Higher-dimensional targeting

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(Received 31 August 1992)

This paper describes a procedure to steer rapidly successive iterates of an initial condition on a chaotic attractor to a small target region about any prespecified point on the attractor using only small controlling perturbations. Such a procedure is called "targeting." Previous work on targeting for chaotic attractors has been in the context of one- and two-dimensional maps. Here it is shown that targeting can also be done in higher-dimensional cases. The method is demonstrated with a mechanical system described by a four-dimensional mapping whose attractor has two positive Lyapunov exponents and a Lyapunov dimension of 2.8. The target is reached by making very small successive changes in a single control parameter. In one typical case, 35 iterates on average are required to reach a target region of diameter 10^{-4}, as compared to roughly 10^{11} iterates without the use of the targeting procedure.

PACS number(s): 05.45.+b

I. INTRODUCTION

A chaotic process has sensitive dependence on initial conditions which prevents long-term predictions of the state of the system. Chaotic dynamics typically exhibit highly irregular behavior and are represented mathematically by so-called "strange attractors" whose small-scale structure is very complex.

Chaotic behavior is manifest in physical processes like turbulent fluid flow [1], the fluttering of a driven flexible beam [2, 3], and the irregular oscillations of a forced damped nonlinear pendulum. Despite the complexities of chaotic behavior, the sensitive dependence on initial conditions can be exploited to direct the system to some desired final state (like a saddle periodic orbit embedded in the attractor) by a carefully chosen sequence of small perturbations to some control parameter. We call this *targeting*.

Ott, Grebogi, and Yorke [4] introduced the idea that control of chaos could in some cases be attained by feedback stabilization of one of the infinite number of unstable periodic orbits that naturally occur in a chaotic attractor. Their method has been used to control a driven, flexible beam about a saddle fixed point in a laboratory experiment whose dynamical behavior was well approximated by a two-dimensional map [5].

Romeiras *et al.* [6] recently extended these ideas and applied them to stabilize saddle periodic points in an attractor in four dimensions arising from a map that describes a kicked double rotor. They showed that control can be achieved (perhaps after several thousand iterations) by using only one control parameter, even when the attractor has two positive Lyapunov exponents (the Lyapunov dimension [7] of the attractor is 2.8).

In this paper we discuss the targeting type of control problem for a chaotic system. We assume that we are given some initial condition on the attractor, and we wish to rapidly direct the resulting trajectory to a small region about some specified point on the chaotic attractor. Because of the inherent exponential sensitivity of chaotic time evolutions to perturbations, one expects that this can be accomplished using only small controlling adjustments of one or more available system parameters.

This was demonstrated theoretically and in numerical experiments for the case of a two-dimensional map by Shinbrot *et al.* [8] and also in a laboratory experiment for which the dynamics were approximately describable by a one-dimensional map [9]. The object of our paper is to present an efficient method of targeting and to demonstrate its applicability in systems of higher dimensionality than previously considered.

As an example, we consider the double-rotor map [10], which is four dimensional. Using our targeting procedure, we find that typical points on the double-rotor attractor can be steered to within 10^{-4} of a given target point on the attractor in an average of 35 iterations. Although there is more than one positive Lyapunov exponent, the control is achieved by making successive changes to a single parameter (here the strength of the kick).

Because the dimension of the double-rotor attractor (for the set of parameters chosen in [6]) is about 2.8, the average distance between nearest neighbors in a subset of N points on the attractor [11] scales as $N^{-1/2.8}$. This implies that about 10^{11} iterations of the map are required on the average to come within 10^{-4} of the target without the control. Since the control procedure described below can steer the initial condition to within 10^{-4} of the target in about 10^2 steps, the method gets to the target about 10^9 times faster than the uncontrolled chaotic process.

For specificity, the description of the numerical method in Sec. III relies on the existence of two positive Lyapunov exponents, but it can be adapted in principle to maps where the attractor has any number of positive exponents and/or is higher dimensional. All calculations described in this paper required only a few minutes on a desktop work station.

Brief descriptions of the double-rotor map and relevant background material on Lyapunov exponents are given in Sec. II. The basic control procedure is outlined in Sec. III. Some refinements to the method, which make the control faster, are described in Sec. IV. Conclusions and results are stated in Sec. V.

II. PRELIMINARIES

A. The double-rotor map

The double rotor is a mapping that describes the effect of a sequence of impulse kicks on two thin, massless rods connected as illustrated in Fig. 1. A derivation of the map is given in [10]; a slightly different version, which we use here, is described in [6].

The first rod, of length ℓ_1, pivots about P_1 (which is fixed), and the second rod, of length $2\ell_2$, pivots about P_2 (which moves). The angles $\theta_1(t)$, $\theta_2(t)$ measure the position of the two rods at time t. A point mass m_1 is attached at P_2, and point masses $m_2/2$ are attached to each end of the second rod (at P_3 and P_4). Friction at P_1 (with coefficient ν_1) slows the first rod at a rate proportional to its angular velocity $\dot\theta_1(t)$; friction at P_2 slows the second rod (and simultaneously accelerates the first rod) at a rate proportional to $\dot\theta_2(t) - \dot\theta_1(t)$. The end of the second rod marked P_3 receives impulse kicks at times $t = T, 2T, \ldots,$ always from the same direction and with strength ρ. Gravity and air resistance are absent.

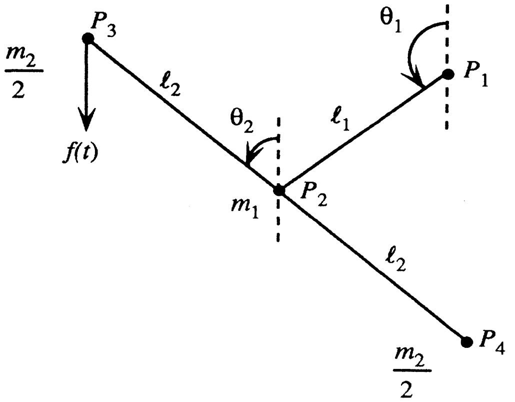

FIG. 1. The double rotor.

The double-rotor map is the four-dimensional map $x_{n+1} = F(x_n)$, defined by

$$x_{n+1} = \begin{pmatrix} \Theta_{n+1} \\ \dot\Theta_{n+1} \end{pmatrix} = \begin{pmatrix} (M\dot\Theta_n + \Theta_n) \bmod 2\pi \\ L\dot\Theta_n + G(\Theta_{n+1}) \end{pmatrix}. \quad (1)$$

Here Θ_n and $\dot\Theta_n$ are two-vectors,

$$\Theta_n = \begin{pmatrix} \theta_1^{(n)} \\ \theta_2^{(n)} \end{pmatrix}, \qquad \dot\Theta_n = \begin{pmatrix} \dot\theta_1^{(n)} \\ \dot\theta_2^{(n)} \end{pmatrix},$$

and

$$G(\Theta) = \begin{pmatrix} c_1 \sin\theta_1 \\ c_2 \sin\theta_2 \end{pmatrix},$$

where the angles θ_1 and θ_2 are taken to lie in $[0, 2\pi]$. The positions of the rods at the instant of the nth kick are given by $\theta_i^{(n)} = \theta_i(nT)$, and the angular velocities of the rods immediately after the nth kick are given by $\dot\theta_i^{(n)} = \dot\theta_i(nT^+)$. L and M are constant 2×2 matrices. For the sake of simplicity we assume $(m_1+m_2)\ell_1^2 = m_2\ell_2^2 \equiv I$. Then

$$L = \sum_{i=1}^{2} W_i e^{\lambda_i T}, \qquad M = \sum_{i=1}^{2} W_i (e^{\lambda_i T} - 1)/\lambda_i$$

with

$$W_1 = \begin{pmatrix} a\, b \\ b\, d \end{pmatrix}, \qquad W_2 = \begin{pmatrix} d & -b \\ -b & a \end{pmatrix}$$

where $a = \frac{1}{2}(1 + \nu_1/\Delta)$, $d = \frac{1}{2}(1 - \nu_1/\Delta)$, $b = -\nu_2/\Delta$, $\Delta = \sqrt{\nu_1^2 + 4\nu_2^2}$, $\lambda_{1,2} = -\frac{1}{2}(\nu_1 + 2\nu_2 \pm \Delta)$, and $c_{1,2} = \rho\ell_{1,2}/I$. In all the numerical work described in this paper, we fix the values of the parameters

$$\nu = T = I = m_1 = m_2 = \ell_2 = 1, \qquad \ell_1 = \frac{1}{\sqrt{2}}$$

and use the force ρ as the control parameter, taking as the nominal value $\rho = \bar\rho = 9$.

In the remainder of the paper, we write $x_n = F^n(x_0)$ to mean the n times iterated point x_0, i.e., the point obtained by iterating the map n times starting from x_0. The double-rotor map is invertible, so $F^{-n}(x_0)$ refers to the nth iterate of x under the inverse map. The notation $F(x)$ means that the map is applied with the kick set to its nominal value (here $\bar\rho = 9$); the notation $F(x, \rho)$ means the map applied to x with the kick set to ρ.

B. The local Lyapunov basis

A numerical estimate of the Lyapunov exponents of the double-rotor attractor can be obtained using the algorithm of Benettin *et al.* [12] For the parameter values given above, the attractor has two positive Lyapunov exponents $L_1 = 1.21$ and $L_2 = 0.26$, and two negative Lyapunov exponents, $L_3 = -1.74$ and $L_4 = -2.72$. The Lyapunov dimension [7] of the attractor is 2.8.

We assume that if there are two positive and two negative Lyapunov exponents associated with x, then there typically exists an ordered orthonormal basis $\{u_0, u_1, s_0, s_1\}$ of $\mathbb{R}^4$ (depending on x) such that the plane spanned by u_0 and u_1 is tangent to the unstable manifold at x, and the plane spanned by s_0 and s_1 is tangent to the stable manifold at x. (We will call this set the *Lyapunov basis* of x.)

The stable (unstable) manifold of x is the set of points y with the property that $\|F^n(x) - F^n(y)\| \to 0$ $[\|F^{-n}(x) - F^{-n}(y)\| \to 0]$ as $n \to \infty$. In other words, the trajectories of points on the stable manifold of x on the average approach the trajectory of x under forward iteration of the map. (The same is true for the backward iterates of points on the unstable manifold.) If x has two positive and two negative Lyapunov exponents, then both its stable and unstable manifolds are two dimensional.

III. BASIC TARGETING PROCEDURE

In this section we outline the basic idea behind the control procedure. Some refinements to the basic procedure will be discussed in Sec. IV.

Assume that we are given a typical point x_T on the attractor, which is to be targeted by the control. By "typical" we mean that x_T is not a periodic point and has two-dimensional stable and unstable manifolds associated with it. Let $x_0, x_1, \ldots, x_{T-1}$ be the T preimages of x_T, as shown in Fig. 2. We will call this set of points, together with the Lyapunov basis associated with each point, a *path* to the target. (If the point to be targeted is a periodic point, then we replace it by a surrogate nonperiodic target point x_T chosen sufficiently near to the actual target.)

Suppose that the map is iterated from an arbitrary initial condition in the basin of attraction for the double-rotor attractor, and that one of the iterates (call it y_0) falls within a suitably small neighborhood of the point x_0 in the path to the target. If y_0 falls close to one of the other points x_i $(0 < i \leq T-2)$, then we relabel the point x_0 and replace T by $T - i$. Without the control, the points $y_1, y_2, \ldots$, rapidly diverge from the trajectory $x_1, x_2, \ldots$, starting at x_0.

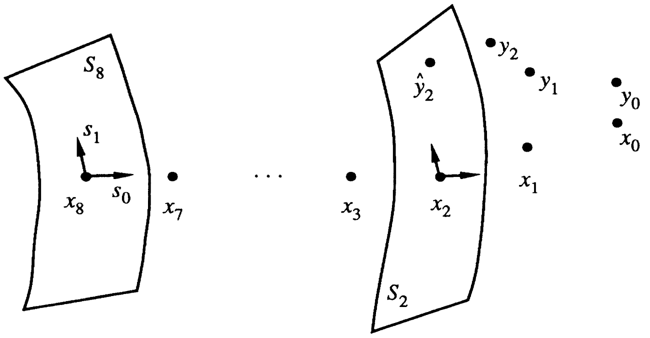

FIG. 2. Schematic diagram of the basic control procedure. Two successive perturbations of the kick are applied at y_0 to steer it to the stable manifold associated with the point x_2, which is a preimage of the target point.

The idea behind the control procedure is to apply two successive perturbations to the kick at y_0 and y_1 in order to steer y_0 to a new point $\hat{y}_2$ that lies on the stable manifold S_2 associated with x_2. (That is, S_2 is the stable manifold of x_2 for the unperturbed map at the nominal kick value $\rho = \bar{\rho}$.) We try to find two values ρ_0 and ρ_1 close to the nominal value of the kick so that $\hat{y}_2 = F(F(y_0, \rho_0), \rho_1) \in S_2$. Whenever the procedure succeeds, the trajectory starting at the new point $\hat{y}_2$ approaches the one leading to the target. Figure 2 illustrates the idea.

The approach of the trajectory starting from y_2 to the one leading to the target may not be uniform. However, on the average, it should approach the target trajectory at a rate proportional to $e^{L_3 t}$, where L_3 is the largest negative Lyapunov exponent and t is the number of iterations, counting from y_2.

Note also that if two different parameters could be varied independently (for example, the kick and the coefficient of friction at one of the pivots), then we would try to determine small perturbations of each parameter in order to hit the stable manifold of x_1, the next iterate in the path to the target, rather than the stable manifold of x_2, which is two iterates down.

[Recall that two two-planes generically intersect at a single point in $\mathbb{R}^4$. We make two successive perturbations to the kick because the vectors $g_0 = \partial F(F(y_0, \rho_0), \rho_1)/\partial \rho_0$ and $g_1 = \partial F(F(y_0, \rho_0), \rho_1)/\partial \rho_1$ typically span a two-plane through y_2 that intersects the two-dimensional stable manifold S_2 of x_2 at a unique point $\hat{y}_2$.]

The difficulty in finding the intersection point $\hat{y}_2$ is that the stable manifold S_2 associated with the point x_2 is not well approximated by a plane except in a small neighborhood of x_2. Although $\|x_0 - y_0\|$ may be small (perhaps of order 10^{-2}), $\|x_2 - \hat{y}_2\|$ generally is not (it may be of order 1). In other words, the intersection of the plane through y_2 spanned by the gradient vectors g_0 and g_1 with the stable manifold S_2 may be relatively far from x_2.

We can approximate S_2 away from x_2 by looking at the inverse images of the stable manifold of a point that is further down the path. Consider, for example, S_8, the stable manifold through x_8. Under the inverse map, S_8 is an expanding set. Suppose we take a point z near x_8 on the tangent plane to S_8. Although z does not lie exactly on S_8, the inverse images $F^{-1}(z), F^{-2}(z), \ldots$ approach the corresponding sets $S_7, S_6, \ldots$ because under the inverse map, errors damp out along the directions spanned by the Lyapunov vectors associated with the positive exponents.

Let s_0 and s_1 be the Lyapunov basis vectors associated with the negative Lyapunov exponents at x_8 (i.e., s_0 and

s_1 span the tangent plane to S_8 at x_8). If σ_0 and σ_1 are small numbers, then $z = x_8 + \sigma_0 s_0 + \sigma_1 s_1$ should be close to S_8, and its inverse images should approach the corresponding stable sets quickly. In particular, $F^{-6}(z)$ should lie very close to the stable set S_2 associated with x_2 even though $\|F^{-6}(z) - x_2\|$ may not be small.

The basic control step is to determine values σ_0 and σ_1, together with two values of the kick ρ_0 and ρ_1, such that

$$\begin{aligned} F^{-6}(x_8 + \sigma_0 s_0 + \sigma_1 s_1) &= F(F(y_0, \rho_0), \rho_1) \\ &= \hat{y}_2. \end{aligned} \tag{2}$$

Equation (2) often can be solved numerically using Newton's method. Figure 2 is a schematic illustration of the situation.

The contracting nature of the stable sets means that σ_0 and σ_1 typically are small. For the double-rotor map used here, σ_0 and σ_1 typically are of order 10^{-6} even when $\|\hat{y}_2 - x_2\|$ is of order 1. Thus, a successful solution of Eq. (2) determines two consecutive kicks such that the trajectory starting from the new point $\hat{y}_2 = F(F(y_0, \rho_0), \rho_1)$ falls within 10^{-6} of the trajectory leading to the target after six iterations. [The values of σ_0 and σ_1 in a typical solution to Eq. (2) depend on the negative Lyapunov exponents associated with the points on the path. More negative exponents lead to smaller values of σ_0 and σ_1.]

There is nothing special about the choice of S_8 and the use of six inverse iterations to find an intersection point on S_2. For example, if the target point is x_6, then we try to find a point on S_2 whose fourth iterate lands close to x_6.

If the target point is at, say, x_{20}, then one can look at the inverse image of points on S_9 or S_{10} instead of S_8. Going further down the path (say to S_9 or S_{10}) typically yields a point whose appropriate inverse image is closer to the stable set S_2. However, it also makes the numerical solution of Eq. (2) more ill conditioned. [If we look at the inverse images on S_8, then it is necessary to evaluate the matrix product $DF^{-1}(x_8)DF^{-1}(x_7)\cdots DF^{-1}(x_3)$. If we look at S_{10} instead, then we must evaluate $DF^{-1}(x_{10})DF^{-1}(x_9)\cdots DF^{-1}(x_3)$, and so on. These matrix products become more singular as more terms are added.] Thus there is a tradeoff between numerical precision and approximation errors arising from the dynamics. For the parameters of the double-rotor map used in this investigation, we have found that six inverse iterations is a good compromise between the accuracy in finding, say, $\hat{y}_2$ and the accuracy in iterating to the target. (Of course, fewer iterations of the inverse map are used as the controlled trajectory gets within six iterates of the target.)

Because of small errors in the initial approximation of S_8 and numerical roundoff errors, the control described above must be repeated from time to time in order to keep the new trajectory close to the path leading to the target. For instance, the two kicks ρ_0 and ρ_1 might be applied successively to y_0; afterward, the system might be set to the nominal value of the kick $\bar{\rho}$ for the next six iterations. The resulting point will be about 10^{-6} away from the target trajectory. The control step can be repeated at x_8 in order to steer the trajectory close to x_{16}, etc. Only very small perturbations of the kick are required at x_8 to accomplish this, because the controlled trajectory is usually within 10^{-6} or so of the path to the target at this point.

It is not always possible to solve Eq. (2). Sometimes Newton's method diverges because good starting values of the parameters cannot be obtained by linearizing Eq. (2). In such cases, it is not possible to initiate a control at the point y_0 in order to bring the trajectory close to the target. If the procedure fails, then we must wait until the trajectory again approaches a neighborhood of x_0 and try again.

For the parameter values described above, we have successfully found solutions to Eq. (2) about 90% of the time when $\|x_0 - y_0\| \leq 0.01$ and about 50% of the time when $\|x_0 - y_0\| \leq 0.05$. Moreover, the controlled trajectory rapidly approaches the trajectory leading to the target once a solution to Eq. (2) is determined. Typically, the distance between x_{16} and $\hat{y}_{16}$ (after two iterations of the control procedure) is close to the numerical precision of the computer (about 10^{-14}).

The basic control method described above can be applied when there is only one positive Lyapunov exponent. In that case, one must determine only a single perturbation ρ_0 to the kick so that the new point $\hat{y}_1 = F(y_0, \rho_0)$ intersects the stable manifold S_1 of x_1 (the next point in the path) because S_1 is three dimensional.

The method can be extended to other maps in different dimensions in a straightforward way. For example, if the attractor sits in a six-dimensional space and has three positive Lyapunov exponents, then the basic control procedure requires three successive changes to a single parameter to hit the three-dimensional stable manifold of the appropriate point in the path. If three parameters can be varied independently, then one tries to hit the stable manifold of x_1, and so on.

We note that the double-rotor map for the parameters we investigate is not hyperbolic. In particular, there exist saddle fixed points in the attractor which have one unstable direction and three stable directions. This is in contrast to our determination that orbits for typical points on the attractor (i.e., almost every point with respect to the natural measure) have two positive and two negative Lyapunov exponents. In spite of this nonhyperbolic situation, we do not find any problem in our numerical experiments. As we iterate, the stable manifold of each point in the paths to the target appears to be two dimensional. (This is not surprising since the set of points for which this is not true has zero natural measure.)

Finally, we remark that it is possible in principle to steer a given initial condition to the target using a sequence of four perturbations of the kick in order to hit a point on the trajectory leading to the target. For example, if $x_0, x_1, x_2, \ldots$ is a trajectory leading to the target and if y_0 is a point near x_0, then one can try to apply a sequence of kicks $\rho_0, \rho_1, \rho_2, \rho_3$ at y_0, y_1, y_2, y_3 in order to hit x_4. In practice, we have not been able to use this approach, because it is not possible to get an accurate linearization of the problem unless the corresponding trajectories are extremely close together.

IV. REFINEMENTS

The procedure described above works well, but it has the disadvantage that the map must be iterated a large number of times before reaching a neighborhood of one of the points in the path leading to the target. A long path increases the likelihood that a given iterate lies near a point on it, but then many control steps are required to reach the target.

Our objective is to steer a typical iterate to the target point in as few steps as possible. In this section we describe some refinements to the basic control procedure that allow us to do this in an average of 35 steps.

We build a hierarchy or "tree" of paths leading to the target as follows. Suppose we have already selected a target point x_T and have a path $x_0, x_1, \dots, x_{T-1}$ on the attractor leading to it. (In the results described below, the path has 20 points, so $T = 20$.) We iterate the map (possibly from an arbitrary initial condition in the basin of attraction) until we obtain a point z_n that lies in a suitably small neighborhood of one of the points in the target path. We store z_n, together with a path of T points now leading to z_n. (The Lyapunov basis of each point must also be stored.)

Of course, it is possible to apply the targeting procedure to the point z_n in order to steer the trajectory starting from z_n to a small neighborhood of the target point. The reason for building the tree before applying the targeting procedure is to increase the probability that a given iterate lies near a path leading to the target point. In this respect, we imagine that it is possible to observe the dynamics for a time before initiating the targeting procedure. This new path is part of the second level of a tree whose root is the path leading to the target. If we have an iterate that falls within a suitable neighborhood of one of the points $z_{n-T}, z_{n-T+1}, \dots$, then we can apply the control procedure described in Sec. III using z_n as an interim target. The idea is to steer the iterate to z_n, so that the controlled iterate now lies in a neighborhood of one of the points leading to the target x_T. Then the control procedure is repeated (starting in a small neighborhood of z_n) with x_T as the target. If both the root and the secondary path have 20 points, then the control steers the iterate to the target in no more than 40 steps. Figure 3 is a schematic illustration of this approach.

Other paths are added to the tree in similar fashion. As the map is iterated, we check to see whether the current point falls suitably close to one of the points in a previously stored path. If it does, then the point is added to the tree, together with its $T = 20$ preimages (and their associated Lyapunov bases). The tree can be made as large and as deep as required, depending on the amount of computer memory available. In the results described here, we limit the tree to a total of 500 paths, each of length 20 (for a total of 10^4 points) and three levels (so there are no more than 60 steps from any point in the tree to the final target). Thus, the tree is not full; that is, not every point has a path at a lower level in the tree leading to a point in its neighborhood. The tree is built with fewer than 10^6 total iterations of the double-rotor map and occupies about three megabytes of computer memory.

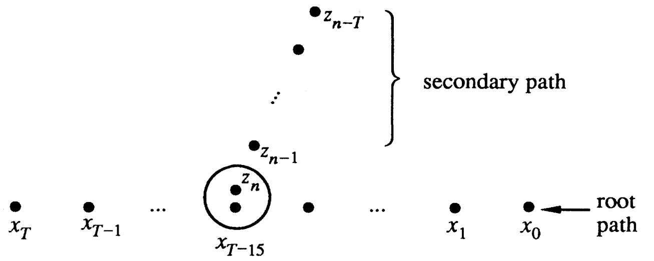

FIG. 3. Schematic illustration of the hierarchy of paths leading to the target point.

Once the tree is built, it is possible to steer points to the target very quickly, as follows. Let z_0 be a point on the attractor. If z_0 is not close to any of the points in the path tree, then we create a new set of points A by making n small random perturbations to the kick. Here $A = \{z_1^{(i)} : z_1^{(i)} = F(z_0, \rho_0 + \eta_i), 1 \le i \le n\}$, where η_i is a random variable in a small interval around 0. Typically we take η_i from a uniform distribution in the interval $[-0.05, 0.05]$.

We now check whether any of the points in A lies near any of the points in the path tree. If so, then we attempt the control procedure. If it is successful, then we have steered the point z_0 to the target in no more than 61 steps (the first step consists of the random kick, followed by no more than 60 steps of the control procedure). Each of the points in A can be iterated (using the nominal value of the kick) until one of them can be steered successfully to the target.

V. RESULTS

The control procedure has been tested using a variety of initial and target points. The first numerical experiment is done in a two-step process. In the first step, the tree is constructed as described above (with a total of 10^4 points in 500 paths on three levels) and stored for later use.

The second step begins by reading in the tree. An arbitrary initial condition is selected and iterated 10^4 times to allow transients to die out. In this step, the targeting procedure is attempted whenever an iterate z_n falls within 0.05 of a point in the tree. Otherwise, we construct a set A consisting of 100 points $\{F(z_n, \bar{\rho} + \eta_i)\}_{i=1}^{100}$ as described above. (Here η_i is a uniformly distributed random variable in $[-0.05, 0.05]$.) The targeting procedure is attempted from any point in A that falls within 0.05 of a point in the tree. If it is successful, then we record the distance of the controlled point from the target and the number of map iterations required to get there.

The points in A are iterated until we find one that lies within 0.05 of a point in the tree. In no case is it necessary to iterate any point in A more than five times before a successful control can be achieved.

On the average, the control takes 30 steps, so that the initial point can be steered to the target in a total of 35 steps (five iterations from the random kick and 30 steps using the control). In this numerical study, no more than 65 steps are required (60 steps using the control and five iterations of the random perturbation). In each case, the point can be steered to within 2×10^{-4} of the target point, and frequently much closer (the median distance is 10^{-6} from the target). The changes to the kick required to initiate the control are typically of order 10^{-1}; once the point gets close to the trajectory leading to the target, the perturbations become much smaller (typically of order 10^{-4} or less).

The targeting procedure takes relatively few iterates because we have built a moderately large tree containing paths that lead quickly to the target. This tree requires about 10^6 total iterations to build.

One may not be willing to observe the system for this length of time in order to build a tree of paths leading to the target. As an alternative, one can still achieve rapid control by simply taking a larger set A of randomly perturbed points. For example, we took a single path of 50 points leading to an arbitrarily chosen target point. We then took another initial condition z_0 at random from the attractor and attempted the targeting procedure as follows.

For each iterate z_n in the trajectory starting at z_0, we made 1000 small random perturbations of the kick. That is, the set A contained 1000 points of the form $F(z_n, \bar{\rho}+\eta_i)$ where η_i is a uniformly distributed random variable in $[-0.05, 0.05]$. In this experiment, we found that typical points on this trajectory could be steered to a small neighborhood of the target in an average of 35 iterates (and in no case more than 114 iterates), about as quickly as in the first experiment discussed above.

If the set A is reduced to 100 points instead, then the average control time rises to 181 iterates (with a maximum of 909 iterates). The increase is due to the greater length of time required before one of the perturbed points approaches a neighborhood of one of the points in the path to the target. Nevertheless, the control procedure is still reasonably fast, even when a tree of paths to the target is not used.

The numerical method requires modest computer resources. It takes about 10 min and three megabytes of memory to build the path tree described in Sec. IV on a desktop workstation. Once the tree is built, about 10 min of computer time are required to attempt the control procedure from 10^4 different initial conditions on the attractor.

VI. CONCLUSIONS

The numerical method employed here attempts to direct iterates on a chaotic attractor to a prespecified target point on the attractor. This is done by finding a sequence of small perturbations to an available system parameter (or parameters) to direct the trajectory of a given initial condition to the stable manifold of a point on a trajectory leading to the target. The number of iterates required to reach a neighborhood of the target point can be reduced by building a hierarchy of paths leading to the target.

The method can be adapted in principle to chaotic attractors with any number of positive Lyapunov exponents. For the double-rotor map, which has four variables, and in the case we study, the attractor has two positive and two negative Lyapunov exponents. On the average, initial conditions can be steered to a small neighborhood (of size 10^{-4} or less) of an arbitrarily chosen target point in 35 iterates using a hierarchy of paths leading to the target point.

ACKNOWLEDGMENTS

E. K. is supported by the National Science Foundation's Computational and Applied Mathematics Program under Grant No. DMS-9017174. The work of C. G., E. O., and J. A. Y. is supported by the Department of Energy (Office of Scientific Computing, Office of Energy Research).

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