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Domain-growth scaling in systems with long-range interactions

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The growth kinetics of a system quenched into the ordered phase from high temperatures is considered for systems with power-law interactions of the form $1/r^{d+\sigma}$, with $0<\sigma<2$. For $\sigma>1$, the characteristic scale $L(t)$, which describes the growth of order at late times, is predicted to obey the conventional Lifshitz-Slyozov and Lifshitz-Cahn-Allen laws, $L(t)\sim t^{1/3}$ and $t^{1/2}$ for conserved and nonconserved scalar order parameters, respectively. For $\sigma<1$, the results $L(t)\sim t^{1/(2+\sigma)}$ and $t^{1/(1+\sigma)}$, respectively, are obtained. For a vector order parameter, we find $L(t)\sim t^{1/(2+\sigma)}$ and $t^{1/\sigma}$ for conserved and nonconserved fields, respectively, for all $\sigma<2$.

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I. INTRODUCTION

When a system described by a scalar order parameter is quenched from the homogeneous high-temperature phase into the two-phase region, domains of the two pure phases are formed and coarsen with time [1]. The late stages of domain growth are well described by a scaling phenomenology [2]: the equal-time two-point correlation function has the form

$$C(\mathbf{r},t)\equiv\langle\phi(\mathbf{x},t)\phi(\mathbf{x}+\mathbf{r},t)\rangle=f(r/L(t)) , \tag{1}$$

where ϕ is the order-parameter field, $L(t)$ is a characteristic length scale ("domain scale") at time t after the quench, and $f(x)$ is a scaling function. The angled brackets in (1) indicate an average over initial conditions and thermal noise.

For systems with purely short-ranged interactions, the form of the growth law for $L(t)$ is well understood: $L(t)\sim t^{1/3}$ for a conserved order parameter [3–5], while $L(t)\sim t^{1/2}$ for a nonconserved order parameter [6,7]. For nonconserved fields, the $t^{1/2}$ growth follows simply from the observation [7] that the domain-wall velocity is proportional to the local curvature. A typical velocity is of order dL/dt while a typical curvature is of order $1/L$. Equating these gives $L(t)\sim t^{1/2}$. For conserved fields, the $t^{1/3}$ growth law can also be derived from intuitive arguments [3]. For this case, however, a renormalization-group (RG) approach, based on the assumption of scaling, is available [4,5] and will be employed.

In this paper we generalize these domain-growth laws to systems with long-ranged interactions, falling off with distance as $r^{-(d+\sigma)}$, where d is the spatial dimension of the system. A suitable coarse-grained Hamiltonian functional is

$$H[\phi]=H_{\rm LR}[\phi]+H_{\rm SR}[\phi] , \tag{2}$$

where

$$H_{\rm LR}[\phi]=(J_{\rm LR}/2)\times\int d^dx\int d^dx'[\phi(\mathbf{x})-\phi(\mathbf{x}')]^2/|\mathbf{x}-\mathbf{x}'|^{d+\sigma} \tag{3}$$

is the long-range part of the Hamiltonian. The short-range part is typically taken to have the Ginzburg-Landau form

$$H_{\rm SR}[\phi]=\int d^dx[(\nabla\phi)^2/2+V(\phi)] , \tag{4}$$

where $V(\phi)$ has a local maximum at $\phi=0$ and global minima at $\phi=\pm\phi_0$. The precise form of $H_{\rm SR}$ is, however, unimportant in what follows.

For a conserved order parameter, the RG treatment, discussed in detail below, leads to the prediction [4,5] $z=d+2-y$ for the "dynamic exponent at the $T=0$ fixed point," where y is the scaling dimension of the Hamiltonian (or minus the scaling dimension of the temperature T) at the $T=0$ fixed point that controls phase ordering. Invariance of the domain morphology under simultaneous rescaling of length and time, $L\to bL$, $t\to b^zt$, implies the growth law $L(t)\sim t^{1/z}$. The growth exponent is simply $1/z$, so determining the growth law reduces to finding y. For short-range interactions, $y=y_{\rm SR}=d-1$ follows from elementary arguments. It will be shown that the long-range part of the Hamiltonian scales with exponent $y_{\rm LR}=d-\sigma$ [essentially, this is just power counting on Eq. (3)]. Thus *long-range interactions are irrelevant for* $\sigma>1$, and $z=z_{\rm SR}=3$. For $\sigma<1$, however, $J_{\rm LR}$ is relevant and $z=z_{\rm LR}=2+\sigma$. We conclude that $L(t)\sim t^{1/3}$ for $\sigma>1$, and $\sim t^{1/(2+\sigma)}$ for $\sigma<1$.

For nonconserved fields, no exact RG treatment is available [5]. However, the growth law may be simply derived as follows. The wall energies in a "domain volume" L^d, associated with the short- and long-range parts of the interaction, scale as $L^{y_{\rm SR}}$ and $L^{y_{\rm LR}}$, i.e., as L^{d-1} (which is just the wall area) and as $L^{d-\sigma}$, respectively. The force per unit area acting on the domain walls scales as the energy density, giving a force per unit area of order L^{-1} and $L^{-\sigma}$ for the short- and long-range parts of the interaction. Again, the long-range interactions are only relevant for $\sigma<1$. For $\sigma>1$, the wall curvature provides the dominant driving force. Equating the driving force to the typical wall velocity dL/dt gives $L(t)\sim t^{1/2}$ for $\sigma>1$ and $L(t)\sim t^{1/(1+\sigma)}$ for $\sigma<1$.

The paper is organized as follows. Section II contains a brief outline of the RG treatment for the conserved case. Further details can be found in Refs. [4,5]. It is noted that the RG approach gives (up to an overall constant) the *amplitude* in the expression for $L(t)$ as well as the growth exponent. The T dependence of the amplitude for $T\to T_C$ (where T_C is the critical temperature) is particularly relevant, in view of the requirement that it be consistent with conventional static and dynamic scaling

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near T_C. All the results derived below satisfy this check. The nonconserved case is discussed in Sec. III. It is emphasized that, for long-range interactions, the growth law cannot be derived from a naive dimensional analysis of the equation of motion. The corresponding results for a vector order parameter, both conserved and nonconserved, are given in Sec. IV. The paper concludes with a summary of the results.

II. CONSERVED SCALAR ORDER PARAMETER

We start from the equation of motion

$$d\phi_{\mathbf{k}}/dt = -\lambda|\mathbf{k}|^{\mu}(\delta H[\phi]/\delta\phi_{-\mathbf{k}}) , \quad (5)$$

where $\phi_{\mathbf{k}}$ is a Fourier component of the order-parameter field, and $\mu=2$ for a conserved order parameter. Since keeping μ arbitrary presents no extra difficulties, we will do so and set $\mu=2$ in final results. Note that, for $0\le\mu\le 2$, Eq. (5) interpolates between (standard) conserved [8] and nonconserved fields (although the field is, of course, conserved for any $\mu>0$). We have omitted a Langevin noise term in (5), since it can be shown to be irrelevant to the asymptotic scaling behavior [4,5]: the late stage scaling is controlled by a $T=0$ fixed point. After dividing through by λk^{μ}, we obtain

$$(1/\lambda k^{\mu})(d\phi_{\mathbf{k}}/dt) = -(\delta H[\phi]/\delta\phi_{-\mathbf{k}}) . \quad (6)$$

The utility of RG methods is suggested by the empirically observed scaling form (1) for the two-point correlation function, and the observation that the whole "domain morphology" seems to scale with $L(t)$.

In the RG approach [4,5], we imagine eliminating modes $\phi_{\mathbf{k}}$ with $\Lambda/b<k<\Lambda$, and then rescaling momenta, times, and fields according to $\mathbf{k}=\mathbf{k}'/b$, $t=b^z t'$, and $\phi_{\mathbf{k}'/b}(b^z t')=b^{d/2}\phi'_{\mathbf{k}'}(t')$. In addition, the coupling constants defining the coarse-grained fixed-point Hamiltonian pick up a factor b^y from the RG transformation. Dividing through the coarse-grained equation of motion by a factor $b^{y-d/2}$ to reinstate the right-hand side to its previous form, and suppressing the primes, we obtain

$$b^{d-y+\mu-z}(1/\lambda k^{\mu})(d\phi_{\mathbf{k}}/dt) = -(\delta H[\phi]/\delta\phi_{-\mathbf{k}}) . \quad (7)$$

In Eq. (7) we have omitted additional terms, generated by the RG transformation, which do not have the same form as the terms in (6). In particular, one expects a term of the form $(1/\Gamma)d\phi_k/dt$ to be generated. However, there can be no additional terms of the form $k^{-\mu}d\phi_{\mathbf{k}}/dt$ since this is singular at $\mathbf{k}=0$ and such terms cannot be generated by the elimination of "hard" modes. It follows that, just as in critical dynamics [8], the renormalization of λ is trivial, namely,

$$(1/\lambda)' = b^{d-y+\mu-z}(1/\lambda) . \quad (8)$$

Provided $1/\lambda$ is nonzero at the fixed point, therefore, we have immediately

$$z = d+\mu-y . \quad (9)$$

We note that for μ sufficiently small, $1/\lambda$ will iterate to zero at the fixed point of the nonconserved system. This will happen for $\mu<z_{\mathrm{nc}}+y-d$, where the subscript nc indicates the nonconserved fixed point. Then the conservation law will be irrelevant [5] and z will be given the result for the nonconserved system (see below) instead of (9).

It remains to determine the exponent y, which is the scaling dimension of the Hamiltonian at the $T=0$ fixed point. For short-range interactions, one has simply, for a scalar order parameter,

$$y_{\mathrm{SR}} = d-1 , \quad (10)$$

since at $T=0$ the energy cost of reversing a coarse-grained local variable (i.e., creating a domain of reversed spins) in the ground state scales as the surface area. The effect of coarse graining on the long-range part of the Hamiltonian is simply obtained either by power counting in real space or by writing H_{LR} in terms of the Fourier components of the field:

$$H_{\mathrm{LR}}[\phi] = J_{\mathrm{LR}}\sum_{\mathbf{k}} k^{\sigma}\phi_{\mathbf{k}}\phi_{-\mathbf{k}} . \quad (11)$$

Under the RG length and field rescalings, H_{LR} retains the same form but with new coupling constant

$$J'_{\mathrm{LR}} = b^{d-\sigma}J_{\mathrm{LR}} , \quad (12)$$

i.e.,

$$y_{\mathrm{LR}} = d-\sigma . \quad (13)$$

The point here is that, as in the renormalization of $1/\lambda$, there are no nontrivial contributions from coarse graining because J_{LR} multiplies a term singular in k as $k\to 0$. The analogous argument at the critical fixed point [9] is responsible for the result $\eta_{\mathrm{LR}}=2-\sigma$; the extension to the $T=0$ fixed point has been discussed previously in another context [10].

Comparing (10) and (13) it follows that J_{LR} is irrelevant at the short-range fixed point for $\sigma>1$, and relevant for $\sigma<1$. In the latter case, the long-range interactions drive the system to a new long-range fixed point at which the whole Hamiltonian scales with exponent y_{LR}. This follows from the exact recursion relation (12): provided J_{LR} is nonzero at the fixed point, Eq. (12) holds.

Inserting these results into (9) gives the final result

$$z = \begin{cases} \mu+1 , & \sigma>1 \\ \mu+\sigma , & \sigma<1 . \end{cases} \quad (14)$$

For the case $\mu=2$, corresponding to standard diffusive transport, one has $z=3$ and $2+\sigma$ for $\sigma>1$ and $\sigma<1$, corresponding to growth exponents of $\frac{1}{3}$ and $1/(2+\sigma)$, respectively.

The above approach may readily be extended to general temperatures $T<T_C$. Thermal fluctuations on scales up to the equilibrium correlation length ξ lead to a reduction of the coarse-grained Hamiltonian, on scales larger than ξ, by a factor of the surface tension Σ, and of the coarse-grained field variable by a factor of the equilibrium order parameter M. Thermal fluctuations do not renormalize the transport coefficient λ. The asymptotic behavior $[L(t)\gg\xi]$ for general T is obtained by inserting these factors into the $T=0$ Langevin equation (5). Absorbing all the factors into an effective λ yields $\lambda(L)\to(\Sigma/M^2)\lambda$ on scales $L\gg\xi$. The "domain scale" $L(t)$ is thus given by $L(t)\simeq[\lambda(\infty)t]^{1/z}$, or, specializing

to $\mu=2$,

$$L(t)\simeq\begin{cases}(\lambda\Sigma t/M^2)^{1/3}\ , & \sigma>1\\ (\lambda\Sigma_{\rm LR}t/M^2)^{1/(2+\sigma)}\ , & \sigma<1\ .\end{cases} \tag{15}$$

For the case $\sigma<1$ we have to define what we mean by the "surface tension" $\Sigma_{\rm LR}$. For short-range interactions Σ is defined in terms of the free energy cost ΔF associated with a domain wall of area A, separating domains of opposite magnetization in a system of length $L\gg\xi$, by the relation $\Delta F=\Sigma A$. For long-range interactions, with $\sigma<1$, the free energy cost is no longer independent of L for $L\gg\xi$. Instead, one has $\Delta F(L)\sim\Sigma_{\rm LR}L^{d-\sigma}$ for a system of linear dimension L, consistent with (12) and the interpretation of $\Delta F(L)$ as a "coarse-grained coupling at scale L." The same line of reasoning gives, on setting $\Delta F(\xi)=O(1)$, the result $\Sigma_{\rm LR}\sim\xi^{-(d-\sigma)}$ as the generalization to $\sigma<1$ of the usual Widom-Josephson scaling law $\Sigma\sim\xi^{-(d-1)}$.

We can now check that (15) is consistent with *critical* dynamic scaling as $T\to T_C$. Using the above scaling forms for Σ and $\Sigma_{\rm LR}$, together with $M^2\sim\xi^{-(d-2+\eta)}$, gives $L(t)\sim\xi(t/\xi^{z_c})^{1/3}$ for $\sigma>1$, and $L(t)\sim\xi(t/\xi^{z_c})^{1/(2+\sigma)}$ for $\sigma<1$, where $z_c=4-\eta$ is the *critical* dynamic exponent for a conserved order parameter [8]. These forms for $L(t)$ are exactly as expected from critical scaling.

We conclude this section by recalling that, for general μ, the result (14) holds only when (14) predicts a *z larger* than the corresponding result for a *nonconserved* field. Otherwise, the conservation law is technically irrelevant and the nonconserved results are recovered asymptotically [5].

III. NONCONSERVED SCALAR ORDER PARAMETER

The essence of the argument for the nonconserved case was given in the Introduction. In this section we give a little more detail, and generalize the results to all $T<T_C$.

The equation of motion for nonconserved fields is the $\mu=0$ version of (6). In real space it reads

$$(1/\Gamma)(\partial\phi/\partial t)=-(\delta H/\delta\phi)\ . \tag{16}$$

While the thermal noise has been omitted from (16), its effects can be included, as in the preceding section, by coarse graining to a scale larger than the correlation length. At this scale, the temperature is effectively zero, the Hamiltonian acquires a "surface tension" factor Σ, and the field a factor M, the equilibrium order parameter. In contrast to the conserved field, however, the kinetic coefficient Γ also acquires a temperature dependence after coarse graining [5,8].

Following Allen and Cahn [7], we can estimate the left-hand side of (16) in the vicinity of a domain wall by introducing a coordinate g normal to the wall. Then

$$(\partial\phi/\partial t)_g=-(\partial\phi/\partial g)_t(\partial g/\partial t)_\phi\ . \tag{17}$$

Now $(\partial g/\partial t)_\phi$ is just the normal velocity v of the wall. Also $(\partial\phi/\partial g)_t\simeq M/\xi$, the change in ϕ across the interface divided by its width. Thus we estimate

$$(\partial\phi/\partial t)_g\sim(M/\xi)v\ . \tag{18}$$

The right-hand side of (16) can be estimated as the excess energy density divided by the order parameter. The excess energy in volume $L(t)^d$ scales as $\Sigma L(t)^y$, giving contributions of order $\Sigma L(t)^{d-1}$ and $\Sigma_{\rm LR}L(t)^{d-\sigma}$ associated with the short- and long-range parts of the Hamiltonian, respectively. The short- and long-range parts dominate in the scaling limit for $\sigma>1$ and $\sigma<1$, respectively, so the right-hand side of (16) is estimated as (after coarse graining to include thermal fluctuations)

$$-\frac{\delta H}{\delta\phi}\sim\begin{cases}\Sigma/ML\ , & \sigma>1\\ \Sigma_{\rm LR}/ML^\sigma\ , & \sigma<1\ .\end{cases} \tag{19}$$

Using the estimates (18) and (19) in the equation of motion (16) gives

$$\frac{dL}{dt}\sim v\sim\begin{cases}\Gamma\Sigma\xi/M^2L\ , & \sigma>1\\ \Gamma\Sigma_{\rm LR}\xi/M^2L^\sigma\ , & \sigma<1\ ,\end{cases} \tag{20}$$

and finally

$$L(t)\sim\begin{cases}(\Gamma\Sigma\xi t/M^2)^{1/2}\ , & \sigma>1\\ (\Gamma\Sigma_{\rm LR}\xi t/M^2)^{1/(1+\sigma)}\ , & \sigma<1\ .\end{cases} \tag{21}$$

For $T\to T_C$ we can check our results against conventional critical scaling. Using the scaling forms given in Sec. II, and also [8] $\Gamma\sim\xi^{2-\eta-z_c}$, we find $L(t)\sim\xi(t/\xi^{z_c})^{1/z}$ in both regimes, consistent with critical scaling, with $z=2$ for $\sigma>1$ and $z=1+\sigma$ for $\sigma<1$.

Equation (21) contains our main results for nonconserved scalar fields. We note that these results could not have been obtained from a naive dimensional analysis of (16), which reads in Fourier space (up to constants) $d\phi_{\bf k}/dt=-\Gamma(k^2+J_{\rm LR}k^\sigma)\phi_{\bf k}$, plus nonlinear terms. Naive power counting, ignoring the nonlinear terms, would predict $L(t)\sim t^{1/\sigma}$ for $\sigma<2$, and a crossover value of σ equal to 2, both of which are incorrect for scalar fields. In this sense, the fact that just such a naive analysis gives the *correct* growth law for short-range interactions is both fortuitous and misleading.

In order to make the growth laws (21) more physically transparent, it is instructive to consider the collapse of a single spherical domain of, say, negative order parameter, in an infinite sea of positive order parameter. We take the initial radius $R(0)$ of the domain to be very large compared to the width ξ of the domain wall. Then the growth laws (21) suggest that the time for the domain to shrink to zero should scale as $[R(0)]^2$ and $[R(0)]^{1+\sigma}$ for $\sigma>1$ and $\sigma<1$, respectively. These results are verified in the Appendix.

IV. VECTOR FIELDS

For a conserved order parameter, the growth law can be obtained by simply replacing (10) by the equivalent result for vector fields, namely,

$$y_{\rm SR}=d-2\ . \tag{22}$$

This standard result follows from the fact that the energy of an imposed rotation over a scale L is distributed uniformly over this length scale (rather than concentrated in a domain wall), so that the associated energy density is given by the "naive" L^{-2} coming from the gradient term in H_{SR} [11], giving an overall energy scaling as L^{d-2}. On the other hand, y_{LR} is still given by (13). For vector fields, therefore, long-range interactions are relevant for all $\sigma<2$. Substituting (22) in (9), with $\mu=2$, gives the result

$$z=\begin{cases}4\ , & \sigma>2\\ 2+\sigma\ , & \sigma<2\ .\end{cases} \tag{23}$$

Thermal fluctuations can be included much as for scalar fields, scaling H by the "spin-wave stiffness" ρ_s (using magnetic language) and ϕ by M. The result is

$$L(t)\sim\begin{cases}(\lambda\rho_s t/M^2)^{1/4}\ , & \sigma>2\\ (\lambda\rho_{LR} t/M^2)^{1/(2+\sigma)}\ , & \sigma<2\ ,\end{cases} \tag{24}$$

where ρ_{LR}, scaling as $\xi^{-(d-\sigma)}$, is the long-range equivalent of ρ_s. One can verify, using $\rho_s\sim\xi^{-(d-2)}$, that conventional critical scaling is recovered in both cases for $T\to T_C$.

Nonconserved vector fields are more problematical. Even for purely short-ranged interactions, the growth laws have not been determined unambiguously. Numerical simulations for vector systems [12], and experiments on related nematic liquid crystals [13] (described by a tensor order parameter), are broadly consistent with $t^{1/2}$ growth, as predicted by a naive dimensional analysis of the equation of motion. For the special case $n=d=2$, however, Yurke *et al.* [14] have argued for the slightly slower growth law $L(t)\sim(t/\ln t)^{1/2}$. Long-range interactions are relevant for $\sigma<2$. The absence of sharp interfaces for vector fields suggests the growth law $L(t)\sim t^{1/\sigma}$ for $\sigma<2$ (with possible logarithmic corrections for $n=2$), consistent with naive dimensional analysis of the equation of motion.

V. SUMMARY

The effect of long-range interactions on domain growth in phase ordering systems has been determined. Naive dimensional analysis of the equation of motion would suggest that, if the interactions fall off with distance as $1/r^{d+\sigma}$, long-range interactions are relevant for all $\sigma<2$. For scalar fields, however, we find that the long-range part of the interaction is relevant only for $\sigma<1$. This prediction is confirmed (see Appendix) by an explicit calculation for the collapse of a single spherical domain. For vector fields, the long-range interactions are relevant for all $\sigma<2$.

For conserved scalar fields we obtain the growth laws $L(t)\sim t^{1/3}$ for $\sigma>1$ and $L(t)\sim t^{1/(2+\sigma)}$ for $\sigma<1$, while for nonconserved scalar fields, $L(t)\sim t^{1/2}$ and $t^{1/(1+\sigma)}$, respectively. For vector fields with $\sigma<2$ we predict $L(t)\sim t^{1/(2+\sigma)}$ and $t^{1/\sigma}$ for conserved and nonconserved fields, respectively.

Hayakawa, Rácz, and Tsuzuki [15] have recently attempted to calculate the scaling functions for the two-point correlation functions of nonconserved systems with long-range interactions, using the singular perturbation theory approach of Kawasaki, Yalabik, and Gunton (KYG) [16]. This starts from the equation of motion (16), with H given by (2)–(4), i.e.,

$$\partial\phi/\partial t=\nabla^2\phi-(dV/d\phi)-(\delta H_{LR}/\delta\phi)\ . \tag{25}$$

Although originally presented as an approximate diagrammatic technique, the KYG method is tantamount to solving the *linear* equation obtained by omitting the $-dV/d\phi$ term from (25), to yield a "linear" solution ϕ_0, and then setting $\phi=\mathrm{sgn}\phi_0$ (or $\boldsymbol{\phi}=\boldsymbol{\phi}_0/|\boldsymbol{\phi}_0|$ for vector fields). This latter step takes into account, in an approximate way, the effect of the potential $V(\phi)$, which constrains the field to have unit length almost everywhere at late times.

It is clear from dimensional analysis of (25) that the KYG approach gives $L(t)\sim t^{1/\sigma}$ for $\sigma<2$, which is qualitatively incorrect for scalar fields. For purely short-range interactions, the KYG method gives the correct $t^{1/2}$ growth law for scalar systems as a result of two cancelling errors: the detailed discussion of Sec. III shows that both sides of Eq. (16) actually scale as $1/L(t)$ [see Eq. (20)], not as $1/L(t)^2$. The correct scaling is implicit in the approaches of Allen and Cahn [7] and of Ohta, Jasnow, and Kawasaki [17], which focus directly on the motion of domain walls. Neither of these approaches, however, readily generalizes to long-range interactions.

In conclusion, therefore, we note that systems with long-range interactions, as well as being of interest in their own right, provide a useful test bed for approximate theories of the correlation scaling function. Any candidate theory for the correlation function should at least be able to reproduce the growth laws derived here for nonconserved scalar fields.

APPENDIX: COLLAPSE OF A SPHERICAL DOMAIN

We start from the equation of motion (16), and seek a solution with radial symmetry. Inserting the explicit form for H, and choosing $d=3$ for simplicity, gives (absorbing Γ into the timescale)

$$\frac{\partial\phi}{\partial t}=\frac{\partial^2\phi}{\partial r^2}+\frac{2}{r}\frac{\partial\phi}{\partial r}-V'(\phi)+J_{LR}\int d^3r'\,\frac{[\phi(r')-\phi(r)]}{|\mathbf{r}'-\mathbf{r}|^{3+\sigma}}\ . \tag{A1}$$

Carrying out the angular integral, the final term becomes

$$\frac{2\pi J_{LR}}{1+\sigma}\frac{1}{r}\int_0^\infty r'dr'\left\{\frac{1}{|r'-r|^{1+\sigma}}-\frac{1}{|r'+r|^{1+\sigma}}\right\}\times[\phi(r')-\phi(r)]\ . \tag{A2}$$

We seek a solution of (A1) in the form $\phi=f(r-R(t))$, representing a shrinking domain of radius $R(t)$, with the domain wall profile function $f(x)$ satisfying $f(x)\to-1$ for $x\ll-\xi$ and $f(x)\to1$ for $x\gg\xi$ [we have taken the minima of the potential $V(\phi)$ to be at $\phi=\pm1$]. Putting this form into (A1) yields

$$0=f''+[(dR/dt)+(2/r)]f'-V'(f)+\frac{2\pi J_{LR}}{1+\sigma}\frac{1}{r}\int_0^\infty r'dr'\left\{\frac{1}{|r'-r|^{1+\sigma}}-\frac{1}{|r'+r|^{1+\sigma}}\right\}\times[f(r'-R)-f(r-R)]\ , \tag{A3}$$

where primes on f indicate derivatives, and the argument of f is $r-R$ where not explicitly given. From now on we will not keep track of the constants in front of the final term.

To derive an equation for $R(t)$, we multiply the equation through by $f'(r-R)$ and integrate through the interface, using the boundary conditions that f' vanishes far from the interface, and that $V(f)$ and f^2 have the same values on both sides of the interface. In addition, since f' is zero except within a region of size ξ near the wall, we can set $r=R$ throughout. Noting also that the integrals of f' and f'^2 across the interface are just constants, we obtain, up to constants,

$$0=\frac{dR}{dt}+\frac{2}{R}+\frac{1}{R}\int_0^\infty r'dr'\left\{\frac{1}{|r'-R|^{1+\sigma}}-\frac{1}{|r'+R|^{1+\sigma}}\right\}\times\mathrm{sgn}(r'-R)\ , \quad \text{(A4)}$$

where, in the final term, we have used the fact that the profile is essentially a step function to replace $f(r'-R)$ by $\mathrm{sgn}(r'-R)$. The integral in (A4) should therefore be understood as a "principal part" integral: using the full profile function f would cancel the singularity at $r'=R$ (at least for $\sigma<1$; for $\sigma>1$, we need an explicit short-distance cutoff on the long-range interaction).

The final step is to evaluate the integral in (A4). The variable change $r'=Rx$ gives, for $\sigma<1$ (where the integral converges, in a principal part sense) a constant times $R^{1-\sigma}$. It is not difficult to show that the constant is positive. Therefore the equation of motion for R takes the form

$$dR/dt=-(2/R)-(J_{\mathrm{LR}}/R^\sigma)\ ,\quad \sigma<1\ . \quad \text{(A5)}$$

For large R the long-range part dominates. Integrating (A5) gives the collapse time for an initial large radius $R(0)$ as $t\sim[R(0)]^{1+\sigma}$. Here "large" $R(0)$ means large compared to the crossover radius $R^*\sim J_{\mathrm{LR}}^{-1/(1-\sigma)}$ obtained by equating the two terms on the right in (A5).

For $\sigma>1$, the integral in (A4) remains finite in the limit $R\to\infty$. The singularity at $r'=R$ is handled by cutting out the interval $(R-a,R+a)$ from the integration range, where a is a short-distance cutoff. The resulting integral has a limit of order $a^{1-\sigma}$ for $R\to\infty$. The result is that the long-range interaction renormalizes the curvature term in the driving force, but is otherwise irrelevant for large R: the collapse time scales as $t\sim[R(0)]^2$, just as for short-range interactions.

Note added in proof. J. Cardy (private communication) has pointed out that in principle the system size L_s can occur in the domain-growth law as a second scaling variable. A more general scaling form for $L(t)$ than that given here would be $L(t)=t^{1/z}f(t^{1/z}/L_s)$, valid for $t\to\infty$, $L_s\to\infty$, with $t^{1/z}/L_s$ fixed. In this paper we have implicitly assumed a well-defined thermodynamic limit, i.e., that $f(0)$ is finite and nonzero. Cardy proposes a scenario where $f(0)=0$, and suggests [J. L. Cardy and B. Lee (unpublished)] that this possibility is realized for $\sigma<1$ in the nonconserved one-dimensional scalar theory, with $z=1+\sigma$ [as in (21)] and $f(x)\sim x^\alpha$ for $x\to 0$.

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