{\zeta}_{\alpha\beta},\hat{\zeta}_{\alpha\beta}\rangle(\underline{D})^2\right]=-\int d^3\mathbf{r}\,p(\mathbf{r}) , \quad (74)$$

where

$$\langle a(v),b(v)\rangle\equiv\int d^3\mathbf{v}\,f_{hyd}^{(0)}(\mathbf{r},\mathbf{v})a(v)b(v) .$$

We recall (see, e.g., [41]) that the heat conductivity coefficient λ and the viscosity coefficient η are expressed in terms of ξ and ζ as

$$\lambda=-\frac{1}{3T^2}\langle\xi_\alpha,\hat{\xi}_\alpha\rangle ,$$

$$\eta=-\frac{1}{10T}\langle\zeta_{\alpha\beta},\hat{\zeta}_{\alpha\beta}\rangle . \quad (75)$$

An additional insight into the nature of the approximations (59) and (68) of solutions of the Boltzmann kinetic equation can be gained by asking the following question. What are the kinetic equations (modifications of the Boltzmann equations) that have (59) and (68) as exact solutions? In other words, how must Boltzmann vector field be modified in order that $\mathcal{M}_{hyd}^{(0)}$ or $\mathcal{M}_{hyd}^{(1)}$ is an exactly invariant submanifold? The answer will not be, of course, unique. The additional requirement that should single out the kinetic equations is that the modified kinetic equations are required to remain as close as possible to the original Boltzmann equation. Leaving aside the prob-

lem of uniqueness, the answer to the above equation is the following. The kinetic equation that has (59) as its exact solution is (A7), in which $\Phi_{\mathrm{hyd}}^{kt0}$ [see (63)] replaces Φ_{ET}^{kt} in $\mathcal{A}$ [see (A6)] and the time evolution of $\mu, T, \mathbf{u}^*$ is governed by Euler hydrodynamic equations. Similarly, the kinetic equation that has (68) as its exact solution is (A7), in which $\Phi_{\mathrm{hyd}}^{kt1}$ [see (74)] replaces Φ_{ET}^{kt} in $\mathcal{A}$ and the time evolution of $\mu, T, \mathbf{u}^*$ is governed by Navier-Stokes-Fourier hydrodynamic equations with the kinetic coefficients given by (75). We note that in both modifications of the Boltzmann kinetic equation the thermodynamic potentials $\Phi_{\mathrm{hyd}}^{kt0}$ or $\Phi_{\mathrm{hyd}}^{kt1}$ play the role of the Lyapunov functions corresponding to the approach to the invariant submanifolds $\mathcal{M}_{\mathrm{hyd}}^{(0)}$ and $\mathcal{M}_{\mathrm{hyd}}^{(1)}$. The thermodynamic potential Φ_{ET}^{kt} plays the same role in the context of the Boltzmann equation and the invariant submanifold $\mathcal{M}_{\mathrm{ET}}$.

V. CONCLUDING REMARKS

Appearance of new stresses in spatially inhomogeneous liquid crystals [1–3], non-Fickean diffusion [4–6], and migration across streamlines [7–11] in spatially inhomogeneous polymer solutions are examples of experimental observations that cannot be adequately explained in the setting of local classical or extended hydrodynamics. The goal of this paper is to formulate a weakly nonlocal classical and extended hydrodynamics that provides an appropriate setting for an analysis of the above observations. We call hydrodynamics weakly nonlocal if its governing equations involve higher-order derivatives with respect to the position coordinate $\mathbf{r}$. Local hydrodynamics is extended in this paper to a weakly nonlocal hydrodynamics as follows. We note first that local hydrodynamics can be regarded as dynamics generated by a thermodynamic potential in the state space equipped with a Poisson structure and a dissipative potential. The Poisson structure expresses in mathematical terms kinematics in the chosen state space. The Poisson structure is thus fixed if the state space is fixed. The individual features of the fluids under consideration are expressed in the thermodynamic and the dissipation potentials. The thermodynamic potentials are in local theories independent of derivatives with respect to $\mathbf{r}$, and the dissipation potentials depend only on derivatives of zero and first order. We now pass from local hydrodynamics to a weakly nonlocal hydrodynamics by generalizing the thermodynamic and the dissipation potentials. The state spaces and thus the Poisson structures in both local and weakly nonlocal theories are the same. The thermodynamic and the dissipation potentials are allowed to depend, in weakly nonlocal hydrodynamics, on higher-order derivatives with respect to $\mathbf{r}$. In Secs. II and III, these two potentials are left unspecified. The governing equations of the weakly nonlocal hydrodynamic derived in Secs. II and III are thus families of equations parametrized by the two potentials. A method allowing one to derive these potentials from an analysis of kinetic equations is introduced in Sec. IV.

More specifically, results of this paper are the following. In Sec. II we worked out the governing equations of

weakly nonlocal hydrodynamics in the state space of classical hydrodynamics (i.e., the state variables are the five hydrodynamic fields of mass, energy, and momentum). The Poisson structure of this state space is well known. We showed that the weakly nonlocal fluids (i.e., the fluids characterized by thermodynamic potentials depending on derivatives of the hydrodynamic fields with respect to $\mathbf{r}$) behave as viscoelastic and "heat-elastic" fluids. In Sec. III we considered complex fluids and thus we studied the extension from local to weakly nonlocal in the extended hydrodynamic state space. We have chosen a tensor field (called a conformation tensor field) as the extra state variable characterizing the internal structure of the fluids. The Poisson structure of this state space has been derived previously in [47] for one-component fluids, and in Sec. III B of this paper the structure is extended to two-component fluids. Among the new terms that arise in the governing equations of weakly nonlocal complex fluids we recognize the extra stresses that play the role of the extra stresses derived previously in a different state space (in which the director vector field instead of the conformation tensor field serves as the extra state variable) and by using different arguments in [1–3]. In addition to the extra stresses we showed that an extra reversible heat flux arises in weakly nonlocal complex fluids. The equations governing the time evolution of two-component weakly nonlocal complex fluids provide a setting for an analysis of nonlinear non-Fickean diffusion and migration across streamlines. In Sec. IV we have shown how thermodynamic potentials depending on derivatives of hydrodynamic fields with respect to $\mathbf{r}$ arise from an analysis of solutions of the Boltzmann kinetic equation.

APPENDIX: MESOSCOPIC DERIVATION OF THE BOLTZMANN KINETIC EQUATION

The purpose of this appendix is to demonstrate that the mesoscopic arguments that have been used in this paper to introduce hydrodynamic equations can also be used to introduce the Boltzmann kinetic equation. We begin with kinematics in M_{kt}. The one-particle kinematics that is known from classical mechanics induces in M_{kt} [13,55] the Poisson bracket

$$\{A, B\} = \int d^3\mathbf{r} \int d^3\mathbf{v}\, f(\mathbf{r}, \mathbf{v}) \left[\frac{\partial}{\partial r_\alpha}(A_f) \frac{\partial}{\partial v_\alpha}(B_f) \right.$$
$$\left. - \frac{\partial}{\partial r_\alpha}(B_f) \frac{\partial}{\partial v_\alpha}(A_f) \right].$$
$$(A1)$$

By A, B, etc., we denote sufficiently regular functionals of $f(\mathbf{r}, \mathbf{v})$. The functional that generates dynamics is taken

to be the thermodynamic potential (55). So far, we thus proceed exactly in the same way as we did in the context of hydrodynamics. We can also verify by direct calculations that $\{A,N\}=\{A,S\}=0$ for all A (i.e., the functionals N and S are distinguished functionals as in hydrodynamics). The equation that governs the nondissipative time evolution is thus

$$\partial_t f = -\frac{\partial}{\partial r_\alpha}\left[f\frac{\partial}{\partial v_\alpha}(\Phi_{ET}^{kt})_f\right]+\frac{\partial}{\partial v_\alpha}\left[f\frac{\partial}{\partial r_\alpha}(\Phi_{ET}^{kt})_f\right], \tag{A2}$$

and if we use (55)–(57)

$$\partial_t f = -\frac{\partial}{\partial r_\alpha}(v_\alpha f). \tag{A3}$$

This is indeed the nondissipative part of the Boltzmann kinetic equation.

We now proceed to introduce the dissipative part of the Boltzmann kinetic equation (i.e., the Boltzmann collision term). We want to introduce it in the same way as we introduced in this paper the dissipative part of hydrodynamic equations. Let us consider the gas under consideration as composed of an infinite number of components. Gas particles having different velocity are considered as belonging to a different component. Collisions

are regarded as chemical reactions in which the component $\mathbf{v}$ interacts with another component $\mathbf{v}_2$ producing the components $\mathbf{v}'$ and $\mathbf{v}_2'$. We require, of course, that

$$\mathbf{v}+\mathbf{v}_2=\mathbf{v}'+\mathbf{v}_2',$$
$$\mathbf{v}^2+\mathbf{v}_2^2=\mathbf{v}'^2+\mathbf{v}_z'^2. \tag{A4}$$

The mass action law leads to the following time evolution equation:

$$\partial_t f(1)=\int d2\int d1'\int d2'W(f;1,2,1',2')$$
$$\times[f(1')f(2')-f(1)f(2)], \tag{A5}$$

where W is the rate constant that is symmetric with respect to $1\rightleftharpoons2$, $(1,2)\rightleftharpoons(1',2')$ and different from zero (positive) only if (A4) holds. We use the notation $1\equiv(\mathbf{r},\mathbf{v})$, $2\equiv(\mathbf{r},\mathbf{v}_2)$, $1'\equiv(\mathbf{r},\mathbf{v}')$ $2'\ni(\mathbf{r},\mathbf{v}_2')$. Equation (A5) is indeed the dissipative part of the Boltzmann kinetic equation written in the form introduced first by Waldmann [56]. It remains to express (A5) in terms of the thermodynamic potential (55) and a dissipative potential. The chemical affinity corresponding to collisions is

$$\mathcal{A}(1,2,1',2')=(\Phi_{ET}^{kt})_{f(1)}+(\Phi_{ET}^{kt})_{f(2)}$$
$$-(\Phi_{ET}^{kt})_{f(1')}-(\Phi_{ET}^{kt})_{f(2')}. \tag{A6}$$

If we introduce now the dissipative potential

$$\Psi(\mathcal{A})=\int d1\int d2\int d1'\int d2'\tilde{W}(f;1,2,1',2')\left[\exp\left[\frac{1}{2T}\mathcal{A}(1,2,1',2')\right]+\exp\left[-\frac{1}{2T}\mathcal{A}(1,2,1',2')\right]-2\right], \tag{A7}$$

we can easily verify that the time evolution equation

$$\partial_t f = -\Psi_{(\Phi_{ET}^{kt})_{f(1)}} \tag{A8}$$

is indeed the same equation as (A5) provided

$$W=\frac{1}{2T}\frac{\tilde{W}}{[f(1)f(2)f(1')f(2')]^{1/2}}. \tag{A9}$$

Note that the dissipative potential (A7) satisfies the re-

quirement (16) and that the dissipation inequality (20), which has the form

$$\partial_t\Phi_{ET}^{kt}=-\int d1\int d2\int d1\int d2'\mathcal{A}(1,2,1',2')$$
$$\times\Psi_{\mathcal{A}(1,2,1',2')}$$
$$\leq 0, \tag{A10}$$

is the statement of the Boltzmann H theorem. The degeneracy of the Poisson bracket (A1) and (A4) imply then the conservation laws (21).

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