Spherical-harmonics decomposition of the Boltzmann equation for charged-particle swarms in the presence of both electric and magnetic fields

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The Boltzmann equation for reacting charged-particle swarms in neutral gases in the presence of both electric and magnetic fields is decomposed into a hierarchy of kinetic equations by expanding the velocity dependence of the phase-space distribution function in terms of spherical harmonics. No limit is set on the number of spherical harmonics and no approximation is made concerning the mass of the charged particles related to that of the neutrals species. The space-time dependence is treated by making the hydrodynamic assumption which is taken to second order in density gradients. Spherical tensors are used throughout. The resulting heirarchy of equations has universal validity and is amenable to a range of numerical solutions. The structure of these equations is discussed and the inadequacies of a Legendre-polynomial expansion are pointed out. The special configurations of the magnetic field parallel and perpendicular to the electric field are discussed in detail.

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I. INTRODUCTION

Since the mid to late 1970s the theoretical analysis of charged-particle transport through neutral gases under the influence of a uniform electrostatic field has advanced considerably. A general overview of this can be obtained from the reviews of Kumar, Skullerud, and Robson [1] and Kumar [2]. It will suffice here to make a few general comments and cite key references. In the case of ions, this advancement followed the introduction of the "two-temperature" moment method by Viehland and Mason [3], before which the theory was in general restricted to weak fields only. Since then, there has been much excellent work on the theory of ion swarms carried out by Viehland and co-workers [4] and Skullerud's group [5]. This situation is comprehensively reviewed in the book by Mason and McDaniel [6]. For electron swarms, the advancement was fueled by the desire to overcome the limitations of the "two-term" approximation which had dominated the theory of electron transport through gases since the time of Holstein [7]. The first systematic multiterm analysis for electron swarms was given by Lin, Robson, and Mason [8] and since then a considerable number of multiterm theories have been published in a relatively short time [9–17]. The situation up to 1986 is discussed by Robson and Ness [17]. Monte Carlo simulations have also played, and continue to play, an important role in swarm physics—for example, in testing the validity of the two-term approximation [18,19] and checking on the accuracy of multiterm solutions [20]. For a discussion of Monte Carlo techniques in swarm physics, the reader is referred to Refs. [21–23].

In contrast, comparatively little work beyond the two-term approximation has been done on transport in the presence of both electric and magnetic fields. This may in part be due to the lack of a systematic approach in solving Boltzmann's equation under these conditions. In the case of electrons, two-term analysis and references to early work are given in the books by Huxley and Crompton [24] and Holt and Haskel [25]. For the Maxwell-Lorentz model, Braglia and Ferrari [26] have discussed the solution of the spatially homogeneous Boltzmann equation for an arbitrary angle between the electric and magnetic fields. More recently, Ikuta and Sugai [27] have applied the "flight time integral" method to investigate electron transport in a model gas for the magnetic field both parallel and perpendicular to the electric field. This theory is, however, only suitable to their particular approach. Biagi [28] claims to have a multiterm Boltzmann analysis for the Lorentz approximation valid for any angle between the electric and magnetic field. However, a Legendre-polynomial expansion is used [29]. In the present work it is shown that such an expansion is strictly valid only in the spatially homogeneous situation for the case of parallel fields. Using a Monte Carlo technique, Brennen, Garvie, and Kelly [30] investigated electron transport in nitrogen for perpendicular fields. Both components of drift and the four components of the diffusion tensor were calculated. In a kinetic theory investigation of ion-cyclotron-resonance collision broadening, Viehland, Mason, and Whealton [31] used the two-temperature moment method to solve the spatially homogeneous Boltzmann equation for ions in the presence of both electric and magnetic fields. Solutions were taken to the second approximation in their truncation scheme [3,31]. Early work on the motion of electron and ion swarms in gases under the influence of electric and magnetic fields, carried out by Allis, is summarized in Ref. [32].

The knowledge of charged-particle transport in the presence of both electric and magnetic fields has a number of practical applications—for example, in high-precision tracking detectors used in high-energy physics, where the configurations of both parallel and perpendicular fields are used in detectors designed to determine both the energy and the momentum of high-energy particles

[33,34]. Application is also found in devices such as high-current switches, cold-cathode rectifiers, and plasma preparation [30]. The configuration of perpendicular fields is discussed from an engineering perspective in the review by Heylen [35], where the effects of the magnetic field on gas breakdown and its applications are considered in some detail. It is believed, however, that the investigation of charged-particle transport in electric and magnetic fields has more fundamental implications. Solution of Boltzmann's equation in conjunction with charged-particle experiments is a well-established procedure for verification or determination of low-energy electron (ion) -molecule cross sections (potentials) [6,36]. The essence of the technique is the comparison of theoretical and experimental transport coefficients. Starting with some initial set of cross sections (or potential), a theoretical calculation is made of the transport coefficients and a comparison is made with experimental values. If the comparison is unsatisfactory, the cross-section set (or potential) is adjusted in a controlled manner and the transport coefficients are recalculated and the comparison is made again. This iterative process is continued until satisfactory agreement between the theory and experiment is obtained. So far, this process has been carried out only when an electric field is present. When, for example, a magnetic field is applied perpendicular to the electric field, there are additional transport coefficients, which could in principle be used to provide additional tests for a cross-section set [37]. This may be of value when questions of uniqueness concerning the cross-section set arise.

The aim of the present work is to develop a systematic approach to the multiterm solution of Boltzmann's equation for "reacting" charged-particle transport in gases in the presence of both electric and magnetic fields to the point where a range of numerical techniques can be implemented. Working with the irreducible-tensor formalism introduced into kinetic theory by Kumar [38], this is done by decomposing the Boltzmann equation into a hierarchy of kinetic equations using spherical harmonics and the gradient expansion. It is found that although the addition of the magnetic field itself is relatively straightforward, the effect it has on the structure of the equations is a significant factor. With regard to the nature of the interactions between the charged particles and the neutral molecules, only the assumption of central forces is made. No assumptions are made concerning the isotropy of the scattering or the mass ratio of the colliding particles. The two configurations of parallel and perpendicular fields are considered in detail. We discuss, however, how the present approach can be applied to an arbitrary configuration of the fields. In the case of perpendicular fields, we compare with the earlier "two-term" theory and point out shortcomings resulting from the use of a Legendre-polynomial expansion. The situation of parallel fields leads to complex equations when considering transport perpendicular to the fields.

II. THEORY

In an earlier paper Robson and Ness [17] (hereafter referred as I) formulated a multiterm spherical-harmonics representation of the phase-space distribution function of "reacting" charged-particle swarms in a gaseous medium under the influence of a uniform electrostatic field in terms of spherical tensors. In the present work we extend this formalism to include a magnetic field. Our starting point is the Boltzmann equation describing the evolution of the phase-space distribution function $f(\mathbf{r},\mathbf{c},t)$ of the charged particles

$$[\partial_t+\mathbf{c}\cdot\partial_{\mathbf{r}}+(\mathbf{a}+\mathbf{c}\times\boldsymbol{\Omega})\cdot\partial_{\mathbf{c}}]f(\mathbf{r},\mathbf{c},t)=-J(f)\ , \tag{1}$$

where

$$\mathbf{a}=e\mathbf{E}/m \tag{2}$$

is the acceleration due to a uniform electric field of strength E and

$$\boldsymbol{\Omega}=e\mathbf{B}/m \tag{3}$$

is the charged-particle cyclotron frequency in a uniform magnetic field of flux density B. The charge and mass of the charged particles are denoted by e and m, respectively, while $\mathbf{r}$ and $\mathbf{c}$ denote the position and velocity, respectively, at some time t. We note the identity

$$(\mathbf{c}\times\boldsymbol{\Omega})\cdot\partial_{\mathbf{c}}=-\boldsymbol{\Omega}\cdot\mathbf{L}\ , \tag{4}$$

where

$$\mathbf{L}=\mathbf{c}\times\partial_{\mathbf{c}} \tag{5}$$

is an operator similar to the angular momentum operator of quantum mechanics. This connection is useful in connection with calculation of the matrix elements, as discussed below.

The spherical-harmonic decomposition of $f(\mathbf{r},\mathbf{c},t)$ is written as

$$f(\mathbf{r},\mathbf{c},t)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}f_m^{(l)}(\mathbf{r},c,t)Y_m^{[l]}(\hat{\mathbf{c}})\ , \tag{6}$$

where $\hat{\mathbf{c}}$ denotes the angles of $\mathbf{c}$. The irreducible-tensor notation used in the present work is discussed in I. Substituting expansion (6) into Eq. (1), multiplying on the left by $Y_m^{(l)}(\hat{\mathbf{c}})$, and integrating over all $\hat{\mathbf{c}}$ yields

$$\sum_{l',m'}\langle lm|\partial_t+\mathbf{c}\cdot\partial_{\mathbf{r}}+\mathbf{a}\cdot\partial_{\mathbf{c}}-\boldsymbol{\Omega}\cdot\mathbf{L}|l'm'\rangle f_{m'}^{(l')}(\mathbf{r},c,t) =-\sum_{l',m'}\langle lm|J|l'm'\rangle f_{m'}^{(l')}(\mathbf{r},c,t)\ , \tag{7}$$

where we have used identity (4) and set

$$\langle lm|\hat{O}|l'm'\rangle\equiv\int Y_m^{(l)}(\hat{\mathbf{c}})\hat{O}Y_{m'}^{[l']}(\hat{\mathbf{c}})d\hat{\mathbf{c}}\ ,$$

with $\hat{O}$ denoting any of the above operators. The explicit representation of the first three terms on the left-hand side (lhs) of Eq. (7) are given in I, where the collision matrix is also discussed. Here, as in I, we do not require the explicit representation of the collision matrix; only the assumption of central forces has been made. The matrix elements of the magnetic-field term are

$$\langle lm|\boldsymbol{\Omega}\cdot\mathbf{L}|l'm'\rangle=\sum_{\mu=-1}^{1}\Omega_\mu^{(1)}\langle lm|L_\mu^{[1]}|l'm'\rangle\ , \tag{8}$$

where

$$L_\mu^{[1]} = \sum_{m_1,m_2} (1m_1 1m_2|1\mu) c_{m_1}^{[1]} \partial_{c_{m_2}}^{[1]} \tag{9}$$

is Eq. (5) in spherical notation [39]. Here, $(1m_1 1m_2|1\mu)$ is a Clebsch-Gordan coefficient, and we have used the tensor coupling rule [17,38]

$$[a^{(l_1)}, b^{(l_2)}]_m^{(l)} = \sum_{m_1,m_2} (l_1 m_1 l_2 m_2 | lm) a_{m_1}^{(l_1)} b_{m_2}^{(l_2)} \ . \tag{10}$$

The Wigner-Eckart theorem [17,38] implies that

$$\langle lm|\boldsymbol{\Omega}\cdot\mathbf{L}|l'm'\rangle = \sum_{\mu=-1}^{1} \Omega_\mu^{(1)} (l'm'1\mu|lm) \langle l \| L^{[1]} \| l' \rangle \ , \tag{11}$$

where $\langle l \| L^{[1]} \| l' \rangle$ is a reduced matrix element, given below. Substitution of Eq. (11) along with Eqs. (I-18) to (I-21) into Eq. (7) gives the spherical-harmonics representation of the Boltzmann equation as

$$\partial_t f_m^{(l)} + \sum_{l',m',\mu} (l'm'1\mu|lm) \langle l \| c^{[1]} \| l' \rangle G_\mu^{(11)} f_{m'}^{(l')} - ia \sum_{l'} (l'm10|lm) \langle l \| \partial_c^{[1]} \| l' \rangle f_m^{(l')} - \sum_{l',m',\mu} \Omega_\mu^{(1)} (l'm'1\mu|lm) \langle l \| L^{[1]} \| l' \rangle f_{m'}^{(l')} = -J^l f_m^{(l)} \ , \tag{12}$$

where the system of coordinates is chosen such that the z axis lies along the electric-field vector **E**. The gradient operator $G_\mu^{(s\lambda)}$ and the collision operator J^l are defined by Eqs. (I-11) [40] and (I-18), respectively. Explicit expressions for the reduced-matrix elements $\langle l \| \partial_c^{[1]} \| l' \rangle$ and $\langle l \| c^{[1]} \| l' \rangle$ are given by Eqs. (I-23) and (I-24), respectively. The reduced-matrix elements of the angular momentum operator are

$$\langle l \| L^{[1]} \| l' \rangle = -\sqrt{l(l+1)}\,\delta_{l'l} \ , \tag{13}$$

so that the magnetic-field term is diagonal in the l index, independent of the configuration of **E** and **B**. Note that the Clebsch-Gordan coefficient in Eq. (11) ensures that in the $l=0$ member of Eq. (12) the magnetic-field term is not present, a reflection of the fact that the acceleration of the charged particles due to the magnetic field does not add energy to the particle. The magnetic field is present for all higher-l members of (12) and therefore influences $f_m^{(1)}$ in the $l=0$ equation. This reflects the fact that although the magnetic field itself does not directly change the energy of a particle, it can do so indirectly. For example, if between collisions the magnetic field turns a charged particle against the electric field, the particle, as a result of the action of the magnetic field, will lose energy.

A. Gradient expansion of $f_m^{(l)}$

In the presence of both an electric and a magnetic field, there are three independent directions in any swarm experiment, determined by the electric-field strength **E**, the magnetic-flux density **B**, and spatial gradients ∇n. Tensors of any rank can be formed from these vectors, either individually or by coupling them together. Thus, any tensor $f_m^{(l)}$ can be represented quite generally by a sum over all possible coupling of tensors formed from **E**, **B**, and ∇n, which produce a tensor of rank l:

$$f_m^{(l)} = \sum_{s=0}^{\infty} \sum_{\lambda=0}^{s} \sum_{\lambda'=0}^{\infty} \sum_{\lambda''=0}^{\infty} \sum_{\lambda'''=0}^{\infty} \bar{f}(l|s\lambda\lambda'\lambda''\lambda''') [[Y^{(\lambda''')}(\hat{\mathbf{E}}), Y^{(\lambda'')}(\hat{\mathbf{B}})]^{(\lambda')}, G^{(s\lambda)}]_m^{(l)} n \ , \tag{14}$$

where $\bar{f}(l|s\lambda\lambda'\lambda''\lambda''')$ are scalar coefficients which vanish unless

$$l+\lambda+\lambda''' = \text{even} \ , \tag{15}$$

a result of parity considerations and independent of the configuration of **E** and **B**. The order of the coupling of the tensors in (14) is arbitrary. For **E** along the z axis, we have

$$Y_{\mu'''}^{(\lambda''')}(\hat{\mathbf{E}}) = (-i)^{\lambda'''} \left[\frac{2\lambda'''+1}{4\pi} \right]^{1/2} \delta_{\mu'''0} \ . \tag{16}$$

Employing the tensor coupling rule (10) in Eq. (14) and using expression (16), we find

$$f_m^{(l)} = \sum_{s=0}^{\infty} \sum_{\lambda=0}^{s} \sum_{\mu=-\lambda}^{\lambda} f(lm|s\lambda\mu) G_\mu^{(s\lambda)} n \ , \tag{17}$$

where

$$f(lm|s\lambda\mu) = \sum_{\lambda'=0}^{\infty} \sum_{\lambda''=0}^{\infty} \sum_{\lambda'''=0}^{\infty} (-i)^{\lambda'''} \left[\frac{2\lambda'''+1}{4\pi} \right]^{1/2} \bar{f}(l|s\lambda\lambda'\lambda''\lambda''') (\lambda' m-\mu \lambda\mu|lm) \times (\lambda''' 0 \lambda'' m-\mu|\lambda' m-\mu) Y_{m-\mu}^{(\lambda'')}(\hat{\mathbf{B}}) = 0 \ \text{ if } |m|>l \ \text{ or } \ |\mu|>\lambda \ . \tag{18}$$

For vanishing magnetic field we have the extra conditions $\lambda''=0$ and $\mu=m$, and Eqs. (17) and (18) above reduce to Eqs.

(I-13) and the equation following it in I.

Substituting expansion (17) into expansion (6) and taking terms up to $s=2$, the phase-space distribution function may be written in the form

$$f(\mathbf{r},\mathbf{c},t)=n(\mathbf{r},t)\sum_{l=0}^{\infty}\sum_{m=-l}^{l} f(lm|000)Y_m^{[l]}(\hat{\mathbf{c}})+\sum_{\mu=-1}^{1}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} f(lm|11\mu)Y_m^{[l]}(\hat{\mathbf{c}})G_\mu^{(11)}n(\mathbf{r},t)$$
$$+\sum_{l=0}^{\infty}\sum_{m=-l}^{l} f(lm|200)Y_m^{[l]}(\hat{\mathbf{c}})G_0^{(20)}n(\mathbf{r},t)+\sum_{\mu=-2}^{2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} f(lm|22\mu)Y_m^{[l]}(\hat{\mathbf{c}})G_\mu^{(22)}n(\mathbf{r},t)\ . \quad (19)$$

Note that, in general, even for the spatially homogeneous case, an expansion in terms of Legendre polynomials is inadequate as m is no longer restricted to zero, but free to range from $-l$ to l. In the special case of the two-term approximation where the sums are truncated at $l=1$, if the polar axis is chosen to lie along the vector $f_m^{(1)}$, then Legendre polynomials may be used. It is, however, incorrect to take such an expansion beyond $l=1$: this point is discussed further later. Also, in the special case of parallel fields where there is a single axis of symmetry, terms to first order in the spatial gradients in Eq. (19) can be reduced to expansions in terms of Legendre and associated Legendre polynomials, as in the case when only an electric field is present [17] [see Eq. (83) below].

B. Equation of continuity

Equation (17) applies to any tensor and the scalar quantity $\partial_t n$ can be expressed in the form

$$\partial_t n=\sum_{s=0}^{\infty}\sum_{\lambda=0}^{s}\sum_{\mu=-\lambda}^{\lambda}\omega(s\lambda\mu)G_\mu^{(s\lambda)}n\ , \quad (20a)$$

i.e.,

$$\partial_t n=\omega(000)n+\sum_{\mu=-1}^{1}\omega(11\mu)G_\mu^{(11)}n+\omega(200)G_0^{(20)}n+\sum_{\mu=-2}^{2}\omega(22\mu)G_\mu^{(22)}n\ , \quad (20b)$$

up to $s=2$. Using the explicit expressions for $G_\mu^{(s\lambda)}$ given in Table I of I, Eq. (20b) can be written as

$$\partial_t n=\omega(000)n+\frac{i}{\sqrt{2}}[\omega(111)-\omega(11-1)]\partial_x n+\frac{1}{\sqrt{2}}[\omega(111)+\omega(11-1)]\partial_y n-i\omega(110)\partial_z n$$
$$+\left[\frac{\omega(200)}{\sqrt{3}}+\frac{\omega(220)}{\sqrt{6}}-\frac{\omega(222)+\omega(22-2)}{2}\right]\partial_x^2 n+\left[\frac{\omega(200)}{\sqrt{3}}+\frac{\omega(220)}{\sqrt{6}}+\frac{\omega(222)+\omega(22-2)}{2}\right]\partial_y^2 n$$
$$+\left[\frac{\omega(200)}{\sqrt{3}}-\sqrt{2/3}\,\omega(220)\right]\partial_z^2 n+i[\omega(222)-\omega(22-2)]\partial_{xy}^2 n$$
$$+[\omega(221)-\omega(22-1)]\partial_{xz}^2 n-i[\omega(221)-\omega(22-1)\partial_{yz}^2 n\ . \quad (21)$$

In the usual notation the continuity equation is

$$\partial_t n=-\alpha n-W_x\partial_x n-W_y\partial_y n-W_z\partial_z n+D_x\partial_x^2 n+D_y\partial_y^2 n$$
$$+D_z\partial_z^2 n+D_1\partial_{xy}^2 n+D_2\partial_{xz}^2 n+D_3\partial_{yz}^2 n\ , \quad (22)$$

where α is the loss rate coefficient, W_x, W_y, and W_z are the three components of the drift velocity; D_x, D_y, and D_z are the diagonal components of the diffusion tensor; and $D_1=D_{xy}+D_{yx}$, $D_2=D_{xz}+D_{zx}$, and $D_3=D_{yz}+D_{zy}$ denote the off-diagonal elements of the diffusion tensor. Comparing Eqs. (21) and (22), we identify

$$\alpha=-\omega(000)\ ,$$
$$W_x=-i[\omega(111)-\omega(11-1)]/\sqrt{2}\ ,$$
$$W_y=-[\omega(111)+\omega(11-1)]/\sqrt{2}\ ,$$
$$W_z=i\omega(110)\ ,$$
$$D_x=[\omega(200)/\sqrt{3}+\omega(220)/\sqrt{6}-(\omega(222)+\omega(22-2))/2]\ ,$$
$$D_y=[\omega(200)/\sqrt{3}+\omega(220)/\sqrt{6}+(\omega(222)+\omega(22-2))/2]\ ,$$
$$D_z=[\omega(200)/\sqrt{3}-\sqrt{2}\omega(220)/\sqrt{3}]\ ,$$
$$D_1=i[\omega(222)-\omega(22-2)]\ ,$$
$$D_2=[\omega(221)-\omega(22-1)]\ ,$$
$$D_3=-i[\omega(221)+\omega(22-1)]\ . \quad (23)$$

C. The hierarchy of equations

Substitution of expansion (17) into Eq. (12) and making use of Eq. (13) and the continuity Eq. (20), we generate a hierarchy of coupled equations for the functions $f(lm|s\lambda\mu)$. After equating coefficients of $G_\mu^{(s\lambda)}n$, we find

$$[J^l+\omega(000)]f(lm|s\lambda\mu)-ia\sum_{l'}(l'm10|lm)\langle l\|\partial_c^{[1]}\|l'\rangle f(l'm|s\lambda\mu)+\sqrt{l(l+1)}\sum_{\nu=-1}^{1}\Omega_\nu^{(1)}(lm-\nu 1\nu|lm)f(lm-\nu|s\lambda\mu)$$
$$=X_{lm}(s\lambda\mu)\ , \qquad (24)$$

where

$$X_{lm}(000)=0\ , \qquad (25a)$$

$$X_{lm}(11\mu)=-\omega(11\mu)f(lm|000)-\sum_{l'}(l'm-\mu 1\mu|lm)\langle l\|c^{[1]}\|l'\rangle f(l'm-\mu|000)\ ,\quad \mu=0\ ,\pm 1 \qquad (25b)$$

$$X_{lm}(200)=-\omega(200)f(lm|000)-\frac{1}{\sqrt{3}}\sum_{\nu=-1}^{1}(-1)^{1-\nu}\omega(11\nu)f(lm|11-\nu)$$
$$-\frac{1}{\sqrt{3}}\sum_{l'}\sum_{\nu=-1}^{1}(-1)^{1-\nu}(l'm-\nu 1\nu|lm)\langle l\|c^{[1]}\|l'\rangle f(l'm-\nu|11-\nu)\ , \qquad (25c)$$

$$X_{lm}(22\mu)=-\omega(22\mu)f(lm|000)-\sum_{\nu=-1}^{1}(1\mu-\nu 1\nu|2\mu)\omega(11\nu)f(lm|11\mu-\nu)$$
$$-\sum_{l'}\sum_{\nu=-1}^{1}(1\mu-\nu 1\nu|2\mu)(l'm-\nu 1\nu|lm)\langle l\|c^{[1]}\|l'\rangle f(l'm-\nu|11\mu-\nu),\quad \mu=0\ ,\pm 1,\pm 2\ . \qquad (25d)$$

Equation (24) comprises a set of ten coupled equations, corresponding to the ten values of $(s\lambda\mu)$, up to $s=2$. For a given value of $(s\lambda\mu)$ the rhs of Eq. (24) is specified by the member of Eqs. (25) corresponding to that value of $(s\lambda\mu)$. Equation (24) with (25a) is the lowest member of the set. Equation (24) with (25b) contains the next three members, corresponding to $\mu=-1,0,1$. Equation (24) with (25c) is the fifth member, while Eq. (24) with (25d) contains the last five members, corresponding to $\mu=-2,-1,0,1,2$. Members with the same value of s do not couple with each other; they only couple to lower-order s members—thus the ordering of equations within a given s "set" is arbitrary. The first member of this chain is an eigenvalue equation for $\omega(000)$, while all remaining members are inhomogeneous equations in which, apart from the quantity $\omega(s\lambda\mu)$, $(s\lambda\mu)\neq(000)$, the right-hand side (rhs) is determined from the solution of lower-order equations in the chain. For $(s\lambda\mu)\neq(000)$, the quantity $\omega(s\lambda\mu)$ on the rhs of the equation for $f(lm|s\lambda\mu)$ must be solved for in a self-consistent manner, as discussed below. The discussion given on page 2074 of I concerning the eigenvalue problem also applies to the above set and there is no need to repeat it here. We note, however, that in applying it to the above hierarchy of equations, the following extensions are necessary:

$$\omega_j(00)\rightarrow\omega_j(000)\ ,$$
$$\omega_j(s\lambda)\rightarrow\omega_j(s\lambda\mu)\ ,$$
$$f_j(lm|s\lambda)\rightarrow f_j(lm|s\lambda\mu)\ ,$$
$$G_m^{(s\lambda)}\rightarrow G_\mu^{(s\lambda)}\ ,$$
$$G_0^{(s\lambda)}\rightarrow G_\mu^{(s\lambda)}$$

and include a summation over μ from $-\lambda$ to λ in Eqs. (I-33), (I-34), and (I-36).

D. Normalization and determination of the $\omega(s\lambda\mu)$

The normalization condition and the procedure for determining the $\omega(s\lambda\mu)$ parallel that given in Sec. II C of I; we find

$$\sqrt{4\pi}\int_0^\infty f(00|s\lambda\mu)c^2dc=\delta_{s0}\delta_{\lambda 0}\delta_{\mu 0}\ , \qquad (26)$$

$$\omega(000)=-\sqrt{4\pi}\int_0^\infty J_R^0[f(00|000)]c^2dc\ , \qquad (27a)$$

$$\omega(11\mu)=\left[\frac{4\pi}{3}\right]^{1/2}\int_0^\infty c^3 f(1\mu|000)dc-\sqrt{4\pi}\int_0^\infty J_R^0[f(00|11\mu)]c^2dc\ , \qquad (27b)$$

$$\omega(200)=-\left[\frac{4\pi}{3}\right]^{1/2}\int_0^\infty c^3\sum_{\nu=-1}^{1}f(1\nu|11\nu)dc-\sqrt{4\pi}\int_0^\infty J_R^0[f(00|200)]c^2dc\ , \qquad (27c)$$

$$\omega(22\mu)=-\left[\frac{4\pi}{3}\right]^{1/2}\int_0^\infty c^3\sum_{\nu=-1}^{1}(-1)^{1-\nu}(1\mu-\nu 1\nu|2\mu)f(1-\nu|11\mu-\nu)dc-\sqrt{4\pi}\int_0^\infty J_R^0[f(00|22\mu)]c^2dc\ , \qquad (27d)$$

where J_R denotes the nonconservative part of the collision operator. Consider Eq. (24) with (25b), and Eq. (27b) which, for a given value of μ, are solved for the quantity $\omega(11\mu)$ and the function $f(lm|11\mu)$. We see that the last term on the rhs of (27b) contains the unknown function $f(00|11\mu)$, which is found by solving Eq. (24) with (25b), but this equation in turn contains the unknown quantity $\omega(11\mu)$. Hence, the pair of Eqs. (24) with (25b) and (27b) must be solved in a self-consistent manner. The same is true for the pair (24) with (25c) and (27c) and the pair (24) with (25d) and (27d). This, of course, does not hold for Eq. (24) with (25a) and Eq. (27a), as Eq. (24) with (25a) is a "true" eigenvalue equation and must be solved as one. Equations (27) may be used in Eqs. (23) to express the transport coefficients in terms of the functions $f(lm|s\lambda\mu)$.

In the absence of nonconservative interactions, $\omega(000)$ and J_R vanish, and three significant simplifications then follow immediately from Eqs. (27). Firstly, the first member of the chain reduces to a homogeneous equation for the function $f(lm|000)$. Secondly, the last term on the rhs of Eqs. (27b) to (27d) vanishes and the quantities $\omega(s\lambda\mu)$ are then completely specified by lower-order members of the chain; hence, all higher-order members reduce to ordinary inhomogeneous equations and there is no longer the need for self-consistent solutions. Finally, we see that the remaining nine quantities $\omega(s\lambda\mu)$, $(s\lambda\mu)\neq(000)$, given by Eqs. (27b) to (27d) can be determined by solving the hierarchy up to $s=1$, i.e., in the absence of reactions, it is sufficient to consider the first four equations in order to determine transport up to diffusion.

With regard to the structure of the matrix of coefficients on the lhs of Eq. (24), we note that (1) the collision term is diagonal in both the l and m indices; (2) the electric-field term is both subdiagonal and superdiagonal in the l index [as $l'=l\pm1$; see Eq. (I-24)] and diagonal in the m index; and (3) the magnetic-field term is diagonal in the l index and tridiagonal in m in general; for particular configurations the m structure can simplify, as we will see. In principle, one can exploit this "block" structure to optimize numerical solution of the set (24). This is discussed in more detail in a subsequent paper [41] where we implement numerical solution. In the absence of a magnetic field, $m=\mu$, and the set of equations (24) and (25) reduce to equations (I-28) to (I-31). This greatly simplifies matters for two reasons: First, to second order in spatial gradients we only require solution of the equations for two values of μ, $\mu=0$ (transport parallel to **E**) and $\mu=1$ (transport perpendicular to **E**). Secondly, the fact that $m=\mu$ means that the m dependence separates into the different equations of the hierarchy, so that in effect there is no m index to consider in solving a particular equation. That is to say, for any given equation, there is only the l index to be concerned with. However, when **B** is present, there is in general no simple relationship between m and μ. Thus, if we truncate expansion (6) at $l=l_{\max}$, say, then for any member of the set of equations (24) and (25), m will range from $-l_{\max}$ to $l_{\max}$. It may of course be sufficient to truncate m at some value $m_{\max}<l_{\max}$ to obtain convergence of the transport coefficients, as indeed we have found to be the case [41]. Nevertheless, in general, when both **E** and **B** are present, for any given member of equations (24) and (25) one has both the l and m indices to consider instead of just the l index, in the implementation of numerical solution. This significantly increases the size of the matrices and CPU time required for solution, when compared to the **E**-only situation. We now consider the special configurations of perpendicular and parallel fields.

III. SPECIAL CONFIGURATIONS OF THE FIELDS

A. Magnetic field perpendicular to electric field

Choosing **B** along the y axis, we have

$$\Omega_0^{(1)}=0\ ,\quad \Omega_{\pm1}^{(1)}=\Omega/\sqrt{2}\ , \tag{28}$$

and the magnetic-field term in Eq. (24) becomes

$$\sqrt{l(l+1)}\sum_{\nu}\Omega_\nu^{(1)}(lm-\nu 1\nu|lm)f(lm-\nu|s\lambda\mu)$$
$$=\frac{\Omega}{2}\left[\sqrt{(l-m)(l+m+1)}f(lm+1|s\lambda\mu)-\sqrt{(l+m)(l-m+1)}f(lm-1|s\lambda\mu)\right]\ . \tag{29}$$

We also have

$$Y_{-\mu}^{(\lambda)}(\hat{\mathbf{B}})=Y_{\mu}^{(\lambda)}(\hat{\mathbf{B}})=(-i)^{\lambda}(-1)^{(\mu+|\mu|)/2}\left[\frac{(2\lambda+1)(\lambda-|\mu|)!}{4\pi(\lambda+|\mu|)!}\right]^{1/2}P_{\lambda}^{|\mu|}(0)e^{i\mu\pi/2}=0 \text{ unless } \lambda+\mu=\text{even}\ . \tag{30}$$

Hence, with this configuration we have the additional constraint in Eq. (18) that

$$\lambda''+\mu+m=\text{even}\ . \tag{31}$$

This constraint reflects the invariants of the physical system (and therefore the Boltzmann equation) under a rotation of π about the z axis. The Clebsch-Gordan coefficients in Eq. (18) plus constraints (15) and (31) then require

$$f(l-m|s\lambda-\mu)=(-1)^{m+\mu}f(lm|s\lambda\mu)\ , \tag{32}$$

which in turn implies

$$\omega(s\lambda-\mu)=(-1)^{\mu}\omega(s\lambda\mu)\ . \tag{33}$$

Substituting (29) into Eq. (24), we have

$$[J^l+\omega(000)]f(lm|s\lambda\mu)-ia\sum_{l'}(l'm\,10|lm)\langle l\|\partial_c^{[1]}\|l'\rangle f(l'm|s\lambda\mu)$$
$$+\frac{\Omega}{\sqrt{2}}[\sqrt{(l-m)(l+m+1)}f(lm+1|s\lambda\mu)-\sqrt{(l+m)(l-m+1)}f(lm-1|s\lambda\mu)]=X_{lm}(s\lambda\mu)\ , \quad (34)$$

where the rhs of (34) as given by Eqs. (25) may be simplified by making use of relationships (32) and (33). With condition (33), Eqs. (23) become

$$\begin{aligned} &W_y=D_1=D_3=0\ ,\\ &\alpha=-\omega(000)\ ,\\ &W_x=-i\sqrt{2}\omega(111)\ ,\\ &W_z=i\omega(110)\ ,\\ &D_x=\omega(200)/\sqrt{3}+\omega(220)/\sqrt{6}-\omega(222)\ ,\\ &D_y=\omega(200)/\sqrt{3}+\omega(220)/\sqrt{6}+\omega(222)\ ,\\ &D_z=\omega(200)/\sqrt{3}-\sqrt{2}\omega(220)/\sqrt{3}\ ,\\ &D_2=D_h=2\omega(221)\ , \end{aligned} \quad (35)$$

where W_x is the $\mathbf{E}\times\mathbf{B}$ component of drift, W_z is the $\mathbf{E}$ component of drift, D_x denotes diffusion along $\mathbf{E}\times\mathbf{B}$, D_y diffusion along $\mathbf{B}$, D_z diffusion along $\mathbf{E}$, and the off-diagonal diffusion coefficient D_h is analogous to the Hall conductivity in plasmas [30]. Relationships (35) can also be expressed as

$$\begin{aligned} &\omega(000)=-\alpha\ ,\\ &\omega(110)=-iW_z\ ,\\ &\omega(111)=iW_x/\sqrt{2}\ ,\\ &\omega(200)=(D_x+D_y+D_z)/\sqrt{3}\ ,\\ &\omega(220)=(D_x+D_y-2D_z)/\sqrt{6}\ ,\\ &\omega(221)=D_h/2\ ,\\ &\omega(222)=(D_y-D_x)/2\ . \end{aligned} \quad (36)$$

Relationship (32) implies that one need only consider positive values of the μ index in solving Eqs. (34) in order to obtain all the information up to second order in the gradient expansion. That is, for $\mathbf{B}$ perpendicular to $\mathbf{E}$, we have seven equations to solve, corresponding to the number of quantities on the lhs of Eqs. (36). In the absence of reactions, we require the solution of three equations in order to obtain both the drift and the diffusion coefficients. In addition, in the case of $\mu=0$, (32) becomes

$$f(l-m|s\lambda0)=(-1)^m f(lm|s\lambda0)\ ,$$

and the equations for which $\mu=0$ need only be solved for positive values of the m index. In the case of $\mu>0$, we must solve Eqs. (34) with both positive and negative values of the m index.

Using (32) in Eqs. (27), we find that the transport coefficients given by Eq. (35) can be written in terms of the functions $f(lm|s\lambda\mu)$ as

$$\alpha=\sqrt{4\pi}\int_0^\infty J_R^0[f(00|000)]c^2dc\ , \quad (37a)$$

$$W_z=i\left[\frac{4\pi}{3}\right]^{1/2}\int_0^\infty c^3 f(10|000)dc-i\sqrt{4\pi}\int_0^\infty J_R^0[f(00|110)]c^2dc\ , \quad (37b)$$

$$W_x=-i\left[\frac{8\pi}{3}\right]^{1/2}\int_0^\infty c^3 f(11|000)dc+i\sqrt{8\pi}\int_0^\infty J_R^0[f(00|111)]c^2dc\ , \quad (37c)$$

$$D_x=-\left[\frac{4\pi}{3}\right]^{1/2}\left\{\int_0^\infty c^3[f(11|111)-f(1-1|111)]dc+\int_0^\infty J_R^0\left[f(00|200)+\frac{f(00|220)}{\sqrt{2}}-\sqrt{3}f(00|222)\right]c^2dc\right\}, \quad (37d)$$

$$D_y=-\left[\frac{4\pi}{3}\right]^{1/2}\left\{\int_0^\infty c^3[f(11|111)+f(1-1|111)]dc+\int_0^\infty J_R^0\left[f(00|200)+\frac{f(00|220)}{\sqrt{2}}+\sqrt{3}f(00|222)\right]c^2dc\right\}, \quad (37e)$$

$$D_z=-\left[\frac{4\pi}{3}\right]^{1/2}\left\{\int_0^\infty c^3 f(10|110)dc+\int_0^\infty J_R^0[f(00|200)-\sqrt{2}f(00|220)]c^2dc\right\}, \quad (37f)$$

$$D_h=\left[\frac{8\pi}{3}\right]^{1/2}\left\{\int_0^\infty c^3[f(11|110)+f(10|111)]dc-\sqrt{6}\int_0^\infty J_R^0[f(00|221)]c^2dc\right\}. \quad (37g)$$

B. Transformation

We make the following transformation of Eqs. (34) in order to present a form that is more amenable to computation and more suitable for comparison with early two-term approximation work; we define

$$F_{lm}^{s\lambda\mu}=i^{l+\lambda}\left[\frac{2^{|\mu|}(2l+1)(l-|m|)!}{4\pi(l+|m|)!}\right]^{1/2} f(lm|s\lambda\mu)\ , \quad (38)$$

and for the sake of convenience we set

$$\begin{aligned} F_{lm}&=F_{lm}^{000}\ ,\\ F_{lm}^{(L)}&=F_{lm}^{110}\ ,\\ F_{lm}^{(T)}&=F_{lm}^{111}\ . \end{aligned} \quad (39)$$

For vanishing magnetic field,

$$\begin{aligned} &F_{lm}\to F_l\ ,\\ &F_{lm}^{(L)}\to F_l^{(L)}\ ,\\ &F_{lm}^{(T)}\to F_l^{(T)}\ ,\\ &F_{lm}^{200}\to F_l^{(2T)}\ ,\\ &F_{lm}^{220}\to -F_l^{(2L)}\ , \end{aligned} \quad (40)$$

where F_l, $F_l^{(L)}$, $F_l^{(T)}$, $F_l^{(2L)}$, and $F_l^{(2T)}$ are defined by Eqs. (I-46) [40]. From Eqs. (32) and (38) it follows that

$$F_{l,-m}^{s\lambda,-\mu}=(-1)^{m+\mu}F_{lm}^{s\lambda\mu}\ . \quad (41)$$

Applying the above transformation to Eqs. (34) and (25), making use of relationship (41), and writing out explicitly all summations, Clebsch-Gordan coefficients, and reduced-matrix elements, we find after some algebra

$$(J^l-\alpha)F_{lm}^{s\lambda\mu}+a[d_{lm}F_{l-1,m}^{s\lambda\mu}+b_{lm}F_{l+1,m}^{s\lambda\mu}] + \frac{\Omega}{2}[g_{lm}F_{l,m+1}^{s\lambda\mu}-g_{l,-m}F_{l,m-1}^{s\lambda\mu}]=H_{lm}^{s\lambda\mu}\ , \quad (42)$$

where

$$d_{lm}=\frac{(l-|m|)}{(2l-1)}\left[\frac{d}{dc}-\frac{(l-1)}{c}\right]\ , \quad (43)$$

$$b_{lm}=\frac{(l+1+|m|)}{(2l+3)}\left[\frac{d}{dc}+\frac{(l+2)}{c}\right]\ , \quad (44)$$

$$g_{lm}=\begin{cases}1\ , & m<0\\ (l-m)(l+1+m)\ , & m\geq 0\ .\end{cases} \quad (45)$$

The $H_{lm}^{s\lambda\mu}$ are

$$H_{lm}\equiv H_{lm}^{000}=0\ , \quad (46a)$$

$$H_{lm}^{(L)}\equiv H_{lm}^{110}=-W_zF_{lm}+c\left[\frac{(l-m)}{(2l-1)}F_{l-1,m}+\frac{(l+1+m)}{(2l+3)}F_{l+1,m}\right],\quad 0\leq m\leq l \quad (46b)$$

$$H_{lm}^{(T)}\equiv H_{lm}^{111}=W_xF_{lm}+c\left[\frac{h_{lm}}{(2l-1)}F_{l-1,m-1}-\frac{k_{lm}}{(2l+3)}F_{l+1,m-1}\right]\ ,\quad -l\leq m\leq l \quad (46c)$$

$$\begin{aligned} H_{lm}^{200}=&-\frac{1}{\sqrt{3}}(D_x+D_y+D_z)F_{lm}-\frac{1}{\sqrt{3}}\left\{W_zF_{lm}^{(L)}-\frac{W_x}{2}[(-1)^mF_{l,-m}^{(T)}+F_{lm}^{(T)}]\right\}\\ &+\frac{c}{\sqrt{3}}\left\{\frac{1}{(2l-1)}\left[(l-m)F_{l-1,m}^{(L)}-\frac{(1-\delta_{m0})}{2}(-1)^mF_{l-1,1-m}^{(T)}-\frac{(1+\delta_{m0})}{2}(l-m)(l-1-m)F_{l-1,m+1}^{(T)}\right]\right.\\ &\quad+\frac{1}{(2l+3)}\left[(l+1+m)F_{l+1,m}^{(L)}+\frac{(1-\delta_{m0})}{2}(-1)^mF_{l+1,1-m}^{(T)}\right.\\ &\quad\left.\left.+\frac{(1+\delta_{m0})}{2}(l+2+m)(l+1+m)F_{l+1,m+1}^{(T)}\right]\right\}\ ,\quad 0\leq m\leq l \end{aligned} \quad (46d)$$

$$\begin{aligned} H_{lm}^{220}=&\frac{1}{\sqrt{6}}(D_x+D_y-2D_z)F_{lm}-\sqrt{\tfrac{2}{3}}W_zF_{lm}^{(L)}-\frac{W_x}{2\sqrt{6}}[(-1)^mF_{l,-m}^{(T)}+F_{lm}^{(T)}]\\ &+c\sqrt{\tfrac{2}{3}}\left\{\frac{1}{(2l-1)}\left[(l-m)F_{l-1,m}^{(L)}+\frac{(1-\delta_{m0})}{4}(-1)^mF_{l-1,1-m}^{(T)}+\frac{(1+\delta_{m0})}{4}(l-m)(l-1-m)F_{l-1,m+1}^{(T)}\right]\right.\\ &\quad+\frac{1}{(2l+3)}\left[(l+1+m)F_{l+1,m}^{(L)}-\frac{(1-\delta_{m0})}{4}(-1)^mF_{l+1,1-m}^{(T)}\right.\\ &\quad\left.\left.-\frac{(1+\delta_{m0})}{4}(l+2+m)(l+1+m)F_{l+1,m+1}^{(T)}\right]\right\}\ ,\quad 0\leq m\leq l \end{aligned} \quad (46e)$$

$$H_{lm}^{221}=\frac{D_h}{2}F_{lm}-\frac{W_z}{\sqrt{2}}F_{lm}^{(T)}+W_xF_{lm}^{(L)}+\frac{c}{\sqrt{2}}\left\{\frac{1}{(2l-1)}[(l-|m|)F_{l-1,m}^{(T)}+h_{lm}F_{l-1,m-1}^{(L)}]+\frac{1}{(2l+3)}[(l+1+|m|)F_{l+1m}^{(T)}+k_{lm}F_{l+1m-1}^{(L)}]\right\},\quad -l\le m\le l \tag{46f}$$

$$H_{lm}^{222}=(D_y-D_x)F_{lm}+W_xF_{lm}^{(T)}+\frac{ch_{lm}}{(2l-1)}F_{l-1,m-1}^{(T)}-\frac{ck_{lm}}{(2l+3)}F_{l+1m-1}^{(T)}\ ,\quad -l\le m\le l \tag{46g}$$

where

$$k_{lm}=\begin{cases}(l+2+|m|)(l+1+|m|)\ , & m\le 0\\ 1\ , & m>0\end{cases} \tag{47}$$

$$h_{lm}=\begin{cases}(l-|m|)(l-1-|m|)\ , & m\le 0\\ 1\ , & m>0\ .\end{cases} \tag{48}$$

Notice that Eqs. (46c) and (46g) contain negative m values of the function F_{lm}, while (46f) contains negative m values of the functions F_{lm} and $F_{lm}^{(L)}$, although from (42) and (46a), and (42) and (46b) we only solve for the positive m values of F_{lm} and $F_{lm}^{(L)}$. Here we make use of relationship (41). The normalization condition is

$$4\pi\int_0^\infty F_{00}^{s\lambda\mu}c^2dc=\delta_{s0}\delta_{\lambda 0}\delta_{\mu 0}\ , \tag{49}$$

and from Eqs. (37) and (38) the transport coefficients are given by

$$\alpha=4\pi\int_0^\infty J_R^0[F_{00}]c^2dc\ , \tag{50a}$$

$$W_z=\frac{4\pi}{3}\int_0^\infty c^3F_{10}dc-4\pi\int_0^\infty J_R^0[F_{00}^{(L)}]c^2dc\ , \tag{50b}$$

$$W_x=-\frac{8\pi}{3}\int_0^\infty c^3F_{11}dc+4\pi\int_0^\infty J_R^0[F_{00}^{(T)}]c^2dc\ , \tag{50c}$$

$$D_x=\frac{4\pi}{3}\int_0^\infty c^3[F_{11}^{(T)}-F_{1,-1}^{(T)}]dc-4\pi\int_0^\infty J_R^0\left[\frac{F_{00}^{200}}{\sqrt{3}}-\frac{F_{00}^{220}}{\sqrt{6}}+\frac{F_{00}^{222}}{2}\right]c^2dc\ , \tag{50d}$$

$$D_y=\frac{4\pi}{3}\int_0^\infty c^3[F_{11}^{(T)}+F_{1,-1}^{(T)}]dc-4\pi\int_0^\infty J_R^0\left[\frac{F_{00}^{200}}{\sqrt{3}}-\frac{F_{00}^{220}}{\sqrt{6}}-\frac{F_{00}^{222}}{2}\right]c^2dc\ , \tag{50e}$$

$$D_z=\frac{4\pi}{3}\int_0^\infty c^3F_{10}^{(L)}dc-\frac{4\pi}{\sqrt{3}}\int_0^\infty J_R^0[F_{00}^{200}+\sqrt{2}F_{00}^{220}]c^2dc, \tag{50f}$$

$$D_h=-\frac{8\pi}{3}\int_0^\infty c^3\left[F_{11}^{(L)}+\frac{F_{10}^{(T)}}{2}\right]dc+4\pi\sqrt{2}\int_0^\infty J_R^0[F_{00}^{221}]c^2dc\ . \tag{50g}$$

In the absence of a magnetic field,

$$F_{11}=F_{00}^{(T)}=F_{1,-1}^{(T)}=F_{00}^{222}=F_{11}^{(L)}=F_{10}^{(T)}=F_{00}^{221}=0\ ,$$

as $m\neq\mu$ and the above expressions reduce to Eqs. (I-53), bearing in mind (40). We now consider a quasi-Lorentz gas.

C. Quasi-Lorentz gas

For a quasi-Lorentz gas we make the approximation [1]

$$J^1[\Phi(c)]=\nu(c)\Phi(c)\ ,$$

where $\Phi(c)$ denotes any function of the charged-particle speed and ν is the momentum-transfer collision frequency. We do not require the explicit form of J^l for $l\neq 1$. In the absence of reactions, the first three members of Eqs. (42) and (46a) are

$$\begin{aligned}&J^0F_{00}+\frac{a}{3}\left[\frac{d}{dc}+\frac{2}{c}\right]F_{10}=0\ ,\\ &\nu F_{10}+a\frac{d}{dc}F_{00}+\frac{2}{5}a\left[\frac{d}{dc}+\frac{3}{c}\right]F_{20}+2\Omega F_{11}=0\ ,\\ &\nu F_{11}+\frac{3}{5}a\left[\frac{d}{dc}+\frac{3}{c}\right]F_{21}-\frac{\Omega}{2}F_{10}=0\ ;\end{aligned} \tag{51}$$

the first three members of Eqs. (42) and (46b) are

$$\begin{aligned}&J^0F_{00}^{(L)}+\frac{a}{3}\left[\frac{d}{dc}+\frac{2}{c}\right]F_{10}^{(L)}=-W_zF_{00}+\frac{c}{3}F_{20}\ ,\\ &\nu F_{10}^{(L)}+a\frac{d}{dc}F_{00}^{(L)}+\frac{2}{5}a\left[\frac{d}{dc}+\frac{3}{c}\right]F_{20}^{(L)}+2\Omega F_{11}^{(L)}=-W_zF_{10}+c\left[F_{00}+\frac{2}{5}F_{20}\right]\ ,\\ &\nu F_{11}^{(L)}+\frac{3}{5}a\left[\frac{d}{dc}+\frac{3}{c}\right]F_{21}^{(L)}-\frac{\Omega}{2}F_{10}^{(L)}=-W_zF_{11}+c\frac{3}{5}F_{21};\end{aligned} \tag{52}$$

and the first four members of Eqs. (42) and (46c) are

$$J^0F_{00}^{(T)}+\frac{a}{3}\left[\frac{d}{dc}+\frac{2}{c}\right]F_{10}^{(T)}=W_xF_{00}+\tfrac{2}{3}cF_{11} ,$$

$$\nu F_{10}^{(T)}+a\frac{d}{dc}F_{00}^{(T)}+\tfrac{2}{5}a\left[\frac{d}{dc}+\frac{3}{c}\right]F_{20}^{(T)}+\Omega[F_{11}^{(T)}-F_{1,-1}^{(T)}]=W_xF_{10}-\tfrac{6}{5}cF_{21} ,$$

$$\nu F_{11}^{(T)}+\frac{3}{5}a\left[\frac{d}{dc}+\frac{3}{c}\right]F_{21}^{(T)}-\frac{\Omega}{2}F_{10}^{(T)}=W_xF_{11}+cF_{00}-\frac{c}{5}F_{20} , \tag{53}$$

$$\nu F_{1,-1}^{(T)}+\frac{3}{5}a\left[\frac{d}{dc}+\frac{3}{c}\right]F_{2,-1}^{(T)}+\frac{\Omega}{2}F_{10}^{(T)}=-W_xF_{11}-\tfrac{12}{5}cF_{22} .$$

From Eqs. (51), F_{10} and F_{11} can be written in the form

$$F_{10}=-\frac{\nu a}{\nu^2+\Omega^2}\left\{\frac{d}{dc}F_{00}+\frac{2}{5}\left[\frac{d}{dc}+\frac{3}{c}\right]F_{20}-\frac{6\Omega}{5\nu}\left[\frac{d}{dc}+\frac{3}{c}\right]F_{21}\right\} , \tag{54a}$$

$$F_{11}=-\frac{\Omega a}{2(\nu^2+\Omega^2)}\left\{\frac{d}{dc}F_{00}+\frac{2}{5}\left[\frac{d}{dc}+\frac{3}{c}\right]F_{20}\right\}-\frac{3}{5}\frac{a\nu}{(\nu^2+\Omega^2)}\left[\frac{d}{dc}+\frac{3}{c}\right]F_{21} . \tag{54b}$$

In a similar fashion we find from Eqs. (52) and (53) that

$$F_{10}^{(L)}=\frac{\nu}{\nu^2+\Omega^2}\left\{-W_z\left[F_{10}-\frac{2\Omega}{\nu}F_{11}\right]+cF_{00}-a\frac{d}{dc}F_{00}^{(L)}+\frac{2}{5}cF_{20}-\frac{2}{5}a\left[\frac{d}{dc}+\frac{3}{c}\right]F_{20}^{(L)}-\frac{6\Omega}{5\nu}cF_{21}+\frac{6\Omega}{5\nu}a\left[\frac{d}{dc}+\frac{3}{c}\right]F_{21}^{(L)}\right\} , \tag{55a}$$

$$F_{11}^{(T)}+F_{1,-1}^{(T)}=\frac{c}{\nu}F_{00}-\frac{3a}{5\nu}\left[\frac{d}{dc}+\frac{3}{c}\right][F_{21}^{(T)}-F_{2,-1}^{(T)}]-\frac{c}{5\nu}[F_{20}-12F_{22}] , \tag{55b}$$

$$F_{11}^{(T)}-F_{1,-1}^{(T)}=\frac{\nu}{\nu^2+\Omega^2}\left\{W_x\left[\frac{\Omega}{\nu}F_{10}+2F_{11}\right]+cF_{00}-\frac{a\Omega}{\nu}\frac{d}{dc}F_{00}^{(T)}-\frac{c}{5}\left[F_{20}+\frac{6\Omega}{\nu}F_{21}+12F_{22}\right]-\frac{a}{5}\left[\frac{2\Omega}{\nu}\left[\frac{d}{dc}+\frac{3}{c}\right]F_{20}^{(T)}+3\left[\frac{d}{dc}+\frac{3}{c}\right](F_{21}^{(T)}-F_{2,-1}^{(T)})\right]\right\} , \tag{55c}$$

$$F_{11}^{(L)}+\frac{F_{10}^{(T)}}{2}=\frac{\nu}{\nu^2+\Omega^2}\left\{-W_z\left[F_{11}+\frac{\Omega}{2\nu}F_{10}\right]-W_x\left[\frac{\Omega}{\nu}F_{11}-\frac{1}{2}F_{10}\right]-\frac{a}{2}\left[\frac{\Omega}{\nu}\frac{d}{dc}F_{00}^{(L)}+\frac{d}{dc}F_{00}^{(T)}\right]+\frac{3\Omega}{10\nu}cF_{20}+\frac{6\Omega}{5\nu}cF_{22}-\frac{a}{5}\left[\frac{\Omega}{\nu}\left[\frac{d}{dc}+\frac{3}{c}\right]F_{20}^{(L)}+\left[\frac{d}{dc}+\frac{3}{c}\right]F_{20}^{(T)}\right]-\frac{3}{5}a\left[\frac{d}{dc}+\frac{3}{c}\right]F_{21}^{(L)}+\frac{3\Omega}{10\nu}a\left[\frac{d}{dc}+\frac{3}{c}\right](F_{21}^{(T)}-F_{2,-1}^{(T)})\right\} . \tag{55d}$$

Making the $l=1$ approximation, we see from the last member of Eq. (51) that

$$F_{11}=\frac{\Omega}{2\nu}F_{10} , \tag{56}$$

and Eqs. (54) reduce to

$$F_{10}=-\frac{\nu a}{(\nu^2+\Omega^2)}\frac{d}{dc}F_{00} , \tag{57a}$$

$$F_{11}=-\frac{\Omega a}{2(\nu^2+\Omega^2)}\frac{d}{dc}F_{00} . \tag{57b}$$

Substitution of Eqs. (57) into Eqs. (50b) and (50c) with $J_R=0$ then gives

$$W_z=-\frac{4\pi}{3}a\int_0^\infty\frac{\nu c^3}{(\nu^2+\Omega^2)}\frac{d}{dc}F_{00}dc , \tag{58a}$$

$$W_x=\frac{4\pi}{3}a\Omega\int_0^\infty\frac{c^3}{(\nu^2+\Omega^2)}\frac{d}{dc}F_{00}dc . \tag{58b}$$

Apart from the lack of a negative sign in the expression for W_x, Eqs. (58) are the same as the equations (8.16) of Huxley and Crompton [24]. The difference in sign for W_x arises because we have chosen **B** along the y axis, whereas Huxley and Crompton choose **B** along the negative y axis [42]. In a similar fashion, we find that making the $l=1$ approximation in Eqs. (55) and substituting the result into Eqs. (50d)–(50g) with J_R set to zero yields

$$D_x=\frac{4\pi}{3}\int_0^\infty \frac{\nu c^3}{(\nu^2+\Omega^2)}\left\{W_x\frac{2\Omega}{\nu}F_{10}+cF_{00}-a\frac{\Omega}{\nu}\frac{d}{dc}F_{00}^{(T)}\right\}dc\ , \quad (59a)$$

$$D_y=\frac{4\pi}{3}\int_0^\infty \frac{c^4}{\nu}F_{00}dc\ , \quad (59b)$$

$$D_z=\frac{4\pi}{3}\int_0^\infty \frac{\nu c^3}{(\nu^2+\Omega^2)}\left\{W_z\frac{(\Omega^2-\nu^2)}{\nu^2}F_{10}+cF_{00}-a\frac{d}{dc}F_{00}^{(L)}\right\}dc\ , \quad (59c)$$

$$D_h=\frac{4\pi}{3}\int_0^\infty \frac{\nu c^3}{(\nu^2+\Omega^2)}\times\left\{\left[2W_z\frac{\Omega}{\nu}+W_x\frac{(\Omega^2-\nu^2)}{\nu^2}\right]F_{10}+a\left[\frac{\Omega}{\nu}\frac{d}{dc}F_{00}^{(L)}+\frac{d}{dc}F_{00}^{(T)}\right]\right\}dc\ . \quad (59d)$$

In writing down Eqs. (59) we have used expression (56), which is valid only for the $l=1$ approximation, to combine the first two terms of (55a), (55c), and (55d). Apart from the first term under the integral in Eqs. (59a), (59c), and (59d), the above expressions for the diffusion coefficients agree [42] with those given by Eqs. (8.45) of Huxley and Crompton [24]. The term in F_{10} is absent in the treatment of Huxley and Crompton because the time derivative of the "vector" function (the equivalent of $f_m^{(1)}$ in the present work) is set to zero. This is equivalent to a zeroth-order truncation in the density-gradient expansion of the vector function [43]. Note that it is not possible in general to extend the Legendre-polymonial expansion of Huxley and Crompton beyond the $l=1$ approximation as terms in F_{2m}, $F_{2m}^{(L)}$, and $F_{2m}^{(T)}$, $m>0$, present in Eqs. (54) and (55) cannot be calculated by this theory. Moreover, these functions depend upon higher l and m members through the coupling of the kinetic equations.

D. Magnetic field parallel to electric field

For **B** along the z axis,

$$\Omega_m^{(1)}=-i\Omega\delta_{m0}\ , \quad (60)$$

and the magnetic-field term in Eq. (24) becomes

$$\sqrt{l(l+1)}\sum_\nu \Omega_\nu^{(1)}(lm-\nu 1\nu|lm)f(lm-\nu|s\lambda\mu)=-i\Omega m f(lm|s\lambda\mu)\ . \quad (61)$$

Thus, the magnetic-field term is now diagonal in both l and m, and we have

$$Y_{\mu''}^{(\lambda'')}(\hat{\mathbf{B}})=(-i)^{\lambda''}\left[\frac{2\lambda''+1}{4\pi}\right]^{1/2}\delta_{\mu''0}\ , \quad (62)$$

which when substituted into (18) leads to the constraints

$$m=\mu \quad (63)$$

and

$$\lambda'+\lambda''+\lambda'''=\text{even}\ . \quad (64)$$

Condition (63) implies

$$f(lm|s\lambda\mu)=f(lm|s\lambda)\delta_{\mu m}\ , \quad (65)$$

$$\omega(s\lambda\mu)=\omega(s\lambda)\delta_{\mu 0}\ , \quad (66)$$

and expansion (17) reduces to

$$f_m^{(l)}=\sum_{s=0}^{\infty}\sum_{\lambda=0}^{s} f(lm|s\lambda)G_m^{(s\lambda)}\ , \quad (67)$$

where

$$f(lm|s\lambda)=\sum_{\lambda',\lambda'',\lambda'''}(-i)^{\lambda''+\lambda'''}\frac{[(2\lambda''+1)(2\lambda'''+1)]^{1/2}}{4\pi}(\lambda'''0\lambda''0|\lambda'0)(\lambda'0\lambda m|lm)\bar{f}(l|s\lambda\lambda'\lambda''\lambda''')=0\ ,\quad m>\min(l,\lambda)\ . \quad (68)$$

Equation (67) above is the same as Eq. (I-13); however, condition (I-14a) does not apply here. Instead, constraints (15) and (64) require

$$f(l-m|s\lambda)=(-1)^{l+\lambda}f^*(lm|s\lambda)\ , \quad (69)$$

$$\omega(s\lambda)=(-1)^\lambda\omega^*(s\lambda)\ , \quad (70)$$

where "*" denotes the complex conjugate. For vanishing magnetic field, $\lambda''=0$ and (64) and (15) combine to give $l+\lambda+\lambda'=\text{even}$, and condition (I-14a) is recovered. Condition (70) required $\omega(00)$, $\omega(20)$, and $\omega(22)$ to be real numbers and $\omega(11)$ to be imaginary.

In spherical notation the continuity equation now has the form

$$\partial_t n=\sum_{s=0}^{\infty}\sum_{\lambda=0}^{s}\omega(s\lambda)G_0^{(s\lambda)}n\ , \quad (71)$$

where

$$\omega(000)\rightarrow\omega(00)=-\sqrt{4\pi}\int_0^\infty J_R^0[f(00|00)]c^2dc\ , \quad (72a)$$

$$\omega(110)\to\omega(11)=\left[\frac{4\pi}{3}\right]^{1/2}\int_0^\infty c^3 f(10|00)dc-\sqrt{4\pi}\int_0^\infty J_R^0[f(00|11)]c^2dc\ , \tag{72b}$$

$$\omega(200)\to\omega(20)=-\frac{\sqrt{4\pi}}{3}\int_0^\infty c^3[f(10|11)+f(11|11)+f(1-1|11)]dc-\sqrt{4\pi}\int_0^\infty J_R^0[f(00|20)]c^2dc\ , \tag{72c}$$

$$\omega(220)\to\omega(22)=-\frac{\sqrt{8\pi}}{3}\int_0^\infty c^3\left[-f(10|11)+\frac{f(11|11)+f(1-1|11)}{2}\right]dc-\sqrt{4\pi}\int_0^\infty J_R^0[f(00|22)]c^2dc\ , \tag{72d}$$

and all other ω quantities in Eqs. (27) vanish. Hence, Eqs. (23) reduce to

$$W_x=W_y=D_1=D_2=D_3=0\ ,$$
$$\alpha=-\omega(00)\ ,$$
$$W=W_z=i\omega(11)\ ,$$
$$D_x=D_y=D_T=\frac{\omega(20)}{\sqrt{3}}+\frac{\omega(22)}{\sqrt{6}}\ ,$$
$$D_z=D_L=\frac{\omega(20)}{\sqrt{3}}-\sqrt{2/3}\,\omega(22)\ . \tag{73}$$

Hence,

$$\alpha=\sqrt{4\pi}\int_0^\infty J_R^0[f(00|00)]c^2dc\ , \tag{74a}$$

$$W=i\left[\frac{4\pi}{3}\right]^{1/2}\int_0^\infty c^3 f(10|00)dc-\sqrt{4\pi}\int_0^\infty J_R^0[f(00|11)]c^2dc\ , \tag{74b}$$

$$D_L=-\left[\frac{4\pi}{3}\right]^{1/2}\left\{\int_0^\infty c^3 f(10|11)dc+\int_0^\infty J_R^0[f(00|20)-\sqrt{2}f(00|22)]c^2dc\right\}\ , \tag{74c}$$

$$D_T=-\left[\frac{4\pi}{3}\right]^{1/2}\left\{\int_0^\infty \frac{c^3}{2}[f(11|11)+f(1-1|11)]dc+\int_0^\infty J_R^0\left[f(00|20)+\frac{1}{\sqrt{2}}f(00|22)\right]c^2dc\right\}\ . \tag{74d}$$

For **B** parallel to **E**, the hierarchy of equations (24) and (25) becomes

$$[J^l+\omega(00)]f(lm|s\lambda)-ia\sum_{l'}(l'm\,10|lm)\langle l\|\partial_c^{[1]}\|l'\rangle f(l'm|s\lambda)-i\Omega m f(lm|s\lambda)=X_l(m|s\lambda)\ , \tag{75}$$

where

$$X_l(0|00)=0\ , \tag{76a}$$

$$X_l(m|11)=-\delta_{m0}\omega(11)f(l0|00)-\sum_{l'}(l'01m|lm)f(l'0|00)\langle l\|c^{[1]}\|l'\rangle\ ,\quad m=0\ ,\pm1 \tag{76b}$$

$$X_l(0|20)=-\omega(20)f(l0|00)-\frac{1}{\sqrt{3}}\omega(11)f(l0|11)-\frac{1}{\sqrt{3}}\sum_{l'}\{-(l'010|l0)f(l'0|11)+(l'11-1|l0)[f(l'1|11)+f(l'-1|11)]\}\langle l\|c^{[1]}\|l'\rangle\ , \tag{76c}$$

$$\begin{aligned}X_l(m|22)=&-\delta_{m0}\omega(22)f(l0|00)-(1m\,10|2m)\omega(11)f(lm|11)\\&-\sum_{l'}\{(l'01m|lm)(101m|2m)f(l'0|11)+(l'11m-1|lm)(111m-1|2m)f(l1|11)\\&+(l'-11m+1|lm)(1-11m+1|2m)f(l-1|11)\}\langle l\|c^{[1]}\|l'\rangle\ ,\quad m=0\ ,\pm1\ ,\pm2\ .\end{aligned} \tag{76d}$$

Here we have denoted the rhs by $X_l(m|s\lambda)$ rather than $X_{lm}(s\lambda)$ to emphasize the fact that due to condition (63), the m dependence separates into the different equations in the hierarchy. Equation (75) with (76a) and (75) with (76b) ($m=0$) are exactly the same as Eqs. (I-28) and (I-29) ($m=0$), respectively. That is, to first order in the spatial gradients, the magnetic field has no effect upon transport parallel to the fields. Equation (75) with (76b) ($m=\pm1$) differ from Eq. (I-31) ($m=\pm1$) by the addition of the term $\pm i\Omega f(l\pm1|11)$ on the lhs. Equation (75) with (76c) and (75) with (76d)

($m=0$) differ from Eqs. I(-30) and I(-31) ($m=0$) on the rhs only, as relationship (69) applies instead of (I-14a). As in the **E**-only situation, the solution of (75) with (76d) is required for only $m=0$, in order to determine the quantity $\omega(22)$ in the presence of reactive processes.

Defining the functions

$$F_{lm}^{s\lambda}=i^{l+\lambda}\left[\frac{2^{|m|}(2l+1)(l-|m|)}{4\pi(l+|m|)}\right]^{1/2} f(lm|s\lambda)\ , \tag{77}$$

analogous to (38), condition (69) becomes

$$F_{l,-m}^{s\lambda}=F_{lm}^{*s\lambda}\ , \tag{78}$$

and the hierarchy of Eqs. (75) and (76) can be expressed in the form

$$(J^l-\alpha)F_{lm}^{s\lambda}+a[d_{lm}F_{l-1,m}^{s\lambda}+b_{lm}F_{l+1,m}^{s\lambda}]+i\Omega m F_{lm}^{s\lambda}=H_l(m|s\lambda)\ , \tag{79}$$

where

$$H_l(0|00)=0\ , \tag{80a}$$

$$H_l(0|11)=-WF_l+c\left[\frac{l}{(2l-1)}F_{l-1}+\frac{(l+1)}{(2l+3)}F_{l+1}\right]\ , \tag{80b}$$

$$H_l(-1||11)=H_l(1|11)=c\left[\frac{F_{l-1}}{(2l-1)}-\frac{F_{l+1}}{(2l+3)}\right]\ ,\quad l\ge 1 \tag{80c}$$

$$H_l(0|20)=-\frac{1}{\sqrt{3}}(2D_T+D_L)F_l+\frac{W}{\sqrt{3}}F_l^{(L)}+\frac{c}{\sqrt{3}}\left\{\frac{l}{(2l-1)}[F_{l-1}^{(L)}-(l-1)\mathrm{Re}(F_{l-1}^{(T)})] + \frac{(l+1)}{(2l+3)}[F_{l+1}^{(L)}+(l+2)\mathrm{Re}(F_{l+1}^{(T)})]\right\}\ , \tag{80d}$$

$$H_l(0|22)=\sqrt{\tfrac{2}{3}}(D_T-D_L)F_l-\sqrt{\tfrac{2}{3}}WF_l^{(L)}+\sqrt{\tfrac{2}{3}}c\left\{\frac{l}{(2l-1)}\left[F_{l-1}^{(L)}+\frac{(l-1)}{2}\mathrm{Re}(F_{l-1}^{(T)})\right] + \frac{(l+1)}{(2l+3)}\left[F_{l+1}^{(L)}-\frac{(l+2)}{2}\mathrm{Re}(F_{l+1}^{(T)})\right]\right\}\ , \tag{80e}$$

with $\mathrm{Re}(F_l^{(T)})$ denoting the real part of the function $F_l^{(T)}$, and we have set

$$F_l=F_{l0}^{00}\ ,\quad F_l^{(L)}=F_{l0}^{11}\ ,\quad F_l^{(T)}=F_{l1}^{11}\ . \tag{81}$$

When $m=0$, Eq. (79) is real and condition (78) ensures that the $F_{l0}^{s\lambda}$ are real. For $m\neq 0$, Eq. (79) is complex, and the $F_{lm}^{s\lambda}$ are complex. The functions F_l and $F_l^{(L)}$ are independent of **B** and exactly the same as in I, i.e., as in the **E**-only situation [see Eqs. (I-48), (I-49a) and (I-49b)]. The function $F_l^{(T)}$ is complex and is found by solving the complex equation (79) with (80c) ($m=1$). For vanishing magnetic field the imaginary part of $F_l^{(T)}$ vanishes and Eq. (79) with (80c) reduces to (I-50). Equation (79) with (80c), ($m=-1$) is the complex conjugate of Eq. (79) with (80c), ($m=1$). Equations (79) with (80d) and (79) with (80e) differ from Eqs. (I-48) with (I-49c) and (I-48) with (I-49d), respectively, only on the rhs, where rather than $F_{l\pm1}^{(T)}$, we have $\mathrm{Re}(F_{l\pm1}^{(T)})$. For vanishing magnetic field $F_{l0}^{20}\to F_l^{(2T)}$ and $F_{l0}^{22}\to -F_l^{(2L)}$, where $F_l^{(2T)}$ and $F_l^{(2L)}$ are defined by (I-46d) and (I-46e), respectively [40].

In terms of the functions defined by Eqs. (77) and (81), the transport coefficients can be expressed as

$$\alpha=4\pi\int_0^\infty J_R^0[F_0]c^2dc\ , \tag{82a}$$

$$W=\frac{4\pi}{3}\int_0^\infty c^3F_1dc-4\pi\int_0^\infty J_R^0[F_0^{(L)}]c^2dc\ , \tag{82b}$$

$$D_L=\frac{4\pi}{3}\int_0^\infty c^3F_1^{(L)}dc-\frac{4\pi}{\sqrt{3}}\int_0^\infty J_R^0[F_{00}^{20}+\sqrt{2}F_{00}^{22}]c^2dc\ , \tag{82c}$$

$$D_T=\frac{4\pi}{3}\int_0^\infty c^3\mathrm{Re}(F_1^{(T)})dc - \frac{4\pi}{\sqrt{3}}\int_0^\infty J_R^0\left[F_{00}^{20}-\frac{1}{\sqrt{2}}F_{00}^{22}\right]c^2dc\ . \tag{82d}$$

From Eqs. (19), (65), (77), (78), and (81), the distribution function expressed in Cartesian notation to first order in spatial gradients has the form

$$f(\mathbf{r},\mathbf{c},t)=n\sum_{l=0}^{\infty}F_l(c)P_l(\cos\theta)-\sum_{l=0}^{\infty}F_l^{(L)}(c)P_l(\cos\theta)\partial_z n$$
$$-\sum_{l=1}^{\infty}\mathrm{Re}(F_l^{(T)})P_l^1(\cos\theta)[\cos\phi\partial_x+\sin\phi\partial_y]n+\sum_{l=1}^{\infty}\mathrm{Im}(F_l^{(T)})P_l^1(\cos\theta)[\sin\phi\partial_x-\cos\phi\partial_y]n\ , \quad (83)$$

where $\mathrm{Im}(F_l^{(T)})$ denotes the imaginary part of $F_l^{(T)}$.

In the absence of reactive effects, we require the solution of the first three members of Eq. (79) in order to determine both drift and diffusion. Explicitly, these are

$$J^l F_l + a[d_{l0}F_{l-1}+b_{l0}F_{l+1}]=0\ , \quad (84)$$

$$J^l F_l^{(L)} + a[d_{l0}F_{l-1}^{(L)}+b_{l0}F_{l+1}^{(L)}]=H_l(0|11)\ , \quad (85)$$

$$J^l F_l^{(T)} + a[d_{l1}F_{l-1}^{(T)}+b_{l1}F_{l+1}^{(T)}]-i\Omega F_l^{(T)} = c\left[\frac{F_{l-1}}{2l-1}-\frac{F_{l+1}}{2l+3}\right] . \quad (86)$$

As noted above, Eqs. (84) and (85), which determine W and D_L, respectively, are exactly the same as in the **E**-only situation and will not be considered further here (see, for example, Refs. [17] and [43] where these equations are discussed). Taking the complex conjugate of Eq. (86) and adding and subtracting it in turn from Eq. (86) yields the following pair of coupled equations for the real and imaginary parts of $F_l^{(T)}$:

$$J^l \mathrm{Re}(F_l^{(T)}) + a[d_{l1}\mathrm{Re}(F_{l-1}^{(T)}+b_{l1}\mathrm{Re}(F_{l+1}^{(T)})] + \Omega\,\mathrm{Im}(F_l^{(T)}) = c\left[\frac{F_{l-1}}{2l-1}-\frac{F_{l+1}}{2l+3}\right] , \quad (87a)$$

$$J^l \mathrm{Im}(F_l^{(T)}) + a[d_{l1}\mathrm{Im}(F_{l-1}^{(T)})+b_{l1}\mathrm{Im}(F_{l+1}^{(T)})] - \Omega\,\mathrm{Re}(F_l^{(T)})=0\ . \quad (87b)$$

Making the $l=1$ approximation for a quasi-Lorentz gas equations (87) can be solved to yield

$$\mathrm{Re}(F_1^{(T)})=\frac{\nu}{\nu^2+\Omega^2}cF_0\ , \quad (88a)$$

$$\mathrm{Im}(F_1^{(T)})=\frac{\Omega}{\nu^2+\Omega^2}cF_0\ . \quad (88b)$$

Hence, from Eq. (82d),

$$D_T=\frac{4\pi}{3}\int_0^{\infty}\frac{\nu}{\nu^2+\Omega^2}c^4F_0\,dc\ ; \quad (89)$$

this result is consistent with Refs. [24] and [25] and, for vanishing Ω, reduces to the usual $l=1$ approximation expression for D_T. Note that, as F_0 is independent of Ω, Eq. (89) shows that as Ω increases, D_T will decrease, a well-known fact [44]. For constant collision frequency, Eq. (89) predicts that

$$\frac{D_T}{D_T(\Omega=0)}=\frac{1}{1+\Omega^2/\nu^2}\ . \quad (90)$$

IV. DISCUSSION

For an arbitrary configuration of electric and magnetic fields, the Boltzmann equation for charged-particle transport in neutral gases has been decomposed into a hierarchy of kinetic equations. This decomposition was done in irreducible-tensor formalism and achieved by performing both a spherical-harmonics and a gradient expansion of the charged-particle phase-space distribution function. The gradient expansion of the number density was taken to second order and no limit was set on the number of spherical-harmonic terms. For the special configurations of the magnetic field perpendicular and parallel to the electric field, the hierarchy of equations has been presented in a form suitable for the implementation of numerical solution. We also demonstrated, for these two configurations of the fields, that for the quasi-Lorentz model in the $l=1$ approximation, the present approach gives results in agreement with earlier work [24,25].

In a subsequent paper, for the case of perpendicular fields, we numerically solve the set of equations (42) for electron swarms by further expanding the energy dependence of the functions $F_{lm}^{s\lambda\mu}$ in terms of Sonine polynomials. Some results of these calculations, for real gases, have already been presented, and they agree well with experiment [45]. This particular method, however, is only one of a number of techniques that could be used to effect numerical solution. For the case of parallel fields, it was shown that to first order in spatial gradients, transport parallel to the fields is independent of **B** and that to second order in spatial gradients, the transport equations are only implicitly dependent upon **B** through its effect on the rhs (see Sec. III). The same, however, is not true for transport perpendicular to the fields, where **B** enters explicitly into the matrix of coefficients, and the present formalism presents us with a complex equation to solve in order to determine the diffusion coefficient perpendicular to the fields. Application of numerical solution to Eq. (79) when $m=1$ will then result in a complex-matrix equation to solve. For a general configuration of **E** and **B**, the present formalism will result in complex equations to solve whenever **B** has a component parallel to **E**. This is now discussed in more detail.

Combining expansions (6) and (17), the phase-space distribution function may be expressed as

$$f(\mathbf{r},\mathbf{c},t)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sum_{s=0}^{\infty}\sum_{\lambda=0}^{s}\sum_{\mu=-\lambda}^{\lambda}f(lm|s\lambda\mu)Y_m^{[l]}(\hat{\mathbf{c}}) \times G_\mu^{(s\lambda)}n(\mathbf{r},t)\ . \quad (91)$$

If we insist on $f(\mathbf{r},\mathbf{c},t)$ being real, as it is essentially a probability density in μ space, then taking the complex conjugate of (91) and using the properties

$$Y_m^{(l)}(\hat{\mathbf{c}})=(-1)^{l+m}Y_{-m}^{[l]}(\hat{\mathbf{c}}) \tag{92}$$

and

$$G_\mu^{[s\lambda]}=(-1)^{\lambda+\mu}G_{-\mu}^{(s\lambda)}\ , \tag{93}$$

it follows that

$$f(l-m|s\lambda-\mu)=(-1)^{l+m+\lambda+\mu}f^*(lm|s\lambda\mu) \tag{94}$$

and

$$\omega(s\lambda-\mu)=(-1)^{\lambda+\mu}\omega^*(s\lambda\mu)\ . \tag{95}$$

Relationships (94) and (95) are independent of the configurations of **E** and **B** and also follow from the explicit expression (18) for $f(lm|s\lambda\mu)$, if the $\bar{f}(l|s\lambda\lambda'\lambda''\lambda''')$ are assumed to be real. Condition (95) ensures that the transport coefficients as expressed by Eq. (23) are all real quantities. In terms of the functions defined by Eq. (38), condition (94) is

$$F_{l,-m}^{s\lambda,-\mu}=(-1)^{m+\mu}F_{lm}^{*s\lambda\mu}\ . \tag{96}$$

For any general configuration of fields, the magnetic field may be decomposed into components parallel and perpendicular to the electric field. The component of **B** perpendicular to **E** may be used to define the y axis in the same manner as we have chosen **E** to define the z axis. Thus, without loss of generality, we may consider **B** to lie in the y-z plane for some general configuration of fields. In the case of perpendicular fields, Eqs. (94) and (32) require

$$f(lm|s\lambda\mu)=(-1)^{l+\lambda}f^*(lm|s\lambda\mu)\ , \tag{97}$$

which in turn implies that the $f(lm|s\lambda\mu)$ are real if $l+\lambda$ is even and imaginary if $l+\lambda$ is odd. Transformation (38) then ensures that the functions $F_{lm}^{s\lambda\mu}$ are all real, as must be the case if both (96) and (41) are to be satisfied. In general, however, this will not be the case, as the $f(lm|s\lambda\mu)$ are complex and application of transformation (38) will lead to complex $F_{lm}^{s\lambda\mu}$. Only when both m and μ are zero will the $F_{lm}^{s\lambda\mu}$ be real. For the case of parallel fields, we have already seen that transport perpendicular to the fields leads to a complex equation to solve. It follows from Eqs. (94), (95), and (24) that the equation for $f(l-m|s\lambda-\mu)$ is the complex conjugate of the equation for $f(lm|s\lambda\mu)$ and that in order to determine all the $\omega(s\lambda\mu)$ quantities, it is both sufficient and necessary to solve only the positive μ members of the hierarchy (24). To second order in spatial gradients, this will require the solution of seven complex equations, with both l and m index dependence. To first order in spatial gradients, however, it is sufficient to solve the lowest three of these in order to determine both drift and diffusion when reactive processes are absent or insignificant.

For **B** lying in the y-z plane, Eq. (24) written in terms of the functions defined by Eq. (38) is

$$(J^l-\alpha)F_{lm}^{s\lambda\mu}+a[d_{lm}F_{l-1,m}^{s\lambda\mu}+b_{lm}F_{l+1,m}^{s\lambda\mu}] + \frac{\Omega\sin(\psi)}{2}[g_{lm}F_{l,m+1}^{s\lambda\mu}-g_{l,-m}F_{l,m-1}^{s\lambda\mu}] - i\Omega\cos(\psi)mF_{lm}^{s\lambda\mu}=H_{lm}(s\lambda\mu)\ , \tag{98}$$

where ψ denotes the angle between **B** and **E** and

$$H_{lm}(s\lambda\mu)=i^{l+\lambda}\left[\frac{2^{|\mu|}(2l+1)(l-|m|)!}{4\pi(l+|m|)!}\right]^{1/2}X_{lm}(s\lambda\mu)\ . \tag{99}$$

In deriving explicit expressions for the $H_{lm}(s\lambda\mu)$ from Eqs. (25), one should make use of relationships (95) and (96) after application of (38). Numerically speaking, the major effect of the complex nature of Eq. (98) will be the doubling of the dimensions of arrays. Apart from this, however, application of numerical solution, although tedious, should be straightforward.

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[39] The velocity-dependent operator $L_\mu^{[1]}$ used in the present work is related to the operator $W_\mu^{[1]}$ used in Ref. [38] by $W_\mu^{[1]}=(1/\sqrt{2})L_\mu^{[1]}$.
[40] Note that there are a number of misprints in I (Ref. [17], R. E. Robson and K. F. Ness): In Eq. (11),

$$G_m^{(2l)}=(\nabla^{(1)},\nabla^{(1)})_m^{(l)} ,$$

not $(\nabla^{(1)})_m^{(l)}$ as shown. In Eqs. (46d) and (46e), i should be raised to the power l, not $l+1$ as shown. The last term in Eq. (49c) should be multiplied by ω_1. The last term on the lhs of Eq. (50) should be

$$\frac{l-1}{2l-1}a\left[\frac{d}{dc}-\frac{l-1}{c}\right]F_{l-1}^{(T)} .$$

[41] K. F. Ness (unpublished).
[42] The notation used in Ref. [24] is related to that used in the present work by

$$\omega=-\Omega ,$$

$$D_{\omega x}=D_x , \quad D=D_y , \quad D_{\omega z}=D_z , \quad D_{\omega zx}=D_h ,$$

$$f_0^*=F_{00} , \quad a_1 f_0^*=F_{00}^{(T)} , \quad b_1 f_0^*=-F_{00}^{(L)} .$$

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