PHYSICAL REVIEW E VOLUME 47, NUMBER 4 APRIL 1993

# Synchronization of chaotic trajectories using control

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(Received 7 December 1992)

We demonstrate that two identical chaotic systems can be made to synchronize by applying small, judiciously chosen, temporal-parameter perturbations to one of them. This idea is illustrated with a numerical example. Other issues related to synchronization are also discussed.

PACS number(s): 05.45.+b

## I. INTRODUCTION

Chaos is characterized by a sensitive dependence of a system's dynamical variables on the initial conditions. Trajectories starting with slightly different initial conditions diverge from each other exponentially. Consequently, synchronization seems unlikely even for two perfectly identical chaotic systems if trajectories start from initial conditions that differ slightly. Moreover, in practical applications the existence of noise (both external and internal) and system imperfect identification makes the hope of synchronizing two chaotic systems even more remote. Nonetheless, it has been established that [1] synchronization of chaotic dynamical systems is not only possible but it is believed to have potential applications in communication [1] and in providing insight into some neural and biological processes [2].

Previous studies demonstrated [1] that for a certain class of chaotic systems, synchronization can indeed be achieved. Consider a system that can be divided into two subsystems. If one of the subsystems has only negative Lyapunov exponents, trajectories from two such subsystems can then be synchronized provided that the other subsystem (whose largest Lyapunov exponent is positive) is used as a common driving system [1]. This strategy was verified numerically on various chaotic systems including the Lorentz and Rössler systems and it was also experimentally realized on some electrical circuits [1].

In this paper, we address the following question: given two almost identical chaotic systems, can one make a chaotic trajectory of one system to synchronize with a chaotic trajectory of the other system *by control*? Here we do not require that the system under study be divided into subsystems. Moreover, we allow for both noise and a small amount of system parameter mismatch.

Our approach to synchronize chaotic systems is based on the idea of controlling chaos by Ott, Grebogi, and Yorke (OGY) [3]. While the original OGY method was proposed to stabilize *unstable periodic orbits* embedded in the chaotic attractor, we extend it to stabilize a *chaotic trajectory* of one system around a *chaotic trajectory* of the other system to achieve synchronization of the two systems. It should be noted that the idea of stabilizing chaotic orbits by using the OGY method were also proposed by Mehta and Henderson [4]. Their approach is to construct an artificial dynamical system evolving errors between the system's output and the target chaotic orbit. If the artificial system has a zero fixed point, parameter perturbations based on the OGY algorithm are then applied to stabilize the artificial system around its zero fixed point, which means that the original system's output is brought to the desired chaotic orbit [4]. They illustrated their method by using one-dimensional maps. Construction of the artificial map for more general dynamical systems may be nontrivial. In our method, on the other hand, parameter perturbations are applied *directly to the original dynamical system* and the method makes use of the geometrical structure of a chaotic trajectory.

## II. SYNCHRONIZATION

To synchronize two chaotic systems that we call $A$ and $B$, we imagine that some parameter of one system (assume $B$) is externally adjustable. Our strategy is illustrated schematically in Fig. 1, where we assume that some state variables of both systems $A$ and $B$ can be measured. Based on this measurement and our knowledge about the system (we can, for example, observe and learn the system first), when it is determined that the state variables of $A$ and $B$ are close, we calculate a small parameter perturbation based on the OGY algorithm and apply it to system $B$. Two systems can then be synchronized, although their trajectories are still chaotic. Under the influence of external noise, there is a finite probability that two already synchronized trajectories may lose synchronization. However, with probability one (due to the ergodicity of chaotic trajectories), after a finite amount of transient time, trajectories of $A$ and $B$ will get close and can then be synchronized again. In this sense, our synchronization method is robust against small external noise.

 

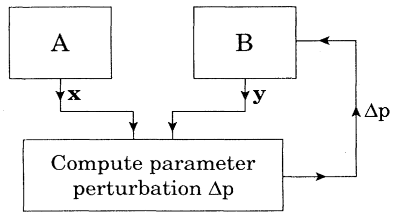


FIG. 1. Schematic illustrations of our strategy to synchronize two chaotic systems. Some dynamical variables of two systems are measured, based on which temporal-parameter perturbations are calculated and applied to the system $B$. We assume that before the synchronization, some information about the geometrical structure of the chaotic attractor (e.g., the Jacobian matrices along a long chaotic trajectory that practically covers the whole attractor) has been obtained.

We consider two *almost identical* chaotic systems that can be described by two-dimensional maps on the Poincaré surface of section

$$\mathbf{x}_{n+1}=\mathbf{F}(\mathbf{x}_n,p_0) \ [A], \quad \mathbf{y}_{n+1}=\mathbf{F}(\mathbf{y}_n,p) \ [B] , \qquad (1)$$

where $\mathbf{x}_n,\mathbf{y}_n \in \mathbb{R}^2$, $\mathbf{F}$ is a smooth function in its variables, $p_0$ for system $A$ is a fixed parameter value, and $p$ for system $B$ is an externally controllable parameter. For the purpose of synchronization, we require that the dynamics should not be substantially different for systems $A$ and $B$. In other words, any parameter perturbations should be small. Thus, we require

$$|p-p_0|<\delta , \qquad (2)$$

where $\delta$ is a small number defining the range of parameter variation. Suppose that two systems start with different initial conditions. In general, the resulting chaotic trajectories are completely uncorrelated. However, due to ergodicity, with probability one two trajectories can get arbitrarily close to each other at some later time $n_c$. Without control, two trajectories will separate from each other exponentially again. Our objective is to program the parameter $p$ in such a way that $|\mathbf{y}_n-\mathbf{x}_n|\rightarrow 0$ for $n \geq n_c$, which means that $A$ and $B$ are synchronized for $n \geq n_c$.

The linearized dynamics in the neighborhood of the "target" trajectory $\{\mathbf{x}_n\}$ is

$$\mathbf{y}_{n+1}-\mathbf{x}_{n+1}(p_0)=\mathbf{J}[\mathbf{y}_n-\mathbf{x}_n(p_0)]+\mathbf{V}(\Delta p)_n , \qquad (3)$$

where $p_n=p_0+(\Delta p)_n$, $(\Delta p)_n \leq \delta$, $\mathbf{J}$ is the $2\times 2$ Jacobian matrix, and $\mathbf{V}$ is a two-dimensional column vector

$$\mathbf{J}=\mathbf{D}_{\mathbf{y}}\mathbf{F}(\mathbf{y},p)|_{\mathbf{y}=\mathbf{x},p=p_0}, \quad \mathbf{V}=\mathbf{D}_p F(\mathbf{y},p)|_{\mathbf{y}=\mathbf{x},p=p_0} . \qquad (4)$$

A property of a chaotic trajectory is the existence of both stable and unstable directions at almost each trajectory point [5,6], which can be seen as follows. Let us choose a small circle of radius $\epsilon$ around some orbit point $\mathbf{x}_n$. The image of a small circle under $\mathbf{F}^{-1}$ is an ellipse at $\mathbf{x}_{(n-1)}$. This deformation from a circle to an ellipse means that distances along the major axis of the ellipse at $\mathbf{x}_{(n-1)}$ contract as a result of the map $\mathbf{F}$. Similarly, a small circle at $\mathbf{x}_{(n-1)}$ maps into an ellipse at $\mathbf{x}_n$ under $\mathbf{F}$, which means that distances along the inverse image of the major axis of the ellipse at $\mathbf{x}_n$ expand under $\mathbf{F}$. Therefore, the forward image of the major axis of the ellipse at $\mathbf{x}_{(n-1)}$ and the major axis of the ellipse at $\mathbf{x}_n$ (image of the small circle at $\mathbf{x}_{(n-1)}$) approximate the stable and unstable directions at $\mathbf{x}_n$, respectively. See Ref. [6] for a systematic algorithm to compute stable and unstable directions for general two-dimensional maps.

Let $\mathbf{e}_{s(n)}$ and $\mathbf{e}_{u(n)}$ be the stable and unstable directions at $\mathbf{x}_n$ and $\mathbf{f}_{s(n)}$ and $\mathbf{f}_{u(n)}$ be two vectors that satisfy $\mathbf{f}_{u(n)}\cdot\mathbf{e}_{u(n)}=\mathbf{f}_{s(n)}\cdot\mathbf{e}_{s(n)}=1$ and $\mathbf{f}_{u(n)}\cdot\mathbf{e}_{s(n)}=\mathbf{f}_{s(n)}\cdot\mathbf{e}_{u(n)}=0$. To stabilize $\{\mathbf{y}_n\}$ around $\{\mathbf{x}_n\}$, we require the next iteration of $\mathbf{y}_n$ after falling into a small neighborhood around $\mathbf{x}_n$ to lie on the stable direction at $\mathbf{x}_{(n+1)}(p_0)$, i.e.,

$$[\mathbf{y}_{n+1}-\mathbf{x}_{(n+1)}(p_0)]\cdot\mathbf{f}_{u(n+1)}=0 . \qquad (5)$$

Substituting Eq. (3) into Eq. (5), we obtain the following expression for the parameter perturbations:

$$(\Delta p)_n=\frac{\{\mathbf{J}\cdot[\mathbf{y}_n-\mathbf{x}_n(p_0)]\}\cdot\mathbf{f}_{u(n+1)}}{-\mathbf{V}\cdot\mathbf{f}_{u(n+1)}} . \qquad (6)$$

It is understood in Eq. (6) that if $(\Delta p)_n>\delta$, we set $(\Delta p)_n=0$.

One advantage of the OGY method is that it does not require any knowledge of system equations [3], although it is necessary to "learn" the system to obtain enough knowledge about the unstable periodic orbits to be stabilized before the control. Here by "knowledge" we mean the Jacobian matrices $\mathbf{J}$ (note that $\mathbf{f}_{u(n)}$ can be calculated in terms of $\mathbf{J}$ [6]) and vector $\mathbf{V}$ in Eq. (6). A time series with the use of delay coordinates is always enough to program the necessary parameter perturbations to stabilize a chaotic trajectory around the unstable periodic orbit. In our synchronization problem, the orbit to be stabilized is chaotic (say, with period $\infty$). In principle, one can still run the system for long enough time to estimate both $\mathbf{J}$ and $\mathbf{V}$ for many trajectory points which practically cover the whole chaotic attractor. While extrapolating both $\mathbf{J}$ and $\mathbf{V}$ from a time series is relatively easy for an unstable periodic orbit [7], it may be very difficult to do so for a long chaotic trajectory. Hence, at present we cannot guarantee that our synchronization scheme is meaningful if we do not know the system well enough. Note that the previous methods [1,4] also require the knowledge of the equations of the system.

## III. NUMERICAL EXAMPLE

We illustrate our synchronization algorithm by using the standard Hénon map: $(x,y)\rightarrow(a-x^2+0.3y,x)$, where $a$ is our control parameter. Consider two such Hénon systems. One has a fixed parameter value ($a=a_0=1.4$) which serves the target and in the other system we adjust $a$ in a small range (1.39,1.41) according to Eq. (6). At time $t=0$, we start two systems with different initial conditions: $(x_1,y_1)=(0.5,-0.8)$ and $(x_2,y_2)=(0.001,0.001)$. Two systems then move on completely uncorrelated chaotic trajectories. At time step

2534, the trajectory points of two systems come close to each other within a circle of radius of 0.01. When this occurs, we turned on the parameter perturbations calculated from Eq. (6). Note that the radius 0.01 above can be changed slightly (without affecting the synchronization) depending on how we define the "synchronization neighborhood" in which two trajectories are considered to be close together. In general, the size of such a neighborhood should be chosen to be proportional to $\delta$, the maximum allowed parameter perturbation [3]. Figure 2(a) shows part of a time series of the uncorrelated and synchronized chaotic trajectories before and after the control is turned on, respectively, where the crosses and diamonds denote values of $x$ for two chaotic trajectories. Clearly, after the control is turned on, crosses and diamonds overlap each other, indicating that the two chaotic Hénon trajectories evolve completely in phase (synchronization), although they are still chaotic. Figure 2(b) shows a time series of $\Delta x(t)=x_2(t)-x_1(t)$, where we see that $\Delta x(t)=0$ after the control is applied.

In the presence of noise, two synchronized trajectories can go uncorrelated again ($x_2$ is "kicked" out of the neighborhood of $x_1$ by the noise). When $\Delta x(t)$ exceeds a critical value, say 0.01, we turn off the control and let the two systems evolve by themselves. Due to ergodicity, the two trajectories will come close again and be synchronized. To model the effect of noise, we add a term $\epsilon\sigma(t)$ to the $x$ component of the two Hénon systems, where $\sigma$ is a random variable with Gaussian probability distribution of zero mean and unit standard deviation, and $\epsilon$ characterizes the noise amplitude. Figures 3(a) and 3(b) show part of the time series of $\Delta x(t)$ for $\epsilon=3.8\times 10^{-4}$ and $\epsilon=4.18\times 10^{-4}$, respectively. Clearly, the smaller the noise amplitude is, the longer the two systems can remain synchronized.

In stabilizing unstable periodic orbits, the average transient time to achieve the control is shown to scale with the maximum allowed parameter perturbation $\delta$ as $\tau\sim\delta^{-\gamma}$, where $\gamma$ is given in terms of the stable and unstable eigenvalues ($\lambda_s$ and $\lambda_u$) of the unstable periodic orbit by [3,8]

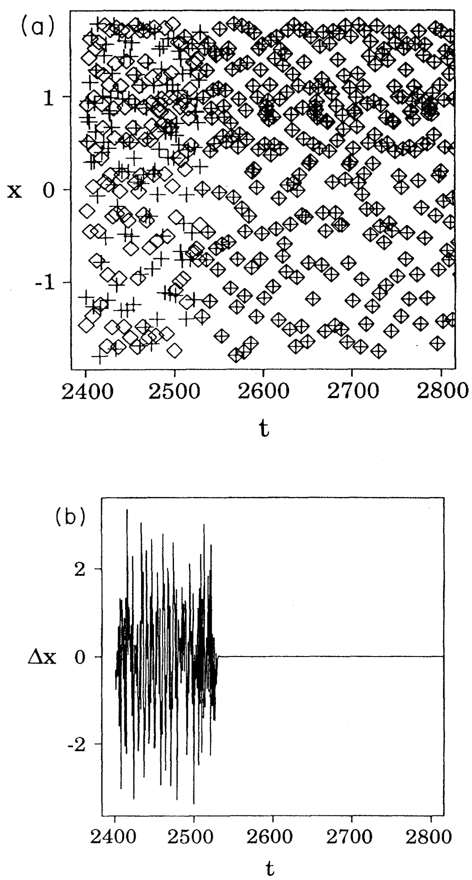


FIG. 2. Synchronizing two Hénon systems $[(x,y)\to(a-x^2+0.3y,x)]$. In system $A$, the parameter $a$ is fixed at $a_0=1.4$. In system $B$, $a$ is allowed to vary in [1.39,1.41]. (a) The uncorrelated and synchronized chaotic trajectories of the two systems before and after the parameter control are turned on, and (b) part of the time series of difference $\Delta x=x_2-x_1$ corresponding to (a). The synchronization neighborhood is chosen to be a circle of radius 0.01 (see text).

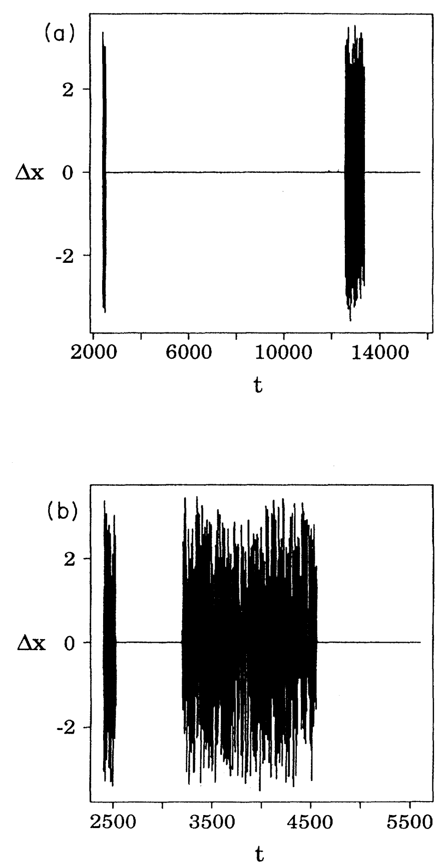


FIG. 3. Influence of noise [of the form $\epsilon\sigma(t)$, where $\sigma(t)$ is a Gaussian random variable having zero mean and unit standard deviation, and $\epsilon$ is the noise amplitude] on synchronized orbits. (a) $\epsilon=3.8\times 10^{-4}$ and (b) $\epsilon=4.18\times 10^{-4}$. It is clear that noise can make the synchronized orbits uncorrelated by kicking one orbit out of the neighborhood of the other orbit.

$$\gamma = 1 - \ln|\lambda_u| / \ln|\lambda_s| \ , \tag{7}$$

if the controlling neighborhood is chose to be a circle. In our case of synchronization, such a scaling relation still holds, as shown in Fig. 4 for the standard Hénon map, where we plot the average time (with respect to 200 random pairs of initial conditions) to achieve synchronization versus $\delta$ on a logarithmic scale. The absolute value of the slope of the line is the scaling exponent $\gamma$, which is approximately 1.225 for Fig. 4. Following the same argument as in Refs. [3,8], it is easy to see that $\gamma$ is still given by Eq. (7) with $\lambda_s$ and $\lambda_u$ denoting the stable and unstable Lyapunov numbers of a *typical chaotic trajectory*. For the standard Hénon map, we found that $\gamma \approx 1.27$ in terms of Eq. (7), which agrees reasonably well with that extrapolated from the straight line in Fig. 4. Note that the average time to achieve synchronization increases drastically as $\delta$ is decreased. For $\delta \sim 10^{-2}$ in the Hénon map, $\tau \sim 10^3$ [see Figs. 2(a), 2(b), and 4]. For stabilizing unstable periodic orbits, it has been demonstrated that the average time to achieve control can be greatly reduced by applying small controls to the orbit outside the control neighborhood. This technique is known as "targeting" [9]. Note that in such a case, the target (the unstable periodic orbit) is always fixed, while in our synchronization problem, the target moves chaotically because both trajectories wander on the chaotic attractor and the actual location at which two trajectories get close together depends sensitively on the pair of initial conditions and the size of the synchronization neighborhood. How to reduce effectively the time to achieve synchronization is at present an unsolved problem.

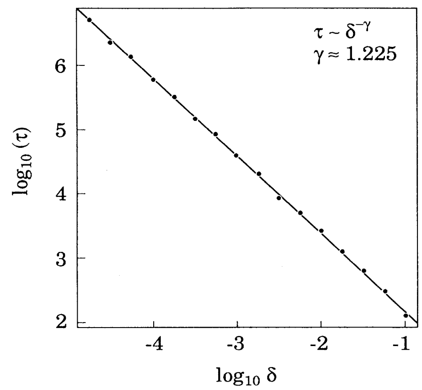


FIG. 4. Average time to achieve synchronization $\tau$ vs the size of the synchronization neighborhood $\delta$ on a log-log plot. Note that $\tau \sim \delta^{-\gamma}$, where $\gamma$ is the absolute value of the straight line in the figure.

## IV. CONCLUSION

We present an algorithm to achieve synchronization of two chaotic systems by applying small, temporally programmed parameter perturbations to one of the systems. Our method makes use of the stable and unstable directions of chaotic trajectories to achieve synchronization. Furthermore, it is robust even in the presence of small noise.


## ACKNOWLEDGMENT

This work is supported by the U.S. Department of Energy (Scientific Computing Staff, Office of Energy Research).



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