Radiation by solitons due to higher-order dispersion

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We consider the Korteweg–de Vries (KdV) and nonlinear Schrödinger (NS) equations with higher-order derivative terms describing dispersive corrections. Conditions of existence of stationary and radiating solitons of the fifth-order KdV equation are obtained. An asymptotic time-dependent solution to the latter equation, describing the soliton radiation, is found. The radiation train may be in front as well as behind the soliton, depending on the sign of dispersion. The change rate of the soliton due to the radiation is calculated. A modification of the WKB method, that permits one to describe in a simple and general way the radiation of KdV and NS, as well as other types of solitons, is developed. From the WKB approach it follows that the soliton radiation is a result of a tunneling transformation of the nonlinearly self-trapped wave into the free-propagating radiation.

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I. INTRODUCTION

A large variety of nonlinear structures have been described by means of reduced equations obtained in different asymptotic limits, such as Korteweg–de Vries (KdV), nonlinear Schrödinger (NS), Kadomtsev-Petviasvili (KP) equations, etc. It appears, however, that by using such equations we lose, sometimes, rather important effects that can change the physical picture.

As an example, we mention a radiation emerging from the self-focusing wave beams in magnetized plasmas [1–3] and other gyrotropic media [4], which eventually leads to defocusing. It has been shown in the mentioned papers that the radiation is caused by a transformation of the nonlinearly self-trapped wave into an outgoing untrapped mode. The radiation mechanism is similar to the tunneling in quantum mechanics and, therefore, it was called the tunneling transformation [1]. This process does not follow from the NS equation because the untrapped mode is lost in the NS equation. It is described by the full system of Maxwell equations [1–4]. Similar results are obtained if one adds higher-order dispersive terms to the NS equation [5]. Evidently, the radiation of nonlinear structures due to the tunneling transformation may take place in any number of dimensions. In particular, one can expect that the tunneling transformation should appear, under certain conditions, in one-dimensional nonlinear systems described by higher-order KdV and NS equations. In the simplest case, they are of the forms

$$\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial u^2}{\partial x}+\beta\frac{\partial^3 u}{\partial x^3}+\gamma\frac{\partial^5 u}{\partial x^5}=0\ , \tag{1}$$

$$i\frac{\partial \psi}{\partial t}+\frac{1}{2}\frac{\partial^2 \psi}{\partial x^2}+|\psi|^2\psi=i\gamma\frac{\partial^3 \psi}{\partial x^3}\ . \tag{2}$$

These equations have been studied in many papers in connection with different physical phenomena. Equation (1) arises in dispersive fluid dynamics (e.g., shallow water and plasma waves), while (2) arises in nonlinear optics and fluid mechanics.

As was first shown numerically by Kawahara [6], Eq. (1), in certain domains of parameters, has stationary soliton solutions. An interesting analytical example of a stationary soliton, satisfying Eq. (1), was given by Nozaki [7]. On the other hand, one can ask what happens with the KdV solitons at small γ. Pomeau, Ramani and Grammaticos [8], using a modification of the asymptotic method of Segur and Kruskal [9], have shown that in this case the solitons emit radiation, and have obtained an expression for the radiation field, which is beyond all orders of the perturbation theory based on expansion in powers of γ.

Radiation of NS solitons, satisfying Eq. (2) at small γ, was first found numerically by Wai *et al.* [10], and then analytically by Wai, Chen, and Lee [11], using again the "beyond all orders" approach based on Ref. [9]. Their results were confirmed by another, but still rather cumbersome, method, by Kuehl and Zhang [12].

In the present paper we show that the radiation of KdV and NS solitons is, actually, a tunneling-transformation effect similar to that found in Refs. [1–5] for multidimensional systems with high-order dispersion. First, however, we consider conditions of existence of radiating and stationary solitons. The most definite conclusions can be obtained for the higher-order KdV equations and, therefore, we restrict our analysis to Eq. (1) (Sec. II). For Eq. (2), the analytical criteria are not so expressive and, perhaps, a numerical investigation, similar to that of Kawahara [6], would be more productive.

In the other part of Sec. II we derive, by a "direct" approach, an asymptotic time-dependent solution to Eq. (1) describing the radiation of the KdV soliton. We show that the radiation train may be in front as well as behind the soliton, depending on the signs of the coefficients in (1). Using the obtained expression, we then calculate in a strict way the change rate of the soliton amplitude due to the emitting radiation. In the stationary limit, our solution turns into the one obtained by Pomeau, Ramani, and

Grammaticos [8], except for an amplitude factor $K \approx 20$, obtained in Ref. [8] numerically. The results obtained are useful for comparison with the tunneling transformation.

In Sec. III, we extend to Eqs. (1) and (2) our theory of the tunneling transformation [1]. This is done by means of a modified WKB approach that leads, in a simple and general way, to expressions describing radiation of the KdV and NS solitons. The derived formulas confirm solutions obtained previously by means of much more cumbersome calculations. Besides that, our approach gives a physical insight into the soliton radiation, showing that this process is a result of the transformation of a nonlinearity self-trapped wave into the free-propagating radiation, similar to what was found in the theory of self-focusing [1–5].

II. RADIATION BY THE KdV SOLITONS DUE TO HIGHER-ORDER DISPERSION

The subject of this section is the nonlinear structures satisfying Eq. (1) with small γ and approaching the KdV solitons at $\gamma \to 0$. For brevity, we call them KdV solitons. It will be shown that, at $\gamma\beta > 0$, they radiate and, therefore, are nonstationary. On the other hand, Eq. (1) may have stationary-soliton solutions and, therefore, one must distinguish between solitons (which may be nonstationary) and stationary solitons. We start with several general comments.

The stationary-soliton solution of Eq. (1) can be written as

$$u = aF(\xi) , \tag{3}$$

where

$$\xi = |a/\beta|^{1/2}[x - x_0(t)] , \tag{4}$$

$$\frac{dx_0}{dt} = a , \quad x_0(0) = 0 , \tag{5}$$

and a = const. Substituting (3) into (1) and integrating one time, we come to the equation

$$\mathrm{sgn}(\gamma\beta)\epsilon^2 \frac{d^4F}{d\xi^4} + \frac{d^2F}{d\xi^2} + \mathrm{sgn}(\beta a)(\tfrac{1}{2}F^2 - F) = 0 , \tag{6}$$

where

$$\epsilon = |\gamma a|^{1/2} |\beta|^{-1} . \tag{7}$$

One can show that soliton solutions of Eq. (6) are even functions: $F(\xi) = F(-\xi)$. Their asymptotic behavior at $|\xi| \to \infty$ is found by neglecting the nonlinear term in (6). Therefore,

$$F(\xi) \approx \sum_\kappa A_\kappa \exp(-\kappa|\xi|) , \quad |\xi| \to \infty , \tag{8}$$

where κ are roots of the quartic equation

$$\mathrm{sgn}(\gamma\beta)\epsilon^2\kappa^4 + \kappa^2 - \mathrm{sgn}(\beta a) = 0 , \tag{9}$$

with $\mathrm{Re}\kappa > 0$ and, generally, A_κ are constants (linear functions of ξ) if κ are simple (multiple) roots. Numerical investigation of Eq. (6) has been performed by Kawahara [6], who came to the following conclusions: soliton solutions of Eq. (6) with the asymptotic behavior (8) exist in the cases

$$(1) \quad \gamma\beta > 0 , \quad a\beta > 0 , \quad \epsilon < \epsilon_0 \sim 1 , \tag{10a}$$

with

$$\kappa = \frac{1}{\sqrt{2}\epsilon}[(1+4\epsilon^2)^{1/2} - 1]^{1/2} , \tag{10b}$$

$$(2) \quad \gamma\beta > 0 , \quad a\beta < 0 , \quad \epsilon > \tfrac{1}{2} , \tag{11a}$$

with

$$\kappa = \frac{1}{2\epsilon}(\sqrt{2\epsilon - 1} \pm i\sqrt{2\epsilon + 1}) ; \tag{11b}$$

$$(3) \quad \gamma\beta < 0 , \quad \beta a > 0 , \quad 0 < \epsilon < \infty , \tag{12a}$$

with

$$\kappa = \frac{1}{2\epsilon}(\sqrt{1+2\epsilon} \pm \sqrt{1-2\epsilon}) . \tag{12b}$$

In case (12), for $\epsilon = \frac{6}{13}$, and negative sign in (12b), an analytical solution of Eq. (1) has been found. It is of the form [7]

$$u = \tfrac{35}{12} a \,\mathrm{sech}^4 \left[\frac{\sqrt{13}}{12} \xi \right] ,$$

with

$$a = -\frac{36\beta^2}{169\gamma} .$$

On the other hand, necessary conditions for the existence of stationary-soliton solutions to Eq. (1) can be derived in a very simple way by considering plane-wave solutions to the linearized Eq. (1). Their phase velocities are

$$V(k) = -\beta k^2 + \gamma k^4 , \tag{13}$$

where k is a wave number. From the plots of $V(k)$ shown in Figs. 1 and 2, it is seen that the fifth-derivative term in Eq. (1) may bring sufficient qualitative effects at $\gamma\beta > 0$. Indeed, in this case the behavior of $V(k)$ changes drastically at $k > k_m$, where $V(k_m) = V_m$ is the minimum (at $\gamma > 0$) or maximum (at $\gamma < 0$) of $V(k)$. From (13), we have

$$k_m = (\beta/2\gamma)^{1/2} , \quad V_m = -\beta^2/4\gamma . \tag{14}$$

Evidently, the velocity of a stationary soliton cannot be equal to the phase velocity of any linear free-propagating mode; i.e., the equation

$$V(k) = a \tag{15}$$

cannot have real roots k. Otherwise, the soliton would resonantly interact with the wave with phase velocity $V(k)$ and, therefore, it cannot be stationary. Then, from Figs. 1 and 2 and Eq. (14), one deduces that the necessary conditions of the existence of stationary-soliton solutions to Eq. (1) coincide with (11a) or (12a).

We see that at conditions (10a), contrary to Ref. [6], stationary solitons of Eq. (1) cannot exist. Instead, there

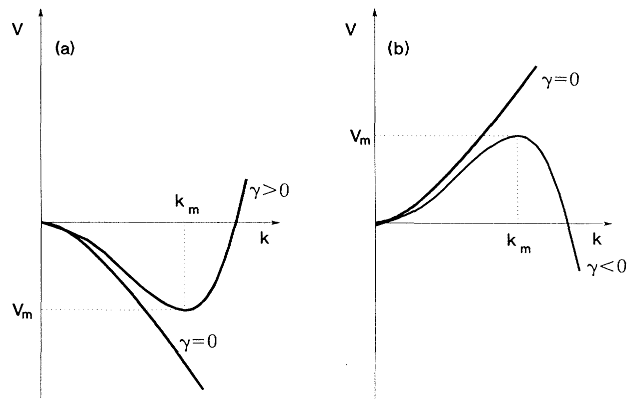

FIG. 1. Plots of $V(k)$ at $\gamma\beta\geq 0$. (a) $\beta>0$; (b) $\beta<0$.

may be radiating solitons, which can have a long lifetime at sufficiently small ϵ.

To obtain quantitative results for the case (10a), we assume that

$$\epsilon^2 \ll 1 , \tag{16}$$

and look for a solution to Eq. (1) of the form

$$u(x,t)=a[f_0(\xi)+f(\xi,\tau)] , \tag{17}$$

with

$$\xi=|a/\beta|^{1/2}[x-x_0(t)] , \quad \frac{dx_0}{dt}=a , \tag{18a}$$

$$\tau=|a/\beta|^{1/2}x_0(t)\operatorname{sgn}\beta , \tag{18b}$$

and

$$f_0(\xi)=3\,\operatorname{sech}^2(\xi/2) . \tag{19}$$

For $a=$const, $af_0(\xi)$ is the soliton solution of the KdV equation, i.e., Eq. (1) with $\gamma=0$. Actually, $a=a(t)$ [contrary to (5), where $a=$const]. Due to (16), $a(t)$ is a slow function of t. Moreover, it will be seen from the final results that $d(\ln a)/dt$ is exponentially small and, therefore, all corrections containing da/dt are neglected throughout this section (even if they are multiplied by powers of t). Then, substituting (17) into (1), we come to the following equation for $f(\xi,\tau)$ in the case (10a)

$$(\operatorname{sgn}\beta)\frac{df}{\partial\tau}-\frac{\partial f}{\partial\xi}+\frac{1}{2}\frac{\partial(f^2)}{\partial\xi}+\frac{\partial(f_0f)}{\partial\xi}+\frac{\partial^3 f}{\partial\xi^3}+\epsilon^2\frac{\partial^5 f}{\partial\xi^5}=-\epsilon^2\frac{\partial^5 f_0}{\partial\xi^5} . \tag{20}$$

The function $f(\xi,\tau)$ is a small quasistationary addition to the KdV soliton profile (19). It has a part $f_r(\xi,\tau)$, describing resonant radiation of the soliton. We are mostly interested in this part. The main contribution to its Fourier transform comes from the harmonics $\exp[i(q\xi-\Omega\tau)]$ with small Ω, because they correspond to plane waves moving with velocities close to the soliton velocity. Taking $|\xi|\to\infty$ and neglecting the nonlinear term in (20), we obtain the dispersion relation in the soliton frame

$$\Omega+q+q^3-\epsilon^2q^5=0 . \tag{21}$$

Introducing the phase velocity in this frame

$$\tilde{V}=\Omega/q , \tag{22}$$

one obtains the following expression:

$$q^2=\frac{1}{2\epsilon^2}\{1\pm[1+4(1+\tilde{V})\epsilon^2]^{1/2}\} . \tag{23}$$

For harmonics close to the resonance,

$$|\tilde{V}|\ll 1 .$$

Therefore, only the plus in (23) gives a real q, and we obtain for the resonant wave number

$$q_r\approx\pm\frac{1}{\epsilon} . \tag{24}$$

The branch in question does not exist at $\epsilon=0$.

Performing the Fourier transform of Eq. (20) with the notations

$$\varphi(q,\tau)=\int_{-\infty}^{\infty}d\xi\, f(\xi,\tau)\exp(-iq\xi) , \tag{25}$$

$$\varphi_0(q)=\int_{-\infty}^{\infty}d\xi\, f_0(\xi)\exp(-iq\xi) , \tag{26}$$

we obtain

$$\pm i\frac{\partial\varphi(q,\tau)}{\partial\tau}-\Omega(q)\varphi(q,\tau)=\varepsilon^2q^5\varphi_0(q)+\frac{q}{2\pi}\left[\int_{-\infty}^{\infty}dq'\,\varphi_0(q-q')\varphi(q',\tau)+\frac{1}{2}\int_{-\infty}^{\infty}dq'\,\varphi(q-q',\tau)\varphi(q',\tau)\right] , \tag{27}$$

where the sign before i coincides with that of β, and

$$\Omega(q)=\epsilon^2q^5-q^3-q . \tag{28}$$

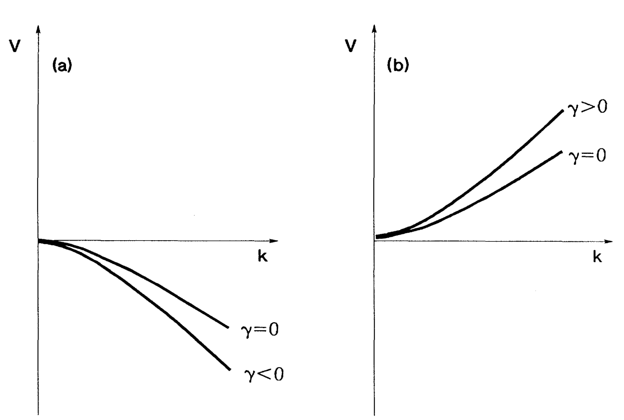

FIG. 2. Plots of $V(k)$ at $\gamma\beta\leq 0$. (a) $\beta>0$; (b) $\beta<0$.

According to (21),

$$\Omega(q_r)=0 \ . \tag{29}$$

Substituting (19) into (26), we have

$$\varphi_0(q)=12\pi q \operatorname{csch}(\pi q) \ . \tag{30}$$

As the initial condition to Eq. (27), we take

$$\varphi(q,0)=0 \ . \tag{31}$$

We are interested in $\varphi(q,\tau)$ near the resonance ($q\approx q_r$). In this region, $\varphi(q,\tau)$ is large but, nevertheless, the nonlinear term in (27) can be neglected. This is justified in the Appendix. Thus, Eq. (27) can be replaced by

$$\pm i\frac{\partial\varphi(q,\tau)}{\partial\tau}-\Omega(q)\varphi(q,\tau) =\epsilon^2 q^5\varphi_0(q)+\frac{q}{2\pi}\int_{-\infty}^{\infty}dq'\,\varphi_0(q-q')\varphi(q',\tau) \ . \tag{32}$$

Introducing the new unknown function

$$\Phi(q,\tau)=\varphi(q,\tau)\exp[\pm i\Omega(q)\tau] \ , \tag{33}$$

we have

$$\pm i\frac{\partial\Phi(q,\tau)}{\partial\tau}-\frac{q}{2\pi}\int_{-\infty}^{\infty}dq'\,\varphi_0(q-q')\Phi(q') \times\exp\{\pm i[\Omega(q)-\Omega(q')]\tau\} =\epsilon^2 q^5\varphi_0(q)\exp[\pm i\Omega(q)\tau] \ . \tag{34}$$

The right-hand side of this equation is a driving term. It is the most essential in the resonant region where, according to (29), $\Omega(q)$ is small. In this region, we may write

$$\Omega(q)\approx\Omega'(q_r)(q-q_r)+\tfrac{1}{2}\Omega''(q_r)(q-q_r)^2 \approx\left[\frac{2}{\epsilon^2}\right](q-q_r)+\left[\frac{7}{\epsilon}\right](q-q_r)^2\approx\left[\frac{2}{\epsilon^2}\right](q-q_r) \ , \tag{35a}$$

where

$$\operatorname{sgn}q_r=\operatorname{sgn}q \ . \tag{35b}$$

Therefore, instead of (34), we write

$$\pm i\frac{\partial\Phi(q,\tau)}{\partial\tau}-\frac{q}{2\pi}\int_{-\infty}^{\infty}dq'\,\varphi_0(q-q')\Phi(q,\tau) \times\exp\left[\pm\frac{2i}{\epsilon^2}(q-q')\tau\right] =\epsilon^2 q^5\varphi_0(q)\exp\left[\pm\frac{2i}{\epsilon^2}(q-q_r)\tau\right] \ , \tag{36}$$

where (35b) is assumed. Equation (36) can be solved by successive approximations. In the lowest approximation, we neglect the second term in the left-hand side of (36) and, taking into account (31) and (33), we have $\varphi(q,\tau)\approx\varphi^{(0)}(q,\tau)$, where

$$\varphi^{(0)}(q,\tau)=\tfrac{1}{2}\epsilon^4 q^5\varphi_0(q)\frac{\exp[\mp i(2\tau/\epsilon^2)(q-q_r)]-1}{q-q_r} \ . \tag{37}$$

From (37) is seen, in particular, that the width of the resonant region is

$$\Delta q=\epsilon^2/2\tau \ . \tag{38}$$

It decreases with time as τ^{-1}.

The complete asymptotic solution to Eq. (32) for $\tau\gg\epsilon^2/2$ [which, according to (38), means $\Delta q\ll 1$] is derived in the Appendix. It has the form

$$\varphi(q,\tau)=\psi_\pm(q)\frac{\exp[\mp i(2\tau/\epsilon^2)(q-q_r)]-1}{q-q_r} \ , \tag{39a}$$

where, for $\epsilon\ll\frac{1}{3}$,

$$\psi_\pm(q)=\tfrac{1}{2}\epsilon^4 q^5\varphi_0(q)\mp 3\pi i\exp(-\pi/\epsilon)|q|\varphi_0(q-q_r) \ , \tag{39b}$$

and the upper (lower) sign is taken for $\beta>0$ ($\beta<0$).

From (25) and (39a), we obtain the following asymptotic expression for the radiation emitted by the soliton in its reference frame:

$$f(\xi,\tau)=\pm\{[\psi_\pm(1/\epsilon)-\psi_\pm(-1/\epsilon)]\sin(\xi/\epsilon) -i[\psi_\pm(1/\epsilon)+\psi_\pm(-1/\epsilon)]\cos(\xi/\epsilon)\} \times g_\pm(\xi,\tau) \ , \tag{40}$$

where

$$g_+(\xi,\tau)=1 \ , \quad 0<\xi<2\tau/\epsilon^2 \ , \qquad =0 \ , \quad \xi<0 \ , \quad \xi>2\tau/\epsilon^2 \ , \tag{41a}$$

$$g_-(\xi,\tau)=1 \ , \quad -2\tau/\epsilon^2<\xi<0 \ , \qquad =0 \ , \quad \xi>0 \ , \quad \xi<-2\tau/\epsilon^2 \ . \tag{41b}$$

The derivation of (40) is given at the end of the Appendix. Substituting (39b) into (40), we come to

$$f(\xi,\tau)=\pm 24\pi\epsilon^{-2}\exp(-\pi/\epsilon)[\sin(\xi/\epsilon)-3\epsilon\cos(\xi/\epsilon) +O(\epsilon^2)]g_\pm(\xi,\tau) \ . \tag{42}$$

One can see that the main term in (42) comes from the lowest approximation (37) and may be obtained if one replaces (37) by

$$\psi^{(0)}(q_r)\frac{\exp[\mp i(2\tau/\epsilon^2)(q-q_r)]-1}{q-q_r} \ , \tag{43a}$$

where

$$\psi^{(0)}(q_r)=\tfrac{1}{2}\epsilon^4 q_r^5\varphi_0(q_r)\approx 12\pi\epsilon^{-2}\exp(-\pi/\epsilon)\operatorname{sgn}q_r \ . \tag{43b}$$

[See (A15) and (A21).] Note the connection of this result with the vanishing of the width of the resonant region at $\tau/\epsilon^2\to\infty$.

In the units used, the group velocity of the resonant radiation is

$$V_g(q_r)=(\text{sgn}\beta)\Omega'(q_r)=\pm 2/\epsilon^2$$

[the sign of $V_g(q_r)$ is in agreement with Fig. 1]. Thus, the factor $g_{\pm}(\xi,\tau)$ in (40) describes the propagation of the radiation edge with the group velocity. Returning to the original variables (x,t), we see that the radiation train is ahead of the soliton for $\beta>0$ [Fig. 1(a)] and behind it for $\beta<0$ [Fig. 1(b)]. At $\tau\to\infty$, from (42) and (41) follows

$$f(\xi,\infty)\approx\pm 24\pi\epsilon^{-2}\exp(-\pi/\epsilon)\sin(\xi/\epsilon)\Theta(\beta\xi)\ , \quad (44)$$

where

$$\Theta(z)=1\ ,\quad z>0\ ,$$
$$=0\ ,\quad z<0\ . \quad (45)$$

At $\beta>0$, this coincides, except for a numerical factor, with the expression found by Pomeau, Ramani, and Grammaticos [8]. (Their amplitude contains an extra factor $K\approx 20$, found numerically.) The attenuation rate of the soliton amplitude, caused by the radiation, can be found from the conservation law

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty}u^2\,dx=0\ ,$$

which follows from Eq. (1). Substituting here (17)–(19) and (42), and taking into account that, at $\tau\gg\epsilon^2/2$, the radiation edge is well separated from the soliton, we have

$$\frac{d\ln a}{d\tau}\approx -32\pi^2\epsilon^{-6}\exp(-2\pi/\epsilon)\ , \quad (46)$$

where ϵ and τ are given by (7) and (18). Equation (46) defines $a(t)$. The change rate, estimated in Ref. [8], is proportional to ϵ^{-4}, instead of ϵ^{-6} in (46).

Thus, we have shown that, at

$$\gamma\beta>0\ ,\quad a\beta>0\ ,$$

the stationary-soliton solutions of Eq. (1) do not exist. Instead, we found radiating solitons that turn, at $\epsilon\to 0$, into the KdV solitons (19). Evidently, the effect of the soliton radiation and, respectively, the change rate of the soliton amplitude, described by (42) and (46), is beyond all orders of the perturbation theory based on expansion in powers of ϵ. In other cases, (11a) and (12a), Eq. (1) has stationary-soliton solutions, found numerically by Kawahara [6].

III. SOLITON RADIATION AS A TUNNELING

In this section we consider the soliton radiation by another approach, from which follows that the radiation is a result of a tunneling of the self-trapped wave field due to its transformation into the free-propagating radiation, similar to what takes place at the self-focusing [1–5].

Starting with Eq. (1), let us consider its solution in the form (17) with initial conditions at $\tau=-\infty$ (instead of $\tau=0$ as in Sec. II): $f(\xi,-\infty)=0$. Then one can expect a quasisteady state at finite τ, with τ entering only through a and, respectively, ϵ. In the lowest approximation we may assume $\epsilon=$const. The nonlinearly self-trapped wave is described by Eq. (19); it is the KdV soliton. The free-propagating radiation in the presence of the soliton is described by the linearized Eq. (20) without the right-hand side, i.e.,

$$(\text{sgn}\beta)\frac{\partial f}{\partial\tau}-\frac{\partial f}{\partial\xi}+\frac{\partial(f_0 f)}{\partial\xi}+\frac{\partial^3 f}{\partial\xi^3}+\epsilon^2\frac{\partial^5 f}{\partial\xi^5}=0\ .$$

We look for a solution to this equation of the form

$$f(\xi,\tau)=e^{-i\Omega\tau}f(\xi)\ .$$

To satisfy the initial condition at $\tau\to-\infty$, we assume that Ω has an infinitesimal positive imaginary part, i.e.,

$$\Omega=\Omega_0+i\delta\ ,\quad \delta\to+0\ .$$

The wave, resonantly interacting with the soliton, must be stationary in the soliton frame. Therefore, we assume $\Omega_0=0$. Then $f(\xi)$ satisfies the equation

$$\frac{d}{d\xi}\left[\epsilon^2\frac{d^4 f(\xi)}{d\xi^4}+\frac{d^2 f(\xi)}{d\xi^2}+[f_0(\xi)-1]f(\xi)\right]+\delta(\text{sgn}\beta)f(\xi)=0\ . \quad (47)$$

Looking for solutions of (47) in the WKB approximation, we come to expressions, proportional to

$$\exp\left[i\int^{\xi}q(\xi')d\xi'\right]\ . \quad (48)$$

From (47) and (48) follows that $q(\xi)$ satisfies the equation

$$\epsilon^2 q^4-q^2+[f_0(\xi)-1-i\mu]=0\ , \quad (49)$$

where $\mu=(\delta/q)\text{sgn}\beta$. In the WKB approach, $q(\xi)$ should be large enough and, therefore, we can assume that μ is real and

$$\mu\to 0\ ,\quad \text{sgn}\mu=\text{sgn}(\beta\,\text{Re}q)\ . \quad (50)$$

The solution of (49) is

$$q^2=\frac{1}{2\epsilon^2}\{1\pm[1+4\epsilon^2(1-f_0+i\mu)]^{1/2}\}\ . \quad (51)$$

The negative sign in (51) gives "regular" roots approaching, at $\epsilon\to 0$ and $\mu\to 0$, the solutions of the quadratic equation

$$q^2-f_0(\xi)+1=0\ .$$

They do not fit the conditions of applicability of the WKB approximation. The branch of (51) with the positive sign gives

$$q_{1,2}=\pm\frac{1}{\epsilon}\{1+\tfrac{1}{2}[1-f_0(\xi)+i\mu]\epsilon^2+O(\epsilon^4)\}\ . \quad (52a)$$

To the lowest order, (51) leads to the resonant wave numbers (24) with a small imaginary addition. In this case the WKB approximation is justified (at least for real ξ). Retaining the principal terms in (51) and taking into account (50), we write

$$q_{1,2}\approx\pm 1/\epsilon+i\delta\,\text{sgn}\beta\ ,\quad \delta\to+0\ . \quad (52b)$$

Observing that

$$f_0(\xi)=3\exp[-2\ln\cosh(\xi/2)]$$
$$=3\exp\left[-\int_0^{\xi}\tanh(\xi'/2)d\xi'\right],$$

and introducing the notation for the square brackets in (17)

$$\Phi(\xi)=f_0(\xi)+f(\xi)\ , \qquad (53)$$

one can write

$$\Phi(\xi)=3\exp\left[i\int_0^{\xi}Q(\xi')d\xi'\right]$$
$$+\left\{C_1\exp\left[i\int_0^{\xi}q_1(\xi')d\xi'\right]\right.$$
$$\left.+C_2\exp\left[i\int_0^{\xi}q_2(\xi')d\xi'\right]\right\}\Theta(\beta\xi)\ , \qquad (54)$$

where

$$Q(\xi)=i\tanh(\xi/2)+O(\epsilon^2)\ , \qquad (55)$$

$q_{1,2}$ are given by (52) with $\delta=0$, and $\Theta(z)$ is defined in (45). The term containing $\Theta(\beta\xi)$ describes the emitted radiation for the present initial condition. Indeed, according to (52b) with a finite but infinitesimal δ, the factors (48) vanish at $\xi\to\infty$ $(\xi\to-\infty)$ for $\beta>0$ $(\beta<0)$, which corresponds to the outgoing waves. On the contrary, these factors tend to infinity at $\xi\to-\infty$ $(\xi\to\infty)$, which corresponds to ingoing radiation. Thus, at $\delta=0$, the outgoing waves should be expressed by the terms containing $\Theta(\beta\xi)$.

To find C_1 and C_2, consider the behavior of the exponents in (54) in the complex plane of ξ, along the contours L_1 and L_2, rounding roots of the equations

$$Q(\xi_1)=q_1(\xi_1)\ ,\quad Q(\xi_2)=q_2(\xi_2)\ , \qquad (56)$$

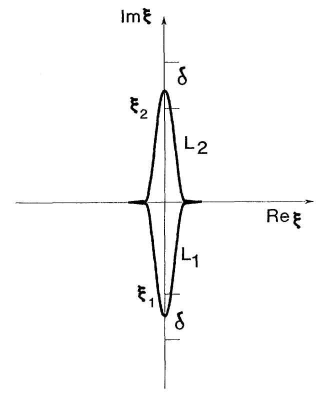

FIG. 3. Contours of integration in the complex plane ξ; ξ_1 and ξ_2 are branch points.

closest to the real axis (Fig. 3). ξ_1 and ξ_2 are branch points, where the first term of (54) turns into the second and third ones and vice versa [1]. It is easy to see that ξ_1 and ξ_2 are located on the imaginary axis. Defining

$$\xi=i\eta\ ,\quad \xi_{1,2}=i\eta_{1,2}\ , \qquad (57)$$

we have, to the lowest order of ϵ,

$$\tan(\eta_{1,2}/2)=\mp 1/\epsilon\ .$$

Thus,

$$\eta_1=-\pi+\delta\ ,\quad \eta_2=\pi-\delta\ , \qquad (58a)$$

$$\delta\approx 2\epsilon\ . \qquad (58b)$$

Integrating in the first term of (54) along the contour L_2, we have

$$3\exp\left[i\int_{L_2}Q(\xi')d\xi'\right]=3\exp\left[i\int_0^{\xi_2}Q(\xi)d\xi+i\int_{\xi_2}^{0}q_2(\xi)d\xi\right]\exp\left[i\int_0^{\xi}q_2(\xi')d\xi'\right]$$
$$=3\exp\left[\int_0^{\eta_2}[q_2(i\eta)-Q(i\eta)]d\eta\right]\exp\left[i\int_0^{\xi}q_2(\xi')d\xi'\right]\ . \qquad (59)$$

Thus, the first term of (54) is transformed into the third one [the second term is lost in this transition because it rapidly vanishes when one moves from the real axis to the upper half plane (cf. Ref. [13]). From (59), it follows that

$$C_2=3\exp\left[\int_0^{\eta_2}[q_2(i\eta)-Q(i\eta)]d\eta\right]\ . \qquad (60)$$

Substituting here

$$q_2\approx-1/\epsilon\ ,\quad Q(i\eta)\approx-\tan(\eta/2)\ , \qquad (61)$$

we have

$$C_2\sim(3e^2/\epsilon^2)\exp(-\pi/\epsilon)\ . \qquad (62)$$

The same result is obtained if, instead of L_2, one integrates along a contour rounding both singularities, $\xi=i\pi$ and $\xi=\xi_2$. A similar integration along the contour L_1 gives

$$C_1=C_2\ . \qquad (63)$$

For $\beta<0$, one should integrate along L_1 and L_2 in the opposite direction.

The sign $\sim$ in (62) means that this relation has been established up to a constant complex factor $Ce^{i\alpha}$, independent of ϵ. This is because we have restricted ourselves by the main powers of ϵ. The next-order terms, being negligibly small for real ξ, become of order 1 near $\xi=i\pi$, and this leads to a different coefficient in (58b). Besides that, an imaginary part in (58b) appears. This leads to an additional complex factor in (62) (cf. with a similar situation in the overbarrier reflection in quantum mechanics [13]). Introducing $C\exp(i\alpha)$ into (62), we come to the following expression for the soliton radiation:

$$f(\xi)=C(6e^2/\epsilon^2)\exp(-\pi/\epsilon)\cos(\xi/\epsilon+\alpha)\Theta(\beta\xi) \ . \tag{64}$$

A comparison with (44) gives

$$C\approx 1.7 \ , \quad \alpha=-\pi/2 \ .$$

Thus, the present approach gives, except for an amplitude factor of order 1 and a phase shift, a correct expression for soliton radiation, showing that the radiation is a result of the transformation of the self-trapped wave into the free-propagating radiation (the tunneling transformation).

In a similar way we can treat Eq. (2). At $\gamma=0$, it has the NS soliton solution of the form

$$\psi(x,t)=a\varphi_0(\xi)\exp(i\tau/2) \ , \tag{65}$$

where $\varphi_0(\xi)$ is a real function,

$$\xi=ax \ , \quad \tau=a^2t \ , \tag{66}$$

and $\varphi_0(\xi)$ satisfies the equation

$$\frac{d^2\varphi_0(\xi)}{d\xi^2}+[2\varphi_0^2(\xi)-1]\varphi_0(\xi)=0 \ , \tag{67}$$

with the regular solution

$$\varphi_0(\xi)=\operatorname{sech}\xi \ , \tag{68}$$

describing the soliton with the amplitude a and width a^{-1}. At $\gamma\neq 0$, we look for a solution of the form

$$\psi(x,t)=a(\tau)\Phi(\xi,\tau)\exp(i\tau/2) \ , \tag{69}$$

with ξ defined in (66), and

$$\tau=\int^t a^2(t')dt' \ .$$

At small γ we can ignore, in the first approximation, all terms with time derivatives of a. Then we have the following equation for $\Phi(\xi,\tau)$:

$$2i\frac{\partial\Phi}{\partial\tau}+\frac{\partial^2\Phi}{\partial\xi^2}+(2|\Phi|^2-1)\Phi=i\epsilon\frac{\partial^3\Phi}{\partial\xi^3} \ , \tag{70}$$

where

$$\epsilon=2\gamma a \ll 1 \ . \tag{71}$$

Without loss of generality, we assume $\epsilon>0$ and write

$$\Phi(\xi,\tau)=\varphi_0(\xi)+\delta\varphi_0(\xi)+\varphi(\xi,\tau) \ , \tag{72}$$

where $\varphi_0(\xi)$ is given by (68), $\delta\varphi_0$ is a small steady correction to the $\operatorname{sech}\xi$ shape of the soliton, satisfying the equation

$$\frac{\partial^2\delta\varphi_0}{\partial\xi^2}+(2\operatorname{sech}^2\xi-1)\delta\varphi_0-i\epsilon\frac{\partial^3\delta\varphi_0}{\partial\xi^3}=i\epsilon\frac{\partial^3\varphi_0}{\partial\xi^3} \ , \tag{73}$$

and $\varphi(\xi,\tau)$ is a free-propagating radiation, produced by the soliton. As before, we take the initial condition $\varphi(\xi,-\infty)=0$ and, therefore, at finite τ one can expect a quasisteady wave function for a resonant radiation. To find it, we assume

$$\varphi=\varphi(\xi)\exp(\mu t) \ , \quad \mu\to+0 \ .$$

Then, from (70)–(73) and (67), we obtain the following equation for $\varphi(\xi)$:

$$\frac{\partial^2\varphi}{\partial\xi^2}+(2\operatorname{sech}^2\xi-1+2i\mu)\varphi=i\epsilon\frac{\partial^3\varphi}{\partial\xi^3} \ . \tag{74}$$

In the WKB approximation, we have

$$\varphi(\xi)=C\exp\left[i\int_0^\xi k(\xi')d\xi'\right] \ , \tag{75}$$

where $k(\xi)$ satisfies the cubic equation

$$\epsilon k^3+k^2-(2\operatorname{sech}^2\xi-1+2i\mu)=0 \ . \tag{76}$$

If $\epsilon\ll 1$, one obtains

$$k(\xi)\approx -1/\epsilon+O(\epsilon)+i\nu \ , \quad \nu\to+0 \ . \tag{77}$$

Two other roots are beyond the WKB approximation.

Now, following the approach used above, we write the soliton solution in the lowest approximation as

$$\varphi_0(\xi)=\exp(-\ln\cosh\xi)=\exp\left[i\int_0^\xi Q(\xi')d\xi'\right] \ , \tag{78}$$

where

$$Q(\xi)=i\tanh\xi \ , \tag{79}$$

and look for the branch point ξ_0, which is a root of the equation

$$Q(\xi_0)=k(\xi_0) \ , \tag{80}$$

closest to the real axis. The root is purely imaginary:

$$\xi_0=i\eta_0 \ , \quad \eta_0=\frac{\pi}{2}-\delta \ , \quad \delta\approx\epsilon \ . \tag{81}$$

Unlike the case of Eq. (1), we now have only one branch point, located in the upper half plane. Transforming the integral in (78) into the one along the contour L_2 (Fig. 2, with $\xi_2\to\xi_0$), we write

$$\exp\left[i\int_{L_2}Q(\xi')d\xi'\right]=\exp\left[-\int_0^{\eta_0}Q(i\eta)d\eta\right]\exp\left[-\int_{\eta_0}^0 k(i\eta)d\eta\right]\exp\left[i\int_0^\xi k(\xi')d\xi'\right]$$

$$=\exp\left[\int_0^{\eta_0}(\tan\eta-1/\epsilon)d\eta\right]\exp\left[i\int_0^\xi k(\xi')d\xi'\right] \ . \tag{82}$$

One sees that the wave function of the nonlinear self-trapped wave (68) is transformed into that of the free radiation (75), with the amplitude

$$C \sim \exp\left[\int_0^{\eta_0}(\tan\eta - 1/\epsilon)d\eta\right] = e\epsilon^{-1}\exp(-\pi/2\epsilon) . \quad (83)$$

Here C, similar to C_2 in (62), is defined up to a constant, independent of ϵ, for the same reason. The integration along a contour rounding both singularities, $\xi = i\pi/2$ and $\xi = \xi_2$, gives (83) again. Taking into account (71), we finally have the wave function of the free radiation in the form ($\gamma > 0$)

$$\varphi(x) = Be(2\gamma a)^{-1}\exp(-\pi/4a\gamma)\exp(-ix/2\gamma a)\Theta(x) , \quad (84)$$

where B is a constant and $\Theta(x)$ is given by (45). Expression (84) coincides with the results obtained by (much more cumbersome) direct methods [11,12]. The time dependence of the amplitude $a(t)$ can be calculated from the conservation of

$$\int_{-\infty}^{\infty} |\Phi(x,t)|^2 dx ,$$

with the same results as in Refs. [11,12] (except for a numerical factor).

IV. DISCUSSION

The higher-order dispersive effects may cause significant qualitative changes in the dynamics of nonlinear structures in different space dimensions. We have shown that the solitons described by the fifth-order KdV and third-order NS equations may radiate due to a transformation of the nonlinearly self-trapped wave into a free-propagating mode having the same phase velocity as the soliton. This is a tunneling effect, similar to the tunneling transformation found in multidimensional systems [1–5]. In one dimension, it resembles the soliton Čerenkov radiation, like that suggested for vortices in inhomogeneous plasmas or rotating fluids [14], but for higher space dimensions this analogy may not hold.

The approach and results of Sec. III, as well as the conclusions following from Eq. (15), may be extended to the solitons of the modified KdV equations with nonlinear terms $u^p \partial u/\partial x$ ($p = 2,3$) [15]. For $p \geq 4$, as well as for the modified NS equations with nonlinear terms $|\psi|^{2s}\psi$, $s \geq 2$, the solitons are unstable with respect to collapse-type phenomena [16]. The role of radiation in these cases will be studied in a separate paper.

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APPENDIX: SOLUTION TO EQ. (32)

Looking for the solution of Eq. (32) by the successive approximation method, we write

$$\varphi(q,\tau) = \sum_{n=0}^{\infty} \varphi^{(n)}(q,\tau) . \quad (A1)$$

To simplify the equations, we first consider the case $\beta > 0$. Then $\varphi^{(0)}$ is given by (37), with a minus in the exponent, and higher approximations are defined by the recursive equation

$$i\frac{\partial\Phi^{(n+1)}(q,\tau)}{\partial\tau} = \frac{q}{2\pi}\int_{-\infty}^{\infty} dq'\varphi_0(q-q')\exp[i(2\tau/\epsilon^2)(q-q')] \times \Phi^{(n)}(q',\tau) , \quad (A2)$$

$$\varphi^{(n)}(q,\tau) = \Phi^{(n)}(q,\tau)\exp[-i(2\tau/\epsilon^2)(q-q_r)] , \quad n = 0,1,2,\ldots . \quad (A3)$$

Assuming

$$\epsilon^2/2\tau \ll 1 , \quad (A4)$$

we look for the solution of (A2) in the form

$$\Phi^{(n)}(q,\tau) = \psi^{(n)}(q)\frac{1-\exp[i(2\tau/\epsilon^2)(q-q_r)]}{q-q_r} . \quad (A5)$$

A simple analysis of (A2), confirmed by the results, shows that the main contribution to the integral comes from q' sufficiently close to q. Then, substituting (A5) into (A2), we have

$$i\frac{\partial\Phi^{(n+1)}(q,\tau)}{\partial\tau} = \frac{q}{2\pi}I_n(q,\tau)\exp\left[i\frac{2\tau}{\epsilon^2}(q-q_r)\right] , \quad (A6)$$

where

$$I_n(q,\tau) = \int_{-\infty}^{\infty} dq'\varphi_0(q-q')\psi^{(n)}(q') \times \frac{\exp[-i(2\tau/\epsilon^2)(q'-q_r)]-1}{q'-q_r} , \quad (A7)$$

and $\mathrm{sgn}q_r = \mathrm{sgn}q$. Introducing a new variable of integration

$$p = (2\tau/\epsilon^2)(q'-q_r) , \quad (A8)$$

we have

$$I_n(q,\tau) = \int_{-\infty}^{\infty} dp\, \varphi_0\left[q-q_r-\frac{\epsilon^2}{2\tau}p\right] \times \psi^{(n)}\left[q_r+\frac{\epsilon^2}{2\tau}p\right]\frac{\exp(-ip)-1}{p} . \quad (A9)$$

Taking into account (A4), we obtain

$$I_n(q,\tau)\approx I_n(q)\equiv -i\pi\varphi_0(q-q_r)\psi^{(n)}(q_r)\ , \tag{A10}$$

where we have used

$$\int_{-\infty}^{\infty} dp\frac{1-\exp(-ip)}{p}=i\int_{-\infty}^{\infty} dp\frac{\sin p}{p}=i\pi\ . \tag{A11}$$

Substituting (A5) and (A10) into (A6), we have

$$\psi^{(n+1)}(q)=\frac{\epsilon^2}{4i}q\varphi_0(q-q_r)\psi^{(n)}(q_r)\ ,\quad n=0,1,2,\ldots\ . \tag{A12}$$

From (A5), (A3), and (37) follows

$$\psi^{(0)}(q)=\tfrac{1}{2}\epsilon^4 q^5\varphi_0(q)\ . \tag{A13}$$

Substituting $q=q_r$ into (A12) and (A13), we obtain

$$\psi^{(n+1)}(q_r)=\frac{\epsilon^2}{4i}q_r\varphi_0(0)\psi^{(n)}(q_r)\ , \tag{A14}$$

$$\psi^{(0)}(q_r)=\tfrac{1}{2}\epsilon^4 q_r^5\varphi_0(q_r)\ . \tag{A15}$$

Defining $\psi(q)$ by

$$\psi(q)=\sum_{n=0}^{\infty}\psi^{(n)}(q)\ , \tag{A16}$$

we write

$$\psi(q)=\psi^{(0)}(q)+\sum_{n=0}^{\infty}\psi^{(n+1)}(q)\ . \tag{A17}$$

Using (A12), we have

$$\psi(q)=\psi^{(0)}(q)+\frac{\epsilon^2}{4i}q\varphi_0(q-q_r)\sum_{n=0}^{\infty}\psi^{(n)}(q_r)\ . \tag{A18}$$

From (A14) and (A15) follows

$$\begin{aligned}\sum_{n=0}^{\infty}\psi^{(n)}(q_r)&=\frac{1}{2\epsilon}\frac{\varphi_0(q_r)}{1+i(\epsilon^2/4)q_r\varphi_0(0)}\\&=\frac{1}{2\epsilon}\frac{\varphi_0(q_r)(1-3i\epsilon\,\mathrm{sgn}q_r)}{1+9\epsilon^2}\mathrm{sgn}q_r\\&\approx\frac{12\pi}{\epsilon^2}\exp\left[-\frac{\pi}{\epsilon}\right]\frac{1-3i\epsilon\,\mathrm{sgn}q_r}{1+9\epsilon^2}\mathrm{sgn}q_r\ ,\end{aligned} \tag{A19}$$

where we have used (24), (35b), and

$$\varphi_0(0)=12\ , \tag{A20}$$

$$\varphi_0(q_r)\approx 24\pi\epsilon^{-1}\exp(-\pi/\epsilon)\ . \tag{A21}$$

Substituting (A19) into (A18) and assuming

$$3\epsilon\ll 1\ , \tag{A22}$$

we have

$$\psi(q)=\psi^{(0)}(q)-3\pi i\exp(-\pi/\epsilon)|q|\varphi_0(q-q_r)\ . \tag{A23}$$

Up to now, we assumed $\beta>0$. For $\beta<0$, we must replace $i\to -i$. Using (A3), (A5), (A16), (A23), and (A1), and considering $\beta\gtrless 0$, we come to expressions (39). To calculate the free radiation field $f(\xi,\tau)$, we substitute (39a) into

$$f(\xi,\tau)=\int_{-\infty}^{\infty}\frac{dq}{2\pi}\varphi(q,\tau)\exp(iq\xi)\ . \tag{A24}$$

Introducing again the integration variable p given by (A8), with q instead of q', we have

$$\begin{aligned}f(\epsilon,\tau)=&\int_{-2\tau/\epsilon^3}^{\infty}\frac{dp}{2\pi}\psi_\pm\left[\frac{1}{\epsilon}+\frac{\epsilon^2}{2\tau}p\right]\frac{\exp\left[ip\left[\frac{\epsilon^2\xi}{2\tau}\mp 1\right]\right]-\exp\left[i\frac{\epsilon^2\xi}{2\tau}p\right]}{p}\exp(i\xi/\epsilon)\\&+\int_{-\infty}^{2\tau/\epsilon^3}\frac{dp}{2\pi}\psi_\pm\left[-\frac{1}{\epsilon}+\frac{\epsilon^2}{2\tau}p\right]\frac{\exp\left[ip\left[\frac{\epsilon^2\xi}{2\tau}\mp 1\right]\right]-\exp\left[i\frac{\epsilon^2\xi}{2\tau}p\right]}{p}\exp(-i\xi/\epsilon)\ .\end{aligned}$$

Due to (A4), this can be transformed to

$$f(\epsilon,\tau)=[\psi_\pm(1/\epsilon)\exp(i\xi/\epsilon)+\psi_\pm(-1/\epsilon)\exp(-i\xi/\epsilon)]J_\pm\ , \tag{A25}$$

where

$$J_\pm=\int_{-\infty}^{\infty}\frac{dp}{2\pi}\frac{\exp\left[ip\left[\frac{\epsilon^2\xi}{2\tau}\mp 1\right]\right]-\exp\left[ip\frac{\epsilon^2\xi}{2\tau}\right]}{p}\ . \tag{A26}$$

Here the upper (lower) sign is taken for $\beta>0$ ($\beta<0$). Calculating (A26) and substituting the result into (A25), we come to expressions (40) and (41). Now, let us estimate the nonlinear term in (27). Denote

$$N(q,\tau)=\tfrac{1}{2}\int_{-\infty}^{\infty}dq'\varphi(q-q',\tau)\varphi(q',\tau)\ . \tag{A27}$$

Substituting here (37), assuming (A4) and considering, for definiteness, $\beta>0$, we have, after some algebra,

$$N(q,\tau)\approx 12\pi\epsilon^{-2}\exp(-\pi/\epsilon)\int_0^\infty dq'\left\{\varphi(q-q',\tau)\frac{\exp[-i(2\tau/\epsilon^2)(q'-|q_r|)]-1}{q'-|q_r|}+\varphi(q+q',\tau)\frac{\exp[i(2\tau/\epsilon^2)(q'-|q_r|)]-1}{q'-|q_r|}\right\}.$$

Substituting here $\varphi(q\mp q')$ from (37) and assuming $q\approx\pm|q_r|$, one can see, after averaging over fast oscillations, that $N(q,\tau)$ is negligible in comparison with the neighboring linear integral in (27), which is approximately equal to I_0, from (A10). At $q=q_r$, we have

$$I_0(q_r)\approx\frac{144\pi^2}{i\epsilon^2}\exp\left[-\frac{\pi}{\epsilon}\right]\mathrm{sgn}q_r\ .$$

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