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Scaling in fluid turbulence: A geometric theory

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We develop a theory that is nonperturbative and free of uncontrolled approximations to understand scaling behavior in turbulence. The main tool is a connection between the dimension of the graphs of the hydrodynamic fields and the scaling exponents of their structure functions. The connection is developed in some generality for both scalar and vector fields, in terms of the geometric invariants of the gradient tensor. We show that fluid mechanics is consistent with fractal graphs for both the scalar and the vector fields, and explain how this leads to the scaling behavior of the structure functions. We derive scaling relations between various scaling exponents, and show that in the case of "strong scaling" (which is defined below) the Kolmogorov solution is unique. Our theory allows additional solutions in which a weaker version of scaling results in a spectrum of scaling exponents. In particular, we identify the dimensionless (but Reynolds-number-dependent) contributions which can lead to deviations from the Kolmogorov exponents (which are derived using dimensional analysis). Results for the dimensions of fractal level sets in hydrodynamic turbulence which are measured in experiments and simulations follow immediately from this theory.

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I. INTRODUCTION

Unquestionably, the property of developed turbulence that attracted most theoretical and experimental attention in the physics community is the scaling behavior of the structure functions of the various hydrodynamic fields. Scaling behavior was predicted by Kolmogorov [1] and Obukhov [2] more than 50 years ago in their celebrated papers, in which they proposed a set of hypotheses concerning the nature of high-Reynolds-number turbulence, which culminated with the prediction that the structure function of the fluid-velocity field should scale over a wide range of scales. Thus, for example, if one denotes the velocity field by $\mathbf{u}(\mathbf{x})$, and the differences $\mathbf{u}(\mathbf{x}+re_i)-\mathbf{u}(\mathbf{x})$ (with e_i being the unit vector in the ith direction) by $\delta_i\mathbf{u}^{(r)}$, then the Kolmogorov approach predicts that

$$\langle \mathrm{Tr}(\delta_i\mathbf{u}^{(r)}\delta_j\mathbf{u}^{(r)})\rangle \sim r^{2\zeta_2}\ , \tag{1.1}$$

where $\langle\ \rangle$ stands for an average over $\mathbf{x}$, Tr is the trace, and ζ_2 is a scaling exponent having the value of $\frac{1}{3}$. Equation (1.1) is expected to hold over an "inertial range" of r values much smaller than an "integral scale" L, and much larger than the "Kolmogorov cutoff length" r_0, which in terms in the Reynolds number Re behaves like $r_0/L \sim \mathrm{Re}^{-3/4}$. The "integral scale" L is understood as the minimal size of the box needed to register the largest available fluctuations in the turbulent field, and the Reynolds number is defined as UL/ν, where ν is the kinematic viscosity of the fluid.

In spite of the simplicity of this result, it has not been derived to date from the equations of fluid mechanics in a satisfactory fashion. A variety of perturbative schemes were suggested in order to understand the existence of scale-invariant solutions to the equations of fluid mechanics [3]. In general, they involve at this stage or another some uncontrolled approximations. Moreover, the hypotheses of Kolmogorov have fallen under experimental attack, and although the value $\frac{1}{3}$ of the scaling exponent ζ_2 seems to be close to the experimental value when measured as in Eq. (1.1), higher-order structure functions (involving more factors of $\delta_i\mathbf{u}^{(r)}$) seem experimentally to deviate from the Kolmogorov predictions [4]. These deviations gave rise to a plethora of models that fall generally under the heading of fractral [5,6] or multifractal [7] models, whose connection to fluid mechanics had been rather dubious.

The aim of this paper is to offer an alternative point of view on the issue of scaling behavior in turbulence. In our view, the fundamental objects of interest are the graphs of the hydrodynamic field, and the geometric properties of these graphs determine the scaling behavior of the structure functions of the fields. We shall understand scaling behavior as a consequence of the wrinkling of the graphs into fractal objects. The self-similarity of the graph, if it exists, reflects itself in the lack of a characteristic scale, which in turn results in scaling behavior. Most importantly, we can estimate the dimensions of the various graphs (to be precisely defined below) using rigorous techniques and employing the equations of fluid mechanics without any uncontrolled approximations. As a consequence, we can derive scaling relations between various scaling exponents, and in particular show that one solution of the scaling relation is the Kolmogorov solution. Interestingly enough, this is not the only solution, and we identify how dimensionless corrections to dimensional analysis can result in a spectrum of scaling exponents, as is described below.

In Sec. II, we discuss the concepts of the volumes of graphs. This section is not necessarily related to hydrodynamics, and is needed as a precursor to the introduction of the connection between the fractal properties of the graph of a field and the scaling exponents of its structure function. This connection is achieved in some generality in Sec. III. It is shown that the connection depends on the tensorial character of the field, and is different for a scalar and a vector field. Sections IV and V employ the results of Secs. II and III in the context of hydrodynamic turbulence. Section IV deals with the case of a scalar, and culminates with a scaling relation between the scaling exponents of the velocity structure functions and the scalar structure function. The dimension of the graph of the scalar, and its scaling exponents, are determined entirely by the analytic structure of the passive scalar equation of motion and the scaling properties of the velocity field itself. These scaling properties are addressed in Sec. V. It turns out that the calculation in the case of the velocity field is much less straightforward than in the case of the passive scalar, due to the more complicated metric properties of the graph of the velocity field, which has to be discussed in six dimensions. It is shown that if one considers the case of "strong scaling," in which there is (by an uncontrolled assumption) only one independent scaling exponent, then the Kolmogorov result is recovered. This is by no means the only solution. The theory allows us to explore other possible solutions, and we can find the conditions that lead to "multiscaling," in which there are deviations from the Kolmogorov predictions for the scaling exponents. These deviations are linked to a property that we refer to as the "geometric factor" in the velocity field and its gradients. Section VI offers a discussion of this paper and some comments about the road ahead. In particular, we stress the differences between our approach and the current "fractal model" [5] of turbulence.

II. THE GEOMETRY OF FRACTAL GRAPHS

In this section, we discuss the tools necessary for the evaluation of the volume of a graph of a tensor that is enclosed in a ball of a chosen radius. The tensor under consideration is a function $\tilde{\mathbf{v}}$ such that

$$\tilde{\mathbf{v}}:\ B \to \mathbb{R}^N , \tag{2.1}$$

with $\mathbb{R}^d \supset B$. We think about B as a ball in Euclidean space, and $\tilde{\mathbf{v}}$ can be a scalar ($N=1$), a vector, or a tensor. We are interested in the volume of the graph of $\tilde{\mathbf{v}}$. At first we consider $\tilde{\mathbf{v}}$ as a time-independent field. In Sec. II B, we examine the changes required when the field $\tilde{\mathbf{v}}$ is the solution of a partial differential equation. The main difference is that we may be interested then in "typical" properties, which are obtained by time averaging [8].

A. Time-independent fields

To define the graph, we associate with $\tilde{\mathbf{v}}$ the function G,

$$G:\ B \to \mathbb{R}^d \times \mathbb{R}^N , \tag{2.2}$$

defined by

$$G(\mathbf{x}) = (\mathbf{x}, \tilde{\mathbf{v}}(\mathbf{x})) . \tag{2.3}$$

It is important to realize that typically the graphs of the hydrodynamic fields are self-affine. This means that they all have a typical scale L on which the largest possible variations in $\tilde{\mathbf{v}}$ are registered. These are denoted as $\tilde{\mathbf{v}}_L$. It is natural to consider the dimensionless field $\mathbf{v}(\mathbf{y}) = \tilde{\mathbf{v}}(\mathbf{x}/L)/\tilde{\mathbf{v}}_L$, and to measure scales with L as a yardstick.

Consider now the image of B under G:

$$G(B) = \{(x,y) | x \in B,\ y = \mathbf{v}(\mathbf{y})\} . \tag{2.4}$$

We want to know the volume of this object. To this aim, we use the area formula of geometric measure theory [Ref. [9], Eq. (3.2.3), and see also Ref. [10]], which says that if $\mathbf{v}$ is Lipshitz, then the (nondimensional) volume of $G(B)$ is

$$H^{(d)}(G(B)) = \frac{1}{L^3} \int_B J(\mathbf{x}) d\mathbf{x} , \tag{2.5}$$

where $H^{(d)}$ is the d-dimensional Hausdorff measure and $d\mathbf{x}$ is the Lebesgue measure in $\mathbb{R}^d$. The integrand J is the square root of the Jacobian of $(\nabla G)^*(\nabla G)$:

$$J^2(\mathbf{x}) = \det \left[\delta_{ij} + L^2 \frac{\partial \mathbf{v}}{\partial x_i} \cdot \frac{\partial \mathbf{v}}{\partial x_j} \right]_{i,j=1,2,\ldots,d} . \tag{2.6}$$

In (2.6), the dot product is a scalar product in R^N.

From the point of view of scaling properties, the crucial objects are the invariants of the matrix

$$[(\nabla \mathbf{v})^* \cdot (\nabla \mathbf{v})]_{ij} = \frac{\partial v^n}{\partial x_i} \frac{\partial v^n}{\partial x_j} , \tag{2.7}$$

which we denote as $I_k(\mathbf{x})$ or $I_k(\mathbf{v})(\mathbf{x})$ when necessary. The invariants depend, of course, on the value of $(\nabla \mathbf{v})(\mathbf{x})$. For instance, in $d=3$, the invariants are

$$I_1 = \sum_{i=1}^{3} \sum_{n=1}^{N} \left| \frac{\partial v^n}{\partial x_i} \right|^2 = \mathrm{Tr}[(\nabla \mathbf{v})^* \cdot (\nabla \mathbf{v})] , \tag{2.8}$$

$$I_2 = \tfrac{1}{2}(\{\mathrm{Tr}[(\nabla \mathbf{v})^* \cdot (\nabla \mathbf{v})]\}^2 - \mathrm{Tr}\{[(\nabla \mathbf{v})^* \cdot (\nabla \mathbf{v})]^2\}) , \tag{2.9}$$

$$I_3 = \det[(\nabla \mathbf{v})^* \cdot (\nabla \mathbf{v})] . \tag{2.10}$$

A convenient way to express the invariants is offered by the eigenvalues of the matrix $[(\nabla \mathbf{v})^* \cdot (\nabla \mathbf{v})]$. Since this matrix is symmetric, it can be diagonalized. Moreover, since it is non-negative, its eigenvalues are non-negative. We denote the eigenvalues by μ_i^2 and order them in nonincreasing order (counted with algebraic multiplicity),

$$\mu_1^2 \geq \mu_2^2 \geq \cdots \geq \mu_d^2 \geq 0 . \tag{2.11}$$

We reiterate that the eigenvalues are dependent on the local value of $(\nabla \mathbf{v})(\mathbf{x})$. In terms of the eigenvalues, the invariants can be written as

$$I_k = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le d} (\mu_{i_1})^2 \cdots (\mu_{i_k})^2 \tag{2.12}$$

and

$$J^2(\mathbf{x}) = 1 + \sum_{k=1}^{d} L^{2k} I_k(\mathbf{x}) \ . \tag{2.13}$$

The actual formula to use in the case of scalar or vector fields depends on how many of the invariant I_k are nonzero. Denote by i_v the number (counting algebraic multiplicities) of nonzero eigenvalues of $[(\nabla \mathbf{v})^* \cdot (\nabla \mathbf{v})]$:

$$i_v = d - \dim(\mathrm{Ker}[(\nabla \mathbf{v})^* \cdot (\nabla \mathbf{v})]) \ . \tag{2.14}$$

We refer to i_v as the index of v (and recall that it is a function of $\mathbf{x}$ via $\nabla \mathbf{v}$). The index equals the number of linearly independent d-dimensional covectors in the list $\nabla \mathbf{v}^{(n)}$, $n = 1, 2, \ldots, N$. Clearly,

$$I_k = 0 \quad \text{if } k > i_v \ , \tag{2.15}$$

and thus (2.13) is in fact a sum up to i_v:

$$J^2(\mathbf{x}) = 1 + \sum_{k=1}^{i_v} L^{2k} I_k(\mathbf{x}) \ . \tag{2.16}$$

For example, in the scalar case $N = 1$, the index is at most 1. The way to see this quickly is to realize that for a scalar, $(\nabla \mathbf{v})$ defines one direction in d-space, and we can find $d - 1$ vectors ξ_i that are orthogonal to $(\nabla \mathbf{v})$. Thus there are at least $d - 1$ zero eigenvalues of $[(\nabla \mathbf{v})^* \cdot (\nabla \mathbf{v})]$. The volume of the graph is then calculated simply as

$$H^{(d)}[G(B)] = \frac{1}{L^3} \int_B (1 + L^2 |\nabla \mathbf{v}|^2)^{1/2} d\mathbf{x} \ . \tag{2.17}$$

If $N = 2$ or if $N = 3$ but the components of $\mathbf{v}$ obey a relation like $|\mathbf{v}| = 1$ (i.e., the field is a unit vector), then the index i_v is at most 2. In this case, I_2 contributes to (2.13). In the most general case, all the invariants are nonzero and need to be taken into account in the calculation of the volume. This is the situation with the velocity field in hydrodynamics, as we shall see below.

B. Time-dependent fields

When the field $\mathbf{v}$ satisfies an equation of motion, the graph $G(B)$ becomes time dependent:

$$G(B,t) = \{(\mathbf{x}, y) | \mathbf{x} \in B, \ y = \mathbf{v}(\mathbf{x}, t)\} \ . \tag{2.18}$$

The volume of this graph can fluctuate, and at certain times can assume atypically large values. We may be interested rather in the typical behavior, which is related to time averaging [8]. The time average of any time-dependent function $f(t)$ is defined as

$$\langle f \rangle = \lim_{T \to \infty} \sup \frac{1}{T} \int_0^T dt\, f(t) \ . \tag{2.19}$$

Thus the average of the volume of the graph is defined as

$$\langle H^{(d)}(G(B,t)) \rangle = \left\langle \int_B J(\mathbf{x}, t) d\mathbf{x} \right\rangle \ , \tag{2.20}$$

with an obvious definition of $J(\mathbf{x}, t)$. We shall see later that this mean volume is associated with a dimension that can be smaller than the "worse" dimension of the fluctuating graph.

III. CHARACTERISTIC INCREMENTS, SCALING INDICES, AND THE DIMENSION OF THE GRAPH

The fields of hydrodynamics are expected to be smooth on the smallest scales (smaller than the dissipative cutoff) but to appear "rough" on scales larger than some cutoff. In this section, we adapt the formalism of Sec. II to deal with this situation. We need to redefine now the invariants discussed above in terms of finite differences. Most of the considerations of this section hold for time-independent fields as well as for time-dependent fields. The time dependence will be used only when averaging is needed.

A. Characteristic increments

From the point of view of scaling behavior, the natural objects of study are the finite differences

$$(\delta_i \mathbf{v}^{(r)})(\mathbf{x}) = \mathbf{v}(\mathbf{x} + r e_i) - \mathbf{v}(\mathbf{x}) \ . \tag{3.1}$$

Instead of the matrix $\partial \mathbf{v} / \partial x_i \cdot \partial \mathbf{v} / \partial x_j$, we consider now the matrix $(1/r^2) T^{(r)ij}$, where

$$T_{ij}^{(r)} = \delta_i \mathbf{v}^{(r)} \cdot \delta_j \mathbf{v}^{(r)} \ . \tag{3.2}$$

This matrix is again symmetric and non-negative. We denote its invariants by

$$I_j^{(r)}(\mathbf{x}), \quad j = 1, 2, \ldots, d \ . \tag{3.3}$$

For instance,

$$I_1^{(r)} = \sum_{i=1}^{d} |\delta_i \mathbf{v}^{(r)}(\mathbf{x})|^2 = \sum_{i=1}^{d} |\mathbf{v}(\mathbf{x} + r e_i) - \mathbf{v}(\mathbf{x})|^2 \ , \tag{3.4}$$

$$I_d^{(r)} = \det[\delta_i \mathbf{v}^{(r)} \cdot \delta_j \mathbf{v}^{(r)}], \quad i, j = 1, \ldots, d \ . \tag{3.5}$$

We call these invariants "characteristic increments," and we will discuss the relation between their scaling properties and the dimensions of the graphs of the tensor $\mathbf{v}$.

B. Scaling

We note that the r approximation to J is [cf. Eq. (2.6)]

$$[J^{(r)}(\mathbf{x})]^2 = \det \left[\delta_{ij} + \left[\frac{L}{r} \right]^2 T_{ij}^{(r)} \right]_{i,j = 1, 2, \ldots, d} \ . \tag{3.6}$$

Equation (2.13) becomes

$$[J^{(r)}(\mathbf{x})]^2 = 1 + \left[\frac{L}{r} \right]^2 I_1^{(r)} + \cdots + \left[\frac{L}{r} \right]^{2d} I_d^{(r)} \ . \tag{3.7}$$

Also,

$$I_k^{(r)} = 0 \quad \text{if } k > i^{(r)_v} \ , \tag{3.8}$$

where $i_v^{(r)}$ is the number (counting algebraic multiplicity) of the nonzero eigenvalues of $T_{ij}^{(r)}$.

Consider now balls B_ρ in $\mathbb{R}^d$ of arbitrary centers and radius ρ, whose volume is $|B_\rho|$. We assume now scaling properties for those characteristic increments $I_j^{(r)}$ that are not identically zero:

$$\left\langle \frac{1}{|B_\rho|} \int_{B_\rho} [I_j^{(r)}(\mathbf{x})]^{1/2} d\mathbf{x} \right\rangle \sim \left[\frac{r}{L} \right]^{j\zeta_j} . \tag{3.9}$$

More precisely, we assume that there exist lengths r_0 and L, constants c_1, c_2, and numbers $0 < \zeta_j \le 1$ so that for *any* ball B_ρ in $\mathbb{R}^d$ of radius ρ (and volume $|B_\rho|$), the inequality

$$c_1 \left[\frac{r}{L} \right]^{j\zeta_j} \le \left\langle \frac{1}{|B_\rho|} \int_{B_\rho} [I_j^{(r)}(\mathbf{x})]^{1/2} d\mathbf{x} \right\rangle \le c_2 \left[\frac{r}{L} \right]^{j\zeta_j} \tag{3.10}$$

holds for r,ρ in the range $L \ge r,\rho \ge r_0$. The meaning of the symbol $\langle\ \rangle$ is an average, and it can take different meanings in different contexts. For our purposes below, we shall take it as an average over time, as in Eq. (2.19). We draw the attention of the reader to the fact that, at this stage, we do not assume that the scaling exponents ζ_j are necessarily smaller than 1. It will be seen that fractal graphs are consistent with ζ_j that are smaller than 1.

C. Dimension of the graph and the connection to the exponents

To find the connection between the scaling indices of the type appearing in Eq. (3.10) and the dimension of the graphs, we return first to the point-wise finite differences

$$(\delta_i \mathbf{v}^{(r)})(\mathbf{x}) = \mathbf{v}(\mathbf{x} + r e_i) - \mathbf{v}(\mathbf{x}), \quad i = 1, \ldots, d . \tag{3.11}$$

In the case $N > 1$, we need also to consider the components of these quantities,

$$(\delta_i \mathbf{v}_\alpha^{(r)})(\mathbf{x}) = \mathbf{v}_\alpha(\mathbf{x} + r e_i) - \mathbf{v}_\alpha(\mathbf{x}), \quad \alpha = 1, \ldots, N . \tag{3.12}$$

If the field under consideration is time independent, we shall take "scaling" to mean a Hölder condition over all components of the field, i.e.,

$$|\mathbf{v}_\alpha(\mathbf{x} + r e_i) - \mathbf{v}_\alpha(\mathbf{x})| \le C r^{\zeta_\infty(\mathbf{V})} \quad \text{for all } i,\alpha \tag{3.13}$$

for all $\mathbf{x}$ and all r, with $0 \le r \le L$, and C and L are positive constants. The constant L has the same meaning as discussed after Eq. (1.1). We should note that, in the physical context, the numerical value of ζ_∞ is smaller than 1 for $r > r_0$ where r_0 is an inner scale, but $\zeta_\infty = 1$ for $r < r_0$ (i.e., the fields become smooth on small scales). This is of course in agreement with (3.13) with ζ_∞ taken to be smaller than 1.

Next we want to relate the value of the exponent ζ_∞ to the geometrical properties of the graph of $\mathbf{v}$. The connection is well known in the case of scalar fields [11], and is not too hard to generalize in the case of tensor fields. We recall the scalar case first.

1. *Scalar fields*

Let us consider a box B_r of size r in $\mathbb{R}^d$ and focus on the piece of the graph of $\mathbf{v}$ above it,

$$G(B_r) = \{ (\mathbf{x}, y) | \mathbf{x} \in B_r,\ y = \mathbf{v}(\mathbf{x}) \} . \tag{3.14}$$

By taking a column of d-dimensional boxes above B_r, we can see that the piece of the graph $G(B_r)$ can be covered by, at most, $(1/r) C r^{\zeta_\infty} + 1$ boxes of side r. Thus, dividing the whole domain B into $1/r^d$ equal pieces, we see that the whole graph can be covered by, at most, $N(r)$ boxes, where for small r

$$N(r) \cong C r^{\zeta_\infty - d - 1} . \tag{3.15}$$

The way to find the dimension of the graph is to consider the D-dimensional Hausdorff measure $H^D(G(B))$ and to recall that by definition it cannot exceed $r^D N(r)$. We can write, therefore,

$$H^D(G(B)) \le C \lim_{r \to 0} r^D r^{\zeta_\infty - d - 1} . \tag{3.16}$$

Clearly, $H^D(G(B)) = 0$ if $D > d + 1 - \zeta_\infty$. So if we denote by D_g the Hausdorff (even the box-counting) dimension of the graph $G(B)$, then

$$D_g \le d + 1 - \zeta_\infty . \tag{3.17}$$

We see that if indeed $\zeta_\infty < 1$, the graph can indeed wrinkle, and its dimension can be higher than the dimension of the space above which it is defined.

If the scaling behavior changes at an inner scale $r = r_0$, all we can do is to produce covers with balls of radii r_0 that have $H^{(D)}$ volumes that scale like a negative power of r_0. As $r_0 \to 0$, these volumes tend to infinity, and the conclusions are the same. We note that the limit $r_0 \to 0$ is equivalent in the hydrodynamic application to the limit $\mathrm{Re} \to \infty$.

2. *Vector fields*

In the case of vector fields, Eq. (3.17) is no longer applicable. In fact, in this case the issue of isotropy becomes important, in the sense that in Eq. (3.13) the exponent ζ_∞ is taken to be the same for all components α. We shall derive the analog of Eq. (3.17) for such an "isotropic" vector field, and then remark on the changes incurred by nonisotropy.

We begin again with a box B_r of size r in $\mathbb{R}^d$ and the piece of the graph of $\mathbf{v}$ above it,

$$G(B_r) = \{ (\mathbf{x}, y) | \mathbf{x} \in B_r,\ y = \mathbf{v}(\mathbf{x}) \} . \tag{3.18}$$

The difference comes when we realize that in order to cover this piece, we need many more boxes than before, but not more than $C r^{N(\zeta_\infty - 1)} + O(1)$ boxes of side r. (The reason for this increase in the number of boxes is that the graph can wiggle in N directions now!) Repeating the arguments leading to Eq. (3.17), we find in this case

$$D_g \le d + N(1 - \zeta_\infty) . \tag{3.19}$$

In particular, for the case of hydrodynamic turbulence, which is discussed below, i.e., $d = N = 3$, the relation reads $D_g \le 6 - 3\zeta_\infty$. We notice again that in the trivial case of $\zeta_\infty = 1$, the smooth graph with $D_g = 3$ is recovered.

In the nonisotropic case, in which the scaling exponent

ζ_∞ can depend on the component α (or, for that matter, on the direction i), the inequality (3.19) can still be derived, using the smallest of the exponents ζ_α. In this case, we do not expect, however, the inequality to be sharp, and this possibility is not treated in any further detail in this paper. From now on, we assume isotropy in the scaling sense.

3. Time-dependent fields

For time-dependent fields, there is no reason to focus primarily on the "worst" possible singularity that is associated with the exponent ζ_∞. We are interested in "typical" dimensions, and therefore we can examine the properties of time averages [8]. Thinking first about the scalar field, consider the average of the number of balls $N(r)$ needed to cover the piece of the graph $G(B_r,t)$. We write

$$\langle N(r,t)\rangle \le \frac{C}{r}\langle |v(\mathbf{x}+re_i)-v(\mathbf{x})|\rangle+1=\frac{C}{r}r^{\zeta_1(v)}+1 . \tag{3.20}$$

Consequently, Eq. (3.16) turns to

$$\langle H^D(G(B,t))\rangle \le C\lim_{r\to 0} r^D r^{\zeta_1-d-1} . \tag{3.21}$$

Denoting by $\bar{D}_g$ the "mean" dimension obtained in this way, we see that Eq. (3.17) turns in this case to

$$\bar{D}_g \le d+1-\zeta_1 . \tag{3.22}$$

Similarly, for vector fields, Eq. (3.19) turns into

$$\bar{D}_g \le d+N(1-\zeta_1) . \tag{3.23}$$

In particular, for the case of hydrodynamic turbulence, which is discussed below, i.e., $d=N=3$, the relation reads $\bar{D}_g\le 6-3\zeta_1(\mathbf{u})$, where $\mathbf{u}$ is the velocity field. We notice again that in the trivial case of $\zeta_1=1$, the smooth graph with $\bar{D}_g=3$ is recovered.

IV. CONNECTION TO FLUID MECHANICS: THE CASE OF THE PASSIVE SCALAR

In this section, we explore the consequences of the formalism developed in Secs. II and III for hydrodynamic turbulence in $d=3$. The two natural fields to consider are a passive scalar θ for which, in the previous notation, $N=1$, and the velocity field $\mathbf{u}$ for which $N=3$. The latter field will be discussed in Sec. V. Here we consider the scalar. We should note that a calculation of the dimension of the level sets of the scalar field in turbulence was achieved recently using the co-area instead of the area formula of the geometric measure theory [9,10]. The level sets are horizontal cuts of the graph, and therefore the two calculations are good checks of each other. We find that the present calculation is more transparent.

A. The calculation of the mean gradient of the scalar

The equation obeyed by θ by

$$(\partial_1+\mathbf{u}\cdot\nabla-\kappa\nabla^2)\theta=f , \tag{4.1}$$

where f is a forcing term which represents a source either in an internal point or at the boundaries of the system. If this forcing is absent, we deal with an initial-value problem for θ, and the results developed below will be valid only for times smaller than a typical diffusive time scale for the equilibration of θ in the whole box. The velocity field $\mathbf{u}(\mathbf{x},t)$ is divergence-free and $\mathbf{x}\in\mathbb{R}^3$. We assume that the initial value of θ is bounded. The square of θ obeys the equation

$$\tfrac{1}{2}(\partial_t+\mathbf{u}\cdot\nabla-\kappa\nabla^2)\theta^2+\kappa|\nabla\theta|^2=f\theta . \tag{4.2}$$

We see that this equation offers a convenient way to calculate the integral over a ball of $|\nabla\theta|^2$, which is what is needed to connect to the considerations of Secs. II and III.

Consider then a ball B_ρ of radius ρ, centered at $\mathbf{x}_0$. We employ [12–14] a cutoff function χ such that $\chi(\mathbf{y})=1$ for $|\mathbf{y}|<1$, $\chi(\mathbf{y})=0$ for $|\mathbf{y}|\ge 1$, and $\chi(\mathbf{y})$ is smooth. Multiplying (4.2) by $(1/\rho)\chi(\mathbf{x}-\mathbf{x}_0/\rho)$ and integrating, we obtain

$$\frac{1}{\rho}\int|\nabla\theta|^2(\mathbf{x})\chi\left[\frac{\mathbf{x}-\mathbf{x}_0}{\rho}\right]d\mathbf{x} =-\frac{1}{2\kappa\rho}\int\chi\left[\frac{\mathbf{x}-\mathbf{x}_0}{\rho}\right][(\partial_t+\mathbf{u}\cdot\nabla-\kappa\nabla^2)\theta^2-f\theta]d\mathbf{x} . \tag{4.3}$$

The right-hand side of (4.3) is a sum of four terms. Using the notation $\mathbf{y}=(\mathbf{x}-\mathbf{x}_0/\rho)$, we write these terms

$$S_1(\mathbf{x}_0,\rho,t)=-\frac{1}{2\kappa\rho}\frac{d}{dt}\int\chi(\mathbf{y})\theta^2 d\mathbf{x} , \tag{4.4}$$

$$S_2(\mathbf{x}_0,\rho,t)=\frac{1}{2\kappa\rho^2}\int\theta^2 u_j(\mathbf{x},t)\left[\frac{\partial\chi}{\partial y_j}\right](\mathbf{y})d\mathbf{x} , \tag{4.5}$$

$$S_3(\mathbf{x}_0,\rho,t)=\frac{1}{2\rho^3}\int\theta^2(\Delta_y\chi)(\mathbf{y})d\mathbf{x} . \tag{4.6}$$

$$S_4(\mathbf{x}_0,\rho,t)=\frac{1}{2\kappa\rho}\int\chi(\mathbf{y})f\theta\, d\mathbf{x} . \tag{4.7}$$

Denoting

$$\Theta=\sup_{\mathbf{x},t}|\theta(\mathbf{x},t)| , \quad F=\sup_{\mathbf{x},t}|f(\mathbf{x},t)|$$

we see that

$$|S_3(\mathbf{x},\rho,t)|\le C_1\Theta^2 , \tag{4.8a}$$

$$|S_4(\mathbf{x}_0,\rho,t)|\le C_2\frac{\Theta F\rho^2}{2\kappa} , \tag{4.8b}$$

with C_1 and C_2 being absolute constants which depend only on $\int_{R^3}|\Delta_y\chi|d\mathbf{y}$ and $\int_{R^3}\chi\, d\mathbf{y}$. At this point, we can take time averages according to Eq. (2.20). Clearly,

$$\langle S_1\rangle=0 , \tag{4.9}$$

and from (4.8)

$$\langle |S_3|\rangle\le C_1\Theta^2 \tag{4.10a}$$

$$\langle |S_4|\rangle\le C_2\frac{\Theta F\rho^2}{2\kappa} . \tag{4.10b}$$

The term S_2 will be written as a sum of two terms, $S_2'+S_2''$:

$$S_2'(\mathbf{x}_0,\rho,t)=\frac{1}{2\kappa\rho^2}\int \theta^2(\mathbf{x},t)[u_j(\mathbf{x},t)-\bar{u}_j(\mathbf{x}_0,t)] \times \left[\frac{\partial\chi}{\partial y_j}\right](\mathbf{y})d\mathbf{x}\ , \tag{4.11}$$

$$S_2''(\mathbf{x}_0,\rho,t)=\frac{1}{2\kappa\rho^2}\bar{u}_j(\mathbf{x}_0,t)\int \theta^2(\mathbf{x},t)\left[\frac{\partial\chi}{\partial y_j}\right](\mathbf{y})d\mathbf{x}\ , \tag{4.12}$$

where

$$\bar{u}_j=\frac{1}{|B_{\rho/4}|}\int_{|\mathbf{x}-\mathbf{x}_0|<\rho/4}u_j(\mathbf{x},t)d\mathbf{x}\ . \tag{4.13}$$

Obviously,

$$S_2''(\mathbf{x}_0,\rho,t)=\frac{\bar{u}_j(\mathbf{x}_0,t)}{2\kappa\rho^2}\int [\theta^2(\mathbf{x},t)-\bar{\theta}^2(\mathbf{x}_0,t)]\left[\frac{\partial\chi}{\partial y_j}\right](\mathbf{y})d\mathbf{x}\ , \tag{4.14}$$

where

$$\bar{\theta}=\frac{1}{|B_{\rho/4}|}\int_{B_{\rho/4}}\theta(\mathbf{x},t)d\mathbf{x}\ . \tag{4.15}$$

Consequently,

$$|S_2''(\mathbf{x}_0,\rho,t)|\le C\frac{|\bar{u}_j|\Theta}{2\kappa\rho^2}\frac{4\pi}{(\rho/4)^3}\int_{\rho/2\le|\mathbf{x}-\mathbf{x}_0|\le\rho}d\mathbf{x}\int_{|\mathbf{z}-\mathbf{x}_0|<\rho/4}d\mathbf{z}[|\theta(\mathbf{x},t)-\theta(\mathbf{z},t)|]\ . \tag{4.16}$$

After a change of order of integration, one gets

$$|S_2''(\mathbf{x}_0,\rho,t)|\le C\frac{|\bar{u}_j|\Theta}{2\kappa\rho^2}\int_{\rho/4}^{5\rho/4}r^2dr\frac{1}{|B_\rho|}\int_{B_\rho}[I^{(r)_1}(\theta)d\mathbf{x}]^{1/2}. \tag{4.17}$$

Upon taking the time average, we find

$$\langle |S_2''(\mathbf{x}_0,\rho)|\rangle\le C\frac{LU\Theta^2}{\kappa}\left[\frac{\rho}{L}\right]^{1+\zeta_1(\theta)}, \tag{4.18}$$

where $U=\sup_{\mathbf{x},t}|\mathbf{u}(\mathbf{x},t)|$, and we assumed that

$$\left\langle\frac{1}{|B_\rho|}\int_{B_\rho}[I_1^{(r)}(\theta)d\mathbf{x}]^{1/2}\right\rangle\le C\Theta\left[\frac{r}{L}\right]^{\zeta_1(\theta)},\qquad \rho/4\le r\le 5\rho/4\ . \tag{4.19}$$

In a similar fashion, if $\mathbf{u}$ satisfies (3.10), i.e., if

$$\left\langle\frac{1}{|B_\rho|}\int_{B_\rho}[I_1^{(r)}(\mathbf{u})(\mathbf{x})d\mathbf{x}]^{1/2}\right\rangle\le CU\left[\frac{r}{L}\right]^{\zeta_1(\mathbf{u})}, \tag{4.20}$$

holds for $\rho/4\le r\le 5\rho/4$, then

$$\langle S_2'\rangle\le C\Theta^2U\left[\frac{\rho}{L}\right]^{1+\zeta_1(\mathbf{u})}. \tag{4.21}$$

From (4.9), (4.10), (4.17), and (4.19), we get

$$\frac{1}{\rho}\left\langle\int_{B_{\rho/2}}|\nabla\theta|^2(\mathbf{x})d\mathbf{x}\right\rangle \le C\Theta^2\frac{UL}{\kappa}\left[1+\left[\frac{\rho}{L}\right]^{[1+\zeta_1(\theta)]}+\left[\frac{\rho}{L}\right]^{[1+\zeta_1(\mathbf{u})]}\right] +\frac{C\Theta FL^2}{\kappa}\left[\frac{\rho}{L}\right]^2. \tag{4.22}$$

From Eqs. (4.19) and (4.20), it is evident that L stands for two different integral scales, that may differ. In (4.19) it is the scale of the largest possible scalar difference across a scale, and in (4.20) the size of the largest eddy. We shall assign L to the smaller of the two, to guarantee that both (4.19) and (4.20) are satisfied simultaneously.

To proceed we need to examine the term that results from the forcing. The ratio F/Θ has the dimension of $(\text{time})^{-1}$. We can therefore write the last term on the right-hand side of (4.22) as $[(C\Theta^2L^2)/(\kappa\tau)](\rho/L)^2$, where τ is the typical forcing time scale. We shall focus here on those cases in wihch $U\gg L/\tau$, or, in other words, when the effects of the wrinkling of the graph by the velocity field are much stronger than the effects of the forcing. With this in mind, the result (4.22) can be connected now to the formalism of Sec. III. The formalism there requires a dimensionless field, which we shall denote as $\tilde{\theta}(\mathbf{x},t)=\theta(\mathbf{x},t)/\Theta$. We recall that when we estimate the mean volume of a graph by covering it with balls of size r_0, we find

$$\left\langle\frac{1}{L^3}\int_{B_L}(1+L^2|\nabla\tilde{\theta}|^2d\mathbf{x})^{1/2}\right\rangle\ge(L/r_0)^{\bar{D}_g-3}\ . \tag{4.23}$$

On the other hand,

$$\left\langle\frac{1}{L^3}\int_{B_L}(1+L^2|\nabla\tilde{\theta}|^2d\mathbf{x})^{1/2}\right\rangle \le\left[\frac{1}{L^3}\left\langle\int_{B_L}1+L^2|\nabla\tilde{\theta}|^2d\mathbf{x}\right\rangle\right]^{1/2}, \tag{4.24}$$

or

$$(L/r_0)^{\bar{D}_g-3}\le\left[1+\frac{1}{L}\left\langle\int_{B_L}|\nabla\tilde{\theta}|^2d\mathbf{x}\right\rangle\right]^{1/2}. \tag{4.25}$$

To complete the calculation, we need to use Eq. (4.22) in Eq. (4.25), right at the value of $\rho=L$. Defining the Péclet number (Pe) $=UL/\kappa$ (which is a product of the Reynolds number and the Prandtl number v/κ), we find finally

$$(L/r_0)^{\bar{D}_g-3} \leq \sqrt{1+C\mathrm{Pe}} \ . \tag{4.26}$$

As a last step of the calculation, we choose r_0, according to

$$(L/r_0)=\mathrm{Pe}^{1/[1+\max(\zeta_1)]} , \tag{4.27}$$

where

$$\max(\zeta_1)=\max[\zeta_1(\theta),\zeta_1(\mathbf{u})] \ . \tag{4.28}$$

Substituting in Eq. (4.26), we get

$$(L/r_0)^{\bar{D}_g-3} \leq \left\{1+C\left[\left[\frac{L}{r_0}\right]^{[1+\max(\zeta_1)]}\right]\right\}^{1/2} . \tag{4.29}$$

The final result is now clear. For $L \gg r_0$, the second term is dominant, and

$$\bar{D}_g \leq 3.5+\max(\zeta)/2 \quad \text{for } \frac{L}{r_0} \gg 1 \ . \tag{4.30}$$

In fact, it is reasonable to expect the inequality to be sharp. The dimension is determined by the powers of r_0, which in turn follow from the analytic structure of Eq. (4.1); the inequalities arise due to our inability to compute the amplitudes, not the powers of r_0. An improvement of the estimate of the amplitudes would not affect the exponents. This is the main result of this section.

B. Discussion of the result (4.30)

As a first comment concerning Eq. (4.30), we point out to the reader that an analogous expression has been obtained recently by the present authors for the dimension of the *level sets* of the passive scalar [13]. The result in that case was $D_c \leq 2.5+\zeta_1(\mathbf{u})/2$. It is pleasing to see that the result for the dimension of the graph is indeed higher by 1, as one could expect on intuitive grounds. The present result offers more, however. In particular, if we assume that the inequality (3.22) is sharp, then it follows immediately that

$$2\zeta_1(\theta)+\max(\zeta_1) \geq 1 \ . \tag{4.31}$$

If $\max(\zeta_1)=\zeta_1(\theta)$, then this inequality reads

$$\zeta_1(\theta) \geq \tfrac{1}{3} \quad [\text{if } \zeta_1(\theta) \geq \zeta_1(\mathbf{u})] \ . \tag{4.32}$$

If $\max(\zeta_1)=\zeta_1(\mathbf{u})$, then the inequality reads

$$2\zeta_1(\theta)+\zeta_1(\mathbf{u}) \geq 1 \ . \tag{4.33}$$

In this case we cannot bound $\zeta_1(\theta)$ numerically, but have to be satisfied with a scaling relation, until we derive an inequality for $\zeta_1(\mathbf{u})$ in Sec. V.

V. CONNECTION TO FLUID MECHANICS: THE CASE OF THE VELOCITY FIELD

The calculation in the case of the velocity field seems initially more cumbersome than in the case of the passive scalar, since now the metric properties of the graph depend on the determinant of the matrix $\partial\tilde{\mathbf{u}}/\partial x_i \cdot \partial\tilde{\mathbf{u}}/\partial x_j$, where $\tilde{\mathbf{u}}$ is the dimensionless field $\mathbf{u}/U$. This determinant is dominated by higher-order products of $\nabla\tilde{\mathbf{u}}$, up to sixth order. The hydrodynamics furnishes a natural calculation of the mean $\langle|\nabla\tilde{\mathbf{u}}|^2\rangle$, similar to the calculation of Sec. IV, but not of higher-order contributions. Notwithstanding, it is very important to try to understand the scaling properties of the velocity field, since they also determine the scaling exponents of the passive scalar, as we have seen in Sec. IV. In addition, from the theoretical point of view, it is interesting to examine the possible sources of deviation from the Kolmogorov predictions for the scaling exponents. We shall see that the point of view developed here allows us to examine these issues in an alternative way. We begin with some general considerations.

A. General considerations

We begin with Eqs. (2.5) and (2.13), which in the present context read

$$H^{(3)}(G(B))=\frac{1}{L^3}\int_B J(\mathbf{x})d\mathbf{x} \ , \tag{5.1}$$

$$J^2(\mathbf{x})=1+\sum_{k=1}^{3} L^{2k}I_k(\mathbf{x}) \ . \tag{5.2}$$

From Eqs. (2.8)–(2.10), we deduce (since all the invariants are positive) that

$$H^{(3)}(G(B)) \geq \int_B \sqrt{\det[(\nabla\tilde{\mathbf{u}})^*\cdot(\nabla\tilde{\mathbf{u}})]}d\mathbf{x} \ . \tag{5.3}$$

The right-hand side (RHS) of Eq. (5.3) is of $O((\nabla\tilde{\mathbf{u}})^3)$, and it is therefore very tempting to replace it with the integral $\int_B|\nabla\tilde{\mathbf{u}}|^3 d\mathbf{x}$, which is a nice quantity that we can deal with, as we shall see in Sec. V B. However, such a replacement is not permitted, and it is in fact equivalent to dimensional analysis. We shall explore the consequences of this replacement just to show that the Kolmogorov predictions are recovered if we do it. Later, in Sec. V C, we shall explain in detail how the difference between the full determinant and this approximant can lead to changes in the scaling exponents as compared to the Kolmogorov predictions.

B. Strong scaling (or dimensional analysis)

By the term "strong scaling" we shall mean, assuming from the start the validity of the two following relations:

$$\langle H^{(3)}(G(B_L))\rangle \sim (L/r_0)^{\bar{D}_g}(r_0/L)^3 \quad (\text{for a suitable } r_0) \ , \tag{5.4}$$

$$\left\langle\int_{B_L}\sqrt{\det[(\nabla\tilde{\mathbf{u}})^*\cdot(\nabla\tilde{\mathbf{u}})]}d\mathbf{x}\right\rangle \geq \left\langle\int_{B_L}|\nabla\tilde{\mathbf{u}}|^3 d\mathbf{x}\right\rangle \ . \tag{5.5}$$

The RHS of Eq. (5.5) can be bounded rigorously according to

$$\left\langle\int_{B_L}|\nabla\tilde{\mathbf{u}}|^3 d\mathbf{x}\right\rangle \geq \left[\frac{1}{r_0}\right]^3\left\langle\int_{B_L}|\tilde{\mathbf{u}}(\mathbf{x}+\mathbf{r}_0)-\tilde{\mathbf{u}}(\mathbf{x})|^3 d\mathbf{x}\right\rangle \ . \tag{5.6}$$

The structure function on the RHS of Eq. (5.6) is the only structure function in hydrodynamics that, with the assumptions of isotropy and homogeneity, obeys an exact scaling relation [14], i.e.,

$$\left\langle \frac{1}{L^3}\int_{B_L} |\tilde{\mathbf{u}}(\mathbf{x}+\mathbf{r}_0)-\tilde{\mathbf{u}}(\mathbf{x})|^3 d\mathbf{x} \right\rangle \sim (r_0/L) \; . \tag{5.7}$$

In the language of scaling exponents, the last relation says that $\zeta_3(\mathbf{u})=\frac{1}{3}$ exactly. Collecting Eqs. (5.4)–(5.7), we find the inequality

$$\left[\frac{L}{r_0}\right]^{\bar{D}_g-3} \geq \left[\frac{L}{r_0}\right]^2 , \tag{5.8}$$

from which one concludes that $\bar{D}_g \geq 5$. Together with Eq. (3.19), we find finally $\zeta_1(\mathbf{u}) \leq \frac{1}{3}$. Since, on the other hand, the Cauchy-Schwartz inequality guarantees that $\zeta_1(\mathbf{u}) \geq \zeta_3(\mathbf{u}) = \frac{1}{3}$, we find finally that

$$\zeta_1(\mathbf{u})=\tfrac{1}{3} \quad \text{strong scaling} \; . \tag{5.9}$$

Even though this result is not surprising, it is pleasing that we recover the standard prediction from our point of view, which is quite different from the usual way in which this result is obtained. It is important to stress again, however, that the way to (5.9) included some uncontrolled approximations, which we need to examine next. We shall see that trying to control these approximations leads to the possibility of new solutions, including multiscaling and deviations from (5.9).

C. Multiscaling in turbulence

Both Eqs. (5.4) and (5.5) are uncontrolled. In reality,

$$\langle H^{(3)}(G(B_L))\rangle \geq (L/r_0)^{\bar{D}_g-3} , \tag{5.10}$$

since every finite covering underestimates the volume of the graph. We do not think, however, that this is a very serious issue. After all, our graphs become smooth on scales smaller than r_0, and for this reason the estimate (5.4) is probably rather safe in our context.

On the other hand, the estimate (5.5) is completely uncontrolled, and can go wrong in a very serious way. We cannot guarantee that there is no Re dependence in the ratio of the two terms in Eq. (5.5). Thus, if we want to proceed using Eq. (5.6), we need to start again, forcing the following inequality to hold:

$$(L/r_0)^{\bar{D}_g-3} \geq \mathrm{Re}^{-\beta}\left\langle \int_{B_L} |\nabla\tilde{\mathbf{u}}|^3 d\mathbf{x}\right\rangle , \tag{5.11}$$

with the exponent β chosen such that (5.11) holds for all Re numbers. Proceeding now with Eq. (5.6), which is rigorous, we end up with

$$(L/r_0)^{\bar{D}_g-3} \geq \mathrm{Re}^{-\beta}\left[\frac{L}{r_0}\right]^2 . \tag{5.12}$$

Making the choice $r_0 = L\,\mathrm{Re}^{-1/[1+\zeta_1(\mathbf{u})]}$, we get

$$(L/r_0)^{\bar{D}_g-3} \geq (L/r_0)^{2-\beta[1+\zeta_1(\mathbf{u})]} , \tag{5.13}$$

or $\bar{D}_g \geq 5-\beta[1+\zeta_1(\mathbf{u})]$. Together with Eq. (3.19), we conclude that

$$\zeta_1(\mathbf{u}) \leq \tfrac{1}{3}+\beta[1+\zeta_1(\mathbf{u})]/3 \; . \tag{5.14}$$

This is the main result of this section.

It should be noted that the choice of r_0 is not arbitrary. It is determined by the exponent $\zeta_1(\mathbf{u})$, which governs the wrinkling of the graph of $\mathbf{u}$. We should make sure, however, that Eq. (5.7) holds down to r_0. Indeed, we could have chosen another r_0, according to the scaling exponent $\zeta_3(\mathbf{u})$, i.e., $\tilde{r}_0 \sim \mathrm{Re}^{-1/[1+\zeta_3(\mathbf{u})]}$. This is a *smaller* cutoff length, which defines the smallest scale for which (5.7) is expected to hold. By choosing r_0 the way we did, we are satisfied that (5.7) is applicable down to the smallest wrinkled scales of the graph.

D. Discussion of the result (5.14)

As we noted already, by the Cauchy-Schwarz inequality, Eq. (5.7) implies $\zeta_1(\mathbf{u}) \geq \frac{1}{3}$ rigorously. Thus, (5.14) means that if indeed β is positive, then $\zeta_1(\mathbf{u})$ can exceed $\frac{1}{3}$. On the other hand, Eq. (4.31) allows then a value of $\zeta_1(\theta)$ that is smaller than $\frac{1}{3}$. The direction of these deviations is as expected on the basis of the (not entirely compelling) data that we have on these exponents [16,17]. We thus see that the issue of deviations from the dimensional-analysis prediction depends crucially on the "geometric factor," which is given by the ratio

$$S(\mathrm{Re}) = \frac{\left\langle \int_{B_L} \sqrt{\det[(\nabla\tilde{\mathbf{u}})^*\cdot(\nabla\tilde{\mathbf{u}})]d\mathbf{x}}\right\rangle}{\left\langle \int_{B_L} |\nabla\tilde{\mathbf{u}}|^3 d\mathbf{x}\right\rangle} \; . \tag{5.15}$$

If S(Re) is independent of Re, or $\beta=0$, we expect the dimensional analysis to hold and to lead to $\zeta_1(\theta)=\zeta_1(\mathbf{u})=\frac{1}{3}$. If, however, there exists nontrivial Re dependence in S(Re) with $\beta>0$, then the numerical values of the scaling exponents are expected to deviate from their Kolmogorov values, as discussed here. Of course, we cannot exclude at this point that S(Re) depends on Re, but only logarithmically [say as 1/log(Re)]. In this case, we shall find only logarithmic corrections to the Kolmogorov scaling laws. Needless to say, it is very interesting to try to find the Re dependence of S(Re) from experiments and simulations, to attempt to put reasonable bounds on β, and to correlate the result with the known values of $\zeta_1(\mathbf{u})$.

A detailed analysis of S(Re) and how to relate it to experimentally accessible measurements is beyond the scope of this paper. We shall therefore only note here that S(Re) can be related to a ratio of eigenvalues of the strain tensor. The strain tensor is the symmetrized gradient of the velocity tensor,

$$e_{i,j} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right] ,$$

and it is diagonalizable. The eigenvalues can be denoted by λ_i, $i=1,2,3$, and they sum up to zero due to incompressibility. It has been shown in another paper [18] that S(Re) can be estimated as

$$S(\mathrm{Re}) \sim \langle \lambda_2 \rangle / \langle \lambda_1 \rangle \,, \tag{5.16}$$

where the eigenvalues are arranged in decreasing order $\lambda_1 > \lambda_2 > \lambda_3$. Simulational evidence indicates that indeed this ratio decreases when Re increases, and the exponent β can be estimated. If this evidence will be supported by additional experimental measurements at higher values of Re, we may have begun to understand here the mechanism and the dynamical reason for the deviations from the Kolmogorov theory. It has been argued [18] that the ratio in (5.16) going to zero with Re means a tendency towards local two-dimensionality in the three-dimensional turbulent flow, with a local alignment of the vorticity direction along the second eigenvector of e_{ij}. Further detailed discussion of this mechanism will be offered elsewhere.

VI. DISCUSSION

The main idea of the present work is that scaling behavior in fluid mechanics is related to the geometric properties of the graphs of the hydrodynamic fields. On the one hand, these graphs are wrinkled by the stretching and folding action of the velocity field, and their dimensions depend on the scaling exponents of the velocity field. On the other hand, the scaling exponents of all the hydrodynamic fields depend on the dimensions of their graphs. Using these facts, we were able to offer an alternative approach to the calculation of the scaling exponents in turbulence.

One merit of our approach is that it allows us to examine contributions that are not seen in dimensional analysis. The reason for this is that, in our geometric approach, the scaling exponents are obtained by comparing terms with divergent contributions of r_0, which is the size of the balls that we use to cover the graphs. This scale tends to zero when the Reynolds number tends to infinity. In this way we can turn a dependence on the Reynolds number into a dependence on r_0/L. Thus, dimensionless contributions can affect our scaling exponents, in a way that can never be achieved within the confines of dimensional analysis.

It is important to stress the differences between our approach and the current "fractal model" [5] of turbulence. In the current model, fractality is invoked to explain *deviations* from the Kolmogorov exponents, stating that the dissipation concentrates on a fractal set. In our approach also the standard scaling exponents are associated with fractal objects, i.e., the graphs of the hydrodynamic fields. Fractality and scaling behavior go hand in hand, irrespective of the numerical values of the scaling exponents (or the numerical values of the dimensions of the graphs). Deviations from the predictions of dimensional analysis can appear because the metric properties of the graph of the velocity field are determined by higher-order velocity gradient factors, whereas the codimension of the graph still determines the low-order scaling exponent ζ_1. This fact introduces the possibility that dimensionless factors depending on Re would change the numerical values of the scaling exponents.

In order to proceed in the venue of estimating the scaling exponents, we need now some information that is not contained in this theory. We need to know the Re dependence of the geometric factor (5.15) (and probably other such quantities). One way that we can see for getting information on such quantities is from experiments and simulations [18]. It appears worthwhile to look for these quantities and find precisely how dimensional analysis is forfeited (if it is) in fluid turbulence.

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