Bound states of envelope solitons

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(Received 17 July 1992)

It has been demonstrated recently that weakly overlapping solitons in the dissipatively perturbed nonlinear Schrödinger (NS) equation may form a set of bound states (BS's). In this work, it is demonstrated that additional "skew" dissipative terms, which occur in various applications, e.g., a term describing the stimulated Raman scattering in a nonlinear optical fiber, destroy all the BS's provided the corresponding coefficient exceeds a certain critical value. Next, taking as an example the dissipationless NS equation with the higher linear dispersion, it is demonstrated that two solitons or a whole array of them may form BS's, interacting with each other via emitted radiation. Then, it is shown that the sine-Gordon (SG) breathers, governed by the standard damped ac-driven equation, may form BS's quite similarly to the NS solitons. At last, interactions of damped NS or SG solitons supported by a parametric ac drive are analyzed, and it is inferred that they, unlike the directly supported solitons, cannot form BS's.

PACS number(s): 42.81.Dp

I. INTRODUCTION

The recent experimental observation of stable bound states (BS's) of solitary pulses of the subcritical traveling-wave convection in a narrow channel [1], as well as a possibility of precise experiments with interacting solitons in nonlinear optical fibers [2], make it relevant to analyze BS's of solitons within the framework of simple models of nonlinear wave propagation. In a general form, a possibility of existence of BS's of well-separated (slightly overlapping) solitons was discussed long ago [3]. Recently, this issue has been analyzed in detail in Ref. [4] within the framework of the model based on the nonlinear Schrödinger (NS) equation perturbed by terms accounting for dissipation and input of energy. In its simplest version, the equation has the form

$$iu_t + u_{xx} + 2|u|^2 u = i\gamma_0 u + i\gamma_1 u_{xx} \ , \tag{1.1}$$

with positive γ_0 and γ_1. This equation admits an exact solitary-pulse solution [5]. As it follows from expansion of this exact solution, or from the direct perturbative analysis of Eq. (1.1) valid when the dimensionless parameter γ_1 is small, and the solitary pulse is close to the NS soliton with the amplitude

$$\eta_0^2 = 3\gamma_0 / 4\gamma_1 \ , \tag{1.2}$$

the asymptotic form of the pulse far from its core is

$$u_{\mathrm{as}} = 4i\eta_0 \exp(-2\eta_0 |x - z_0|) \times \exp(4i\eta_0^2 t - ik|x - z_0| + i\phi_0) \ , \tag{1.3}$$

where z_0 is the coordinate of the pulse's center, the wave number

$$k = \tfrac{4}{3}\gamma_1 \eta_0 \tag{1.4}$$

is solely produced by the perturbing terms in Eq. (1.1), and ϕ_0 is an arbitrary phase constant. The long-wave spatial oscillations generated by the small wave number k give rise to an oscillating tail in the effective potential of interaction of two far separated solitons (the potential is determined by the overlapping integral between the tail of each soliton and its mate's core). The local minima of the interaction potential, corresponding to the values of the distance z between the solitons close to

$$z_n = (\pi/2k)(1+2n), \quad n = 0, 1, 2, \ldots, \tag{1.5}$$

give rise to stable BS's of the two-soliton pair, as well as to multisoliton BS's. Formally, the number of the two-solitons BS's determined by Eq. (1.5) is infinite. However, their binding energies E_n are exponentially small,

$$E_n \sim \exp[-(1+2n)\pi\eta_0 / k] \tag{1.6}$$

[recall that the quantity $\eta_0/k = 3/4\gamma_1$ is large according to Eq. (1.4)]. So, one may expect that only few of the weakly stable BS's should be physically meaningful.

Shortly after this, a similar problem was considered in Ref. [6] within the framework of a system of two NS equations with no dissipation, coupled by two different cubic terms (incoherent and coherent ones, i.e., respectively, independent and dependent of the relative phase of the two complex wave fields). The presence of a small group-velocity difference between the two coupled modes was presumed too (it may be produced, e.g., by the birefringence effect in a bimodal nonlinear optical fiber). It has been demonstrated that the latter factor gives rise to the terms $\sim |x - z_0^{(1,2)}|$ in the asymptotic phases of the *polarized solitons,* i.e., the ones belonging to the first and second modes, respectively [cf. Eq. (1.3)]. This, in turn, implies that the coherent intermode coupling generates an oscillating potential of the interaction between the polarized solitons. It, however, competes with the purely attractive (nonoscillating) potential produced by the incoherent coupling. The analysis presented in Ref. [6] demonstrates that the competition of the two potentials may allow for the existence of a finite number of the BS's.

Many-soliton bound states in the model (1.1) have been found in numerical experiments reported in the work of G. P. Agrawal [Phys. Rev. A **44**, 7493 (1991)].

The objective of the present paper is to develop the analysis of the BS's of the envelope (NS-like) solitons, taking into account some additional physically important factors. This work is stimulated by the experimental discovery of the stable BS's formed by left- and right-traveling pulses in the subcritical traveling-wave convection [1], as well as by the necessity to analyze the possibility of existence of the BS's of optical solitons in the nonlinear fibers, which might be important for applications.

The most significant effect missing in the model of the nonlinear optical fiber based on Eq. (1.1) is the stimulated Raman scattering (SRS) which, in the simplest approximation, can be accounted for by adding the nonlinear dissipative term to Eq. (1.1) [7,8]:

$$iu_t+u_{xx}+2|u|^2u=i\gamma_0u+i\gamma_1u_{xx}+\epsilon(|u|^2)_xu \qquad (1.7)$$

with real ϵ [9]. Generally speaking, the additional term in Eq. (1.7) represent but one of the possible "skew" terms (odd with respect to the reflection $x\rightarrow -x$). It has been demonstrated in Ref. [4] that the simplest conservative skew term, accounting for the higher dispersion, does not affect the existence and stability of the BS's in Eq. (1.1). However, the *dissipative* skew terms may be very important. Note that similar skew dissipative terms occur in model equations governing dynamics of pulses in convection and like nonequilibrium systems [10].

The BS's of solitons in Eq. (1.7) are considered in Sec. II. As the skew terms exerts an effective force upon the soliton [7,8], the analysis is developed in a moving reference frame in which this force is balanced by the friction, so that the solitons are quiescent. The diffusion term in Eq. (1.7) (the one $\sim\gamma_1$), when transformed into the moving reference frame, gives rise to an additional dissipative skew term $\sim\gamma_1u_x$. This linear term alters the form of the soliton's asymptotic tail. Analysis of the effective potential of the interaction between two solitons with the modified tails reveals that, even formally, the number of the BS's becomes finite, and they all disappear if the parameter ϵ in Eq. (17) exceeds a certain critical value. It is relevant to mention that the influence of the SRS term on the two-soliton BS's in the form of the so-called breathers (unlike the BS's considered in Ref. [4], the breathers exist in the unperturbed NS equation, and their binding energy is exactly equal to zero, so that they are commonly regarded as unphysical states) was earlier considered in Refs. [8] and [11]. In these works, it was assumed that the solitons forming the breather had different amplitudes η, so that the forces produced by the SRS term, being proportional to $\epsilon\eta^4$, gave rise to a difference force separating the solitons. However, in the present case the amplitudes η_0 of the solitons which constitute the BS are automatically equal, both being uniquely selected by the balance between the input and dissipation of energy [see, e.g., Eq. (2)].

Another interesting issue is a possibility to bind solitons via a radiative interaction. In particular, anomalously long-range radiative forces acting between the solitons have been detected experimentally in the nonlinear optical fiber [2]. An analytical approach to this problem was developed by Kaup in Ref. [12]. In the present work, two-soliton states bound via their radiation field are considered in Sec. III for the NS equation incorporating the higher dispersion,

$$iu_t+u_{xx}+2|u|^2u=-i\epsilon u_{xxx}\ , \qquad (1.8)$$

where ϵ is real. As was mentioned above, the term on the right-hand side (rhs) of Eq. (1.8), added to Eq. (1.1), did not affect the BS's produced by the dissipative perturbations. However, if one deals with the purely conservative model (1.8), it is known [13] that the perturbation gives rise to emission of radiation at the large wave number k_0 at which the radiation frequency $\omega_{\rm rad}=k^2-\epsilon k^3$ coincides with the soliton's frequency $\omega_{\rm sol}=-4\eta^2$ [see Eq. (1.3)],

$$k_0\approx 1/\epsilon+4\epsilon\eta^2\ , \qquad (1.9)$$

η being the amplitude of the quiescent emitting soliton. This effect is quite similar to that known in the Korteweg–de Vries (KdV) equation with the higher dispersion accounted for by the fifth spatial derivative [14–16]. If one deals with an initial soliton pulse, it will very slowly decay into the radiation. However, it has as well been demonstrated for the KdV model [15,16] that a *stable* soliton may exist on the background of an extended radiation wave. This can be interpreted as an equilibrium between the emission and absorption of radiation by the soliton [16]. Although a similar equilibrium solution has not, as yet, been accurately analyzed in the framework of Eq. (1.8), it seems very plausible that it must exist as well. Then two solitons riding on top of their common radiation "substrate" will feel an effective pinning potential and may thus form a BS when the pinning is stronger than the direct mutual attraction of the solitons. Actually, the separation between the bound (pinned) solitons may take an arbitrary value larger than a critical one, at which the pinning and attraction forces are equal. As a matter of fact, this effect does not crucially depend on the existence of the above-mentioned equilibrium solution coupling a stable soliton to the radiation substrate. In the nonstationary problem, essentially the same effect must take place when the radiation wave emitted by the trailing soliton reaches the leading one having the same frequency. This radiation force could be amenable in part for long-range interaction between the optical solitons detected in Ref. [2], or, at least, can be detectable in precise experiments with the optical solitons in a spectral region near the zero point of the usual dispersion, where the higher dispersion may become important [17]. At last, it is worthy to mention that quite a similar mechanism gives rise to the mutual pinning of KdV solitons in the model including the higher dispersion [18] (in the unperturbed KdV equation, the solitons simply repeal each other).

In Sec. IV I consider a more general model that admits envelope solitons, viz., the sine-Gordon (SG) equation with small damping and driving terms. The envelope solitons of the SG equation are the so-called breathers, i.e., its spatially localized time-periodic solutions. An important example is the model of a damped ac-driven long Josephson junction [19]

$$\phi_{tt}-\phi_{xx}+\sin\phi=-\alpha\phi_t+\beta\phi_{txx}+\epsilon\cos(\omega t) , \quad (1.10)$$

α and β being coefficients of the shunt and surface losses. The ac drive in Eq. (1.10) can support a breather at the frequency $\omega<1$, provided ϵ exceeds a threshold value which is a linear combination of the dissipative constants α and β. The model (1.10) finds a number of other applications in dynamical problems of condensed matter physics, e.g., when one considers an ac-driven weakly damped charge-density-wave conductor in the commensurability regime [20]. The main point of the analysis developed in Sec. IV is that, in the presence of the dissipative terms, the asymptotic wave form of the SG breather becomes spatially oscillating, thus giving way to formation of BS's of the breathers.

A soliton in the damped NS system, or a breather in the SG one, may be supported not only by the *direct* drive, as in Eq. (1.10), but also by a *parametric* drive [21]. In Sec. V it is demonstrated that the parametrically driven envelope solitons, unlike the directly driven ones, *cannot* form BS's. The fundamental cause for this is that when the parametric drive's amplitude is sufficiently large to compensate dissipation, the solitons' tails cannot be oscillating.

II. BOUND STATES OF NS SOLITONS IN THE PRESENCE OF THE SKEW DISSIPATIVE TERMS

The unperturbed soliton solution of the NS equation is taken in its standard form:

$$u_{\text{sol}}=2i\eta\,\text{sech}\{2\eta[x-z_0(t)]\}\exp[\tfrac{1}{2}iVx+i\psi(t)] , \quad (2.1)$$

where

$$\frac{dz_0}{dt}=V, \quad \frac{d\psi}{dt}=4\eta^2-\tfrac{1}{4}V^2 , \quad (2.2)$$

η and V being the amplitude and velocity of the soliton. It is straightforward to derive evolution equations for these parameters, taking into account the small perturbing terms in Eq. (1.7) in the lowest (adiabatic) approximation of the perturbation theory [22,7,8]:

$$\frac{d\eta}{dt}=2\gamma_0\eta-\tfrac{8}{3}\gamma_1\eta^3-\tfrac{1}{2}\gamma_1\eta V^2 , \quad (2.3a)$$

$$\frac{dV}{dt}=-\tfrac{16}{3}\gamma_1\eta^2 V+\epsilon\frac{(16\eta^2)^2}{15} . \quad (2.3b)$$

The dynamical system (2.3) has the single stationary point

$$\eta_0^2=\tfrac{2}{3}(8\epsilon/5)^{-2}\{[\gamma_1^4+\tfrac{9}{4}(8\epsilon/5)^2\gamma_0\gamma_1]^{1/2}-\gamma_1^2\} , \quad (2.4a)$$

$$V_0=(16\epsilon/5\gamma_1)\eta_0^2 , \quad (2.4b)$$

which is always stable.

It is convenient to consider BS's of the solitons in the reference frame in which they are quiescent. To do this, one should make the Galilean transformation

$$x\to x-V_0t, u\to u\exp[i(\tfrac{1}{2}V_0x-\tfrac{1}{4}V_0^4t)] ,$$

to cast Eq. (1.7) into the form

$$iu_t+u_{xx}+2|u|^2u=i(\gamma_0-\tfrac{1}{4}V_0^2\gamma_1)u-\gamma_1V_0u_x +\gamma_1u_{xx}+\epsilon(|u|^2)_xu . \quad (2.5)$$

It is now straightforward to find the asymptotic form of the soliton far from its center, which is located at $x=z_0$. The linearized equation (2.5) yields [cf. Eq. (1.3)]

$$u_{\text{as}}=4i\eta_0\exp[-2\eta_0|x-z_0|+\lambda(x-z_0)] \times\exp(4i\eta_0^2t-ik|x-z_0|+i\phi_0) , \quad (2.6)$$

where η_0 is the equilibrium amplitude (2.4a), and [cf. Eq. (1.4)]

$$k=(4\eta_0)^{-1}(\gamma_0-\tfrac{1}{4}V_0^2\gamma_1+4\eta_0^2\gamma_1) , \quad (2.7)$$

$$\lambda=-\tfrac{1}{2}\gamma_1V_0 ,$$

V_0 being the equilibrium velocity (2.4b). The presence of the term $\lambda(x-z_0)$ in the argument of the first multiplier in Eq. (2.6) is the crucial difference from the expression (1.3) valid in the absence of the skew term in Eq. (1.7). The subsequent analysis closely follows that developed in Ref. [4]. Considering the interaction between two far separated solitons, one approximates the full wave field by their linear superposition

$$u(x,t)=u_{\text{sol}}^{(1)}(x,t)+u_{\text{sol}}^{(2)}(x,t) . \quad (2.8)$$

Next, one inserts Eq. (2.8) into the interaction Hamiltonian

$$H_{\text{int}}=-\int_{-\infty}^{+\infty}|u(x)|dx \quad (2.9)$$

of the NS system. As the interaction between the solitons is dominated by the overlapping of the core of each soliton with the weak tail of another one, it is sufficient to linearize the expression (2.9) with respect to u_2 in the region where the core of the first soliton is located, and vice versa. This procedure leads to the effective interaction potential in the form [4]

$$U(z,\phi)=-4\int_{-\infty}^{-\infty}dx\,|u_{\text{sol}}^{(1)}(x)|^2 \times\text{Re}[u_{\text{sol}}^{(1)}(x)u_{\text{sol}}^{(2)*}(x)]+(1\rightleftarrows 2) , \quad (2.10)$$

z and ϕ being, respectively, the distance between the centers of the two solitons and their phase difference; it is assumed that both solitons have the equilibrium amplitude η_0, and that $z\eta_0\gg1$ (which means that the solitons are far separated). Inserting Eqs. (2.1) for $u_{\text{sol}}^{(1)}$ and (2.6) for $u_{\text{sol}}^{(2)}$ into Eq. (2.10), one finds

$$U(z,\phi)=-128\eta_0^3e^{-2\eta_0z}[e^{(1/2)\gamma_1V_0z}\cos(\phi+kz) +e^{-(1/2)\gamma_1V_0z}\cos(\phi-kz)] . \quad (2.11)$$

The two terms in the square brackets come from the regions where the cores of the two solitons are located. The contributions from these two regions are asymmetric due to the presence of the term $\sim\lambda$ in the argument of the first exponential in Eq. (2.6).

The BS's exist if the interaction potential (2.11) has local minima, which are determined by the equations

$$\frac{\partial U}{\partial \phi}=0 , \quad (2.11a)$$

$$\frac{\partial U}{\partial z}=0 . \quad (2.11b)$$

It is straightforward to see that Eq. (2.11a) amounts to the following one:

$$\tan\phi=-\tanh(\tfrac{1}{2}\gamma_1 V_0 z)\tan(kz) . \quad (2.12)$$

Making use of Eq. (2.12), one can cast Eq. (2.11b) into the form

$$[1+\tan^2(\tfrac{1}{2}\gamma_1 V_0 z)\tan^2(kz)]-(\gamma_1 V_0/4\eta_0)\tanh(\tfrac{1}{2}\gamma_1 V_0 z)[1+\tan^2(kz)]+(k/2\eta_0)\text{sech}^2(\tfrac{1}{2}\gamma_1 V_0 z)\tan(kz)=0 . \quad (2.13)$$

The applicability of the perturbation theory assumes that the coefficients $\gamma_1 V_0/4\eta_0$ and $k/2\eta_0$ in Eq. (2.13) are small; this, in turn, implies that Eq. (2.13) may have real solutions only if

$$\gamma_1 V_0 z \ll 1 . \quad (2.14)$$

With regard to the inequality (2.14), Eq. (2.13) can be transformed once again into

$$(\tfrac{1}{2}\gamma_1 V_0 z)^2\tan^2(kz)+(k/2\eta_0)\tan(kz)+1=0 . \quad (2.15)$$

Because of Eq. (2.14), real roots of Eq. (2.15) must be located near the points where $\tan(kz)$ diverges, i.e., the roots are looked for in the form [cf. Eq. (1.5)]

$$z=z_n\equiv(\pi/2k)(1+2n)+k^{-1}\delta_n , \quad (2.16)$$

$n=0,1,2,\ldots$, with $|\delta_n|\ll\pi/2$. At last, insertion of Eq. (2.16) into Eq. (2.15) gives rise to the following equation for δ_n:

$$(\delta_n/\gamma_1)^2+(k/2\eta_0\gamma_1)(\delta_n/\gamma_1)+[(\pi V_0/4k)(1+2n)]^2=0 . \quad (2.17)$$

This square equation has real roots provided

$$1+2n\le k^2/\pi V_0\eta_0\gamma_1\equiv(5/16\pi)(k^2/\epsilon\eta_0^3) , \quad (2.18)$$

where Eq. (2.4b) was taken into account.

Thus, Eq. (2.18) tells that the potential (2.13), biased by the skew term in the underlying equation (1.7), admits only a finite number of the local minima. With the growth of the parameter ϵ, the last bound state, corresponding to $n=0$ in Eq. (2.18), disappears when its rhs becomes smaller than one, i.e., at

$$\epsilon>\epsilon_{\text{thr}}\equiv(5/16\pi)(k^2/\eta_0^3) . \quad (2.19)$$

It was mentioned above that different dissipative skew terms might appear as well in model equations governing dynamics of nonequilibrium systems [10]. Within the framework of the perturbation theory, the effect of those terms on BS's of solitons can be analyzed as was done above for Eq. (1.7). The general inference following from this analysis is that the skew terms are apt to destroy the BS's.

It is relevant to give some estimates for a real optical fiber to see if the effect considered takes place indeed. For the femtosecond solitons, typical values of the parameters in Eq. (1.7) are [23] $\gamma_0=0.05$, $\gamma_1=0.01$, $\epsilon=0.1$; in this notation, a typical soliton has the width $(2\eta)^{-1}=1$ [in physical units, this is 100 fsec; note that if one regards Eqs. (1.1) and (1.7) as models of the optical fiber, the variables t and x have, respectively, the meaning of the propagation distance and of the so-called reduced time]. Insertion of these values into Eqs. (2.4) and (2.7) yields $\eta_0^2\approx\frac{3}{4}$, $V_0\approx 2$, $k\approx\frac{1}{15}$, and, finally, Eq. (2.19) yields $\epsilon_{\text{thr}}\sim 10^{-3}$. The above-mentioned physical value $\epsilon=0.1$ is much larger than ϵ_{thr} and this may lend an explanation to the fact that the BS's of the optical solitons have not been thus far detected in the optical fibers. Let us, however, emphasize once again that the BS's have been observed in the experiments with the subcritical traveling-wave convection [1], so that in this physical system the skew dissipative terms are not strong enough to destroy the BS's.

III. SOLITONS BOUND BY A RADIATION FIELD

In this section, the possibility of getting a BS of solitons interacting via emission and absorption of radiation will be considered within the framework of Eq. (1.8), although, as will be seen, the same can be done in a rather general context (e.g., for the KdV equation with the fifth derivative [18]). Since the radiative effects in this and similar models are exponentially weak [13–16], the analysis will be developed without regard to preexponential factors that do not give rise to any significant effects.

As was said in Sec. I, the solitons governed by Eq. (1.8) slowly emit radiation at the wave number k_0 given by Eq. (1.9). I assume the presence of two (or of a chain of) solitons in a configuration where the wave emitted by a soliton with the coordinate $z_0^{(1)}$ is absorbed by another one with the coordinate $z_0^{(2)}$. The phases of the two solitons are assumed to be

$$\psi^{(1,2)}=4\eta^2 t+\phi_0^{(1,2)} \quad (3.1)$$

[see Eq. (2.2), where, to simplify the analysis, it is set $V=0$]. The phase of the radiation wave between the solitons is

$$\phi_{\text{rad}}=k_0 x+4\eta^2 t . \quad (3.2)$$

Thus the *relative* phases $\tilde{\phi}$ between the radiation and the solitons are

$$\tilde{\phi}^{(1,2)}(t)\equiv\phi_{\text{rad}}(z_0^{(1,2)},t)-\psi^{(1,2)}(t)=k_0 z_0^{(1,2)}-\phi_0^{(1,2)} . \quad (3.3)$$

The underlying Eq. (1.8) is invariant with respect to the transformation

$$u \to u^*, \quad x \to -x, \quad t \to -t \ . \tag{3.4}$$

As the transformation (3.4) reverses the direction of time, it turns the emission of radiation into absorption. The emitted wave has a certain phase shift δ relative to the emitting soliton [13]. The time-reversing transformation (3.4) tells us that for the absorbing soliton the phase shift must be $-\delta$. In our case, $\tilde{\phi}^{(1)} \equiv \delta$, and $\tilde{\phi}^{(2)} \equiv -\delta$. Thus it follows from Eq. (3.3) that

$$k_0 z_0^{(1)} + \phi_0^{(1)} = -(k_0 z_0^{(2)} + \phi_0^{(2)}) + 2\pi n \ , \tag{3.5}$$

n being an arbitrary integer.

It is well known [3,4] that the energy of the direct interaction between the separated NS solitons has a minimum (and the attraction force takes its maximum value) when their phases are equal. Therefore, in what follows I set $\phi_0^{(1)} = \phi_0^{(2)} \equiv \phi_0$, and then Eq. (3.5) yields

$$\phi_0 = -\tfrac{1}{2} k_0 (z_0^{(1)} + z_0^{(2)}) + \pi n \ . \tag{3.6}$$

The energy of the pinning of solitons by the radiation "substrate" is determined by the term (2.9) in the Hamiltonian of the system considered. Similar to Eq. (2.10), this term must be linearized with respect to the small-amplitude radiation field, and the resulting pinning potential U_{pin} can be estimated as

$$U_{\text{pin}} \sim A\eta^3 \text{Re} \left[e^{i\phi_0} \int_{-\infty}^{+\infty} \text{sech}^3[2\eta(x - z_0)] e^{ik_0 x} dx \right] , \tag{3.7}$$

where A is the amplitude of the radiation wave and use of Eqs. (2.1), (3.1), and (3.2) has been made. It immediately follows from Eq. (3.7) that

$$U_{\text{pin}} \sim A k_0^2 \exp \left[-\frac{\pi k_0}{4\eta} \right] \cos(\phi_0 + k_0 z_0) \ ; \tag{3.8}$$

to obtain the estimate (3.8), it has been taken into account that $k_0 \gg \eta$ [Eq. (1.9)]. At last, assuming the presence of two separated solitons, one should insert Eq. (3.6) into Eq. (3.8) taken for either soliton, to transform U_{pin} into the effective interaction potential:

$$U_{\text{pin}} \sim A k_0^{(2)} \exp \left[-\frac{\pi k_0}{4\eta} \right] \cos[\tfrac{1}{2} k_0 (z_0^{(1)} - z_0^{(2)})] \ . \tag{3.9}$$

The potential U_{attr} of the direct attraction between the solitons with equal phases may be estimated as follows [3,4]:

$$U_{\text{attr}} \sim \eta^3 \exp(-2\eta |z_0^{(1)} - z_0^{(2)}|) \ . \tag{3.10}$$

The amplitude A of the radiation wave emitted by the soliton under the action of the perturbation term in Eq. (1.8) can be estimated, with the exponential accuracy, as [13]

$$A \sim \epsilon \eta^2 \exp \left[-\frac{\pi k_0}{4\eta} \right] . \tag{3.11}$$

Comparing the expressions (3.9) and (3.10) and making use of Eq. (3.11), one concludes that, in the lowest approximation (when only the exponential factors are compared), the pinning dominates over the attraction provided the distance $z \equiv |z_1 - z_2|$ between the solitons exceeds the minimum value $z_{\min}$:

$$z > z_{\min} \equiv \pi / 4\eta^2 \epsilon \ . \tag{3.12}$$

Thus, the pair or a whole array of solitons can be stably pinned by their common radiation field if the distances between them are not less than given by Eq. (3.12). Since the pinning potential (3.9) is spatially periodic, these distances, strictly speaking, may only take discrete values, multiple of $4\pi/k_0$. However, in the situation analyzed here this limitation is not essential, as this spacing is much smaller than the proper size of the soliton $\sim \eta^{-1}$.

IV. BOUND STATES OF BREATHERS IN THE DAMPED ac-DRIVEN SINE-GORDON MODEL

The analysis of BS's of breathers in the model based on Eq. (1.10) can be developed in a straightforward way. The exact breather solution to the unperturbed SG equation is commonly known:

$$\phi_{\text{br}} = 4 \tan^{-1} \{ \omega^{-1} (1 - \omega^2)^{1/2} \cos(\omega t + \delta) \times \text{sech}[(1 - \omega^2)^{1/2} x] \} \ . \tag{4.1}$$

The frequency ω and the phase shift δ are determined by the ac drive if one regards Eq. (4.1) as an approximation solution to the perturbed equation (1.10) [19].

The asymptotic solution of the linearized equation (1.10), which goes over into that given by Eq. (4.1) when the perturbations are absent, can be found in the form [cf. Eqs. (1.3) and (2.6)]

$$\phi_{\text{as}} = c \exp[-(1 - \omega^2)^{1/2} x] \cos(\omega t - k|x|) \ , \tag{4.2}$$

where $c = 8\omega^{-1}(1 - \omega^2)^{1/2}$, and a constant phase shift is neglected. The wave number k is determined by the dissipative parameter α, provided $\beta = 0$:

$$k = \alpha\omega / 2(1 - \omega^2)^{1/2} \ . \tag{4.3}$$

If $\beta \neq 0$, the parameter α in Eq. (4.3) and in what follows below should be replaced by

$$\alpha_{\text{eff}} \equiv \alpha + \beta(1 - \omega^2) \ . \tag{4.4}$$

Proceeding from the asymptotic solution (4.2), it is easy to arrive at the following estimate for the effective potential of the breather-breather interaction [cf. Eq. (2.11)]:

$$U \sim e^{-(1-\omega^2)^{1/2} z} \cos\phi \cos(kz) \ , \tag{4.5}$$

where z and ϕ are, respectively, the distance and phase shift between the breathers. This potential has a set of local minima at the distances [cf. Eqs. (1.5) and (2.16)]

$$z_n \approx \frac{\pi}{2k}(1 + 2n) \equiv \frac{\pi(1 - \omega^2)^{1/2}}{\alpha\omega}(1 + 2n) \ , \tag{4.6}$$

where $n = 0, 1, 2, \ldots$, and the corresponding values of $\cos\phi$ alternate as $(-1)^n$.

It is well known [20] that, in the limit $1 - \omega^2 \to 0$, the damped ac-driven SG model (1.10) goes over into the damped NS model driven at the complementary frequency $1-\omega$. In this limit, the small-amplitude SG breather (4.1) becomes a NS soliton. The BS's of the solitons in this version of the perturbed NS equation have been considered earlier in Ref. [4].

V. ABSENCE OF BOUND STATES OF PARAMETRICALLY DRIVEN SOLITONS

In damped NS and SG systems, dissipation may be compensated not only by the "direct" drive as in Eq. (1.10), but also by the parametric drive [21]. As the simplest example, one can take the perturbed NS equation similar to that analyzed in Ref. [21]:

$$iu_t+u_{xx}+2|u|^2u=-i\gamma_0 u+i\gamma_1 u_{xx}+i\epsilon e^{2i\omega t}u^* , \quad (5.1)$$

where the asterisk stands for the complex conjugation, γ_0 and γ_1 are positive dissipative constants, and the driving frequency ω is assumed positive too. To start the analysis of the BS problem, one should, as above, linearize Eq. (5.1) and look for its asymptotic solutions in the form of the oscillating tails. However, this time the standard form (1.3) used above is no longer relevant, and the solution must be looked for as follows:

$$u_{\rm as}=\exp(-\sqrt{\omega}|x|+i\omega t)(ae^{-ik|x|}+be^{ik|x|}) , \quad (5.2)$$

with some constants a and b. Insertion of Eq. (5.2) into the linearized equation (5.1) leads to a system of linear equations for the amplitudes a and b, whose resolvability condition yields

$$(2\sqrt{\omega}k)^2=(\gamma_0-\omega\gamma_1)^2-\epsilon^2 . \quad (5.3)$$

Thus, the oscillatory tails, corresponding to a real k in Eq. (5.3), exist provided

$$\epsilon^2<(\gamma_0-\omega\gamma_1)^2 . \quad (5.4)$$

On the other hand, it is well known that the soliton can be supported by the ac drive, provided the drive's amplitude ϵ exceeds a certain threshold value $\epsilon_{\rm thr}$. Following the lines of Ref. [21], where $\epsilon_{\rm thr}$ has been found for $\gamma_1=0$, it is easy to find

$$\epsilon_{\rm thr}=\gamma_0+\tfrac{1}{3}\omega\gamma_1 . \quad (5.5)$$

Obviously, the necessary condition $\epsilon^2>\epsilon_{\rm thr}^2$ following from Eq. (5.5) is in contradiction with the inequality (5.4). Thus, the BS's of the parametrically driven NS solitons are not possible.

The NS model based on Eq. (5.1) may be regarded as the small-amplitude limit of the analogous SG model,

$$\phi_{tt}-\phi_{xx}+\sin\phi=-\alpha\phi_t+\beta\phi_{txx}+\epsilon\sin(2\omega t)\sin(\phi/2) , \quad (5.6)$$

which governs, e.g., dynamics of some magnetic systems [21]. In this model, the ac drive may support breathers at the frequency ω. The asymptotic tail of the breather is looked for in the form [cf. Eq. (5.2)]

$$\phi_{\rm as}=\exp[-(1-\omega^2)^{1/2}x|][a\sin(\omega t-k|x|) + b\sin(\omega t+k|x|)] .$$

Straightforward calculations yield [cf. Eq. (5.3)]

$$4(1-\omega^2)k^2=[\alpha-(1-\omega^2)\beta]^2\omega^2-(\epsilon/4)^2 . \quad (5.7)$$

Although an analytical calculation of $\epsilon_{\rm thr}$ for Eq. (5.6) is not possible, using numerical data presented in Ref. [21] demonstrates that the inequality

$$(\epsilon/4)^2<[\alpha-(1-\omega^2)\beta]^2\omega^2$$

[cf. Eq. (5.4)], following from Eq. (5.7), is likely to contradict the condition $\epsilon^2>\epsilon_{\rm thr}^2$. Thus, the parametrically driven breathers should not be capable either to form BS's.

VI. CONCLUSION

In this paper, I have extended the analysis of bound states (BS's) of envelope solitons initiated in Refs. [4] and [6]. It has been demonstrated that (i) the skew dissipative terms are apt to destroy the BS's; (ii) in the absence of dissipation, solitons may form a BS via emitted radiation; (iii) SG breathers in the standard damped ac-driven model may form BS's quite similarly to the NS solitons; and (iv) in parametrically driven models, the envelope solitons are not capable of forming BS's. It might be interesting to extend the analysis developed in this work to bimodal systems governed by coupled NS (or SG) equations. As it has been demonstrated (for a simplest situation) in Ref. [6], in the bimodal systems, unlike the single-mode ones, solitons belonging to different modes may form BS's due to their purely Hamiltonian interaction in the absence of dissipative and radiative effects.

ACKNOWLEDGMENTS

I appreciate support from the Physics Lab I at the Technical University of Denmark, and from the School of Mathematics at the University of New South Wales, where this work has been done.

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