Mean first-passage times for systems driven by gamma and McFadden dichotomous noise

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We consider mean first-passage times (MFPT's) for systems driven by non-Markov gamma and McFadden dichotomous noises. A simplified derivation is given of the underlying integral equations and the theory for ordinary renewal processes is extended to modified and equilibrium renewal processes. The exact results are compared with the MFPT for Markov dichotomous noise and with the results of Monte Carlo simulations.

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I. INTRODUCTION

The study of mean first-passage times (MFPT's) in systems driven by dichotomous noise has been a topic of great interest in recent years [1–6]. The most common system investigated is of the form

$$\dot{x}(t)=f(x)+F(t)\;, \tag{1}$$

where $f(x)$ is a smooth function and $F(t)$ is a dichotomous noise process. Although existing theory has, in large part, been derived for the most general situation in which the noise is non-Markovian, analytical difficulties arise in applying the results to the non-Markov case. The exception to this is the free case $f(x)=0$ because the governing equations greatly simplify and can usually be solved. However, there are almost no cases of bound processes $f(x)\neq 0$ in which a complete solution has been obtained. For example, Masoliver, Lindenberg, and West [2] investigate a first-order system with a linear restoring force driven by a dichotomous noise whose intervals are governed by a rectangular probability density function and are able to obtain the MFPT valid only over restricted parameter ranges. The goal of the present paper is to obtain MFPT's for two cases which are distinctly non-Markovian and which do not appear, to the authors' knowledge, to have been previously studied. These are dichotomous noises constructed from ordinary renewal processes with intervals governed by a gamma probability density and with intervals governed by a McFadden probability density function. Although the theory can be extended to dichotomous noises constructed from general renewal processes, we show how to extend our particular results only to the equilibrium renewal case by means of a simple expression.

Almost all previous work done for systems like (1) with gamma or McFadden dichotomous noises has been in connection with the closely related problems of finding the steady-state probability density function of $x(t)$ or of finding quantities such as the average numbers of level crossings of $x(t)$ [7–11]. The methods and techniques used to obtain these quantities are of interest because of the insight they provide in studying the MFPT problem. In addition, there are similarities in the governing equations and the means for solving them.

One approach is to apply the stochastic trajectory analysis technique (STAT) [2] which has recently been applied successfully to study bistability driven by dichotomous noise [3]. Although we essentially employ the results of STAT, we rederive these results in a much neater and more compact way which does not require detailed examination of all the system trajectories. It is also possible to formulate these types of problems starting with Fokker-Planck or master equations [4–6]; however, it has been our experience that the STAT has an apparent advantage when it comes to determining boundary conditions for the correct solution. Also, the Fokker-Planck or master-equation methods appear to work best for Markov noise processes.

The gamma and McFadden dichotomous noises are discussed in Sec. II. Derivations of the integral equations for the MFPT are presented in Sec. III. The MFPT for a linear system driven by the McFadden dichotomous noise is obtained in Sec. IV and the MFPT for the gamma dichotomous noise in Sec. V. Section VI extends the results to equilibrium processes. Brief mention is made of the associated probability density functions for the steady-state system in Sec. VII and Sec. VIII discusses and summarizes the results.

II. GAMMA AND McFADDEN DICHOTOMOUS NOISES

The only types of dichotomous noise which we consider here are those resulting from a binary process whose times of state occupation are a renewal process [12] whose intervals are independent of one another and independent of the state of the dichotomous noise at the transition times. As a matter of simplicity, the two states of the dichotomous noise are taken to be ± 1 with the generalization to arbitrary states more or less straightforward. Although the generalization to dichotomous noises with different densities for the two states is not quite as obvious, it will not be considered here. In addition, there are dichotomous noises which cannot be described by an underlying renewal process like, for example, the one resulting from hard-limiting of a Gaussian process [13]. Such additional complexities will not be addressed here either.

The only quantities necessary to completely specify the dichotomous noise are then the starting value of $F(t)$ at $t=0$ and the probability density function of the state switching times, $\psi(t)$. When this density has the exponential form $\psi(t)=a\exp(-at)$, we have the case of Markov dichotomous noise for which there are many known results in the literature. We desire a departure from the exponential form but only to a degree which will enable us to maintain mathematical tractability in our analyses. The following two densities provide this flexibility with some measure of success.

A. Gamma dichotomous noise

In this case the intervals of the dichotomous noise have the probability density function

$$\psi(t)=\frac{a^{N+1}t^{N}e^{-at}}{\Gamma(N+1)}, \quad t>0 . \tag{2}$$

For general values of the parameter N, nothing much further can be said. However, when N is an integer, the gamma density has the nice property that it can be regarded as an Nth-order convolution of exponential densities, i.e.,

$$\psi(t)=ae^{-at}*ae^{-at}*\cdots*ae^{-at} , \tag{3}$$

with N convolutions.

If we conceptualize the dichotomous noise as a force acting upon a particle, say in accordance with (1), then the implication of the multiple convolution operation is that the force essentially "kicks" the particle $N+1$ times in the same direction with each "kick" corresponding to an exponential density. Herein lies the added complication and interest accorded by a gamma dichotomous noise. A closely related noise is the McFadden dichotomous noise.

B. McFadden dichotomous noise

McFadden was one of the first to consider the problem of determining the steady-state probability density function of a first-order linear system driven by a non-Markov dichotomous noise [7]. Through what essentially amounted to an educated guess, he was able to completely solve this problem when the interval density of the dichotomous noise has the form

$$\psi(t)=\frac{e^{-at}(1-e^{-t})^{b-a-1}}{B(a,b-a)}, \quad t>0, b\geq a+1 \tag{4}$$

where $B(\ ,\ )$ denotes the beta function. As in the case of the gamma density, when a and b differ by an integer, we can once again represent the overall density as a multiple convolution. If $b=a+N+1$, we have

$$\psi(t)=ae^{-at}*(a+1)e^{-(a+1)t}*\cdots*(a+N)e^{-(a+N)t} . \tag{5}$$

Once again, the hypothetical particle being driven by the

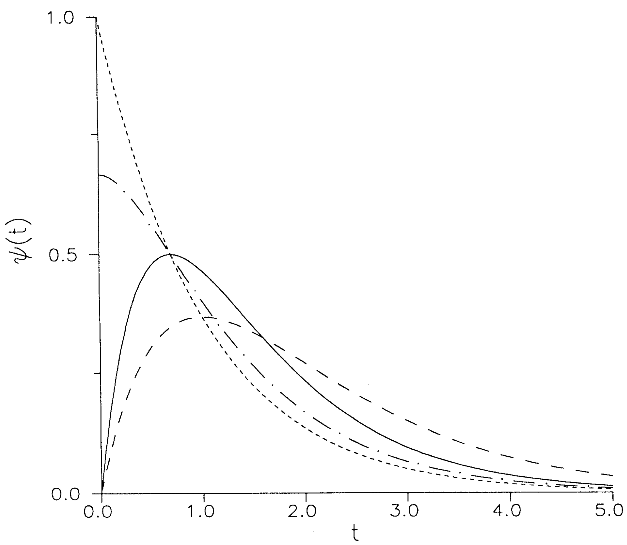

FIG. 1. Illustrations of the exponential (dotted line), gamma (dashed line), and McFadden (solid line) interval probability density functions. The first interval probability density function for the equilibrium McFadden renewal process is also shown (dashed-dotted line).

dichotomous noise undergoes multiple kicks with each kick being an exponential but successive kicks having diminishing mean application times.

The importance of the McFadden noise lies in the mathematical tractability it affords when attempting to apply the STAT theory to calculate the MFPT. As will be seen in Sec. VII, the steady-state probability density function is also much simpler for the McFadden dichotomous noise than it is for the gamma dichotomous noise. In the following, we will treat in detail only the case $N=1$ for the gamma dichotomous noise and the case $b=a+2$ for the McFadden noise. Because the analysis is simpler for the McFadden case, although still complicated, it will be treated first. The gamma and McFadden interval probability density functions are illustrated in Fig. 1 along with the exponential density.

III. INTEGRAL EQUATIONS FOR THE MFPT

The governing integral equations for the MFPT are derived below in a simpler way than originally done in developing STAT. Since the system (1) can start at some arbitrary time which may not correspond to the time of a switch in $F(t)$, the first interval will be treated differently and its probability density function referred to as $\phi(t)$. Then the switching intervals constitute a modified renewal process [12]. There are two special cases of interest. If $\phi(t)=\psi(t)$ we have an *ordinary renewal process* and when

$$\phi(t)=\frac{1}{\mu}\int_t^\infty \psi(\tau)d\tau , \tag{6}$$

where $\mu=$mean time between switches, we have an *equilibrium renewal process* [12].

Let us now define the first-passage time probability density as follows: $p(t;x_0)dt$ is defined as the probability that the process $X(t)$, given that $X(0)=x_0$, crosses z_1 or z_2 in the time range $(t,t+dt)$ without ever having crossed either of these levels during the time span $[0,t]$. The existence of two realizations of $F(t)$ then leads us to define the two conditional probability densities,

$$p^{\pm}(t;x_0)\equiv p(t;x_0|F(0)=\pm 1) . \quad (7)$$

If we assume that $F(0)=\pm 1$ with equal probability, then

$$p(t;x_0)=\tfrac{1}{2}[p^{+}(t;x_0)+p^{-}(t;x_0)] . \quad (8)$$

Assuming also that switching times are governed by a modified renewal process, we can write

$$p^{+}(t;x_0)=\delta(t-\tau_1)\int_{\tau_1}^{\infty}dt'\phi(t') + \int_0^{\tau_1}dt'\phi(t')p_{\rm ord}^{-}(t-t',x_1) , \quad (9)$$

where

$$\tau_1\equiv\int_{x_0}^{z_1}\frac{dx}{f(x)+1} \quad (10)$$

is the *ballistic time* to reach the upper boundary under $F(t)=+1$, and $x_1\equiv x(t'|x_0)$ is the distance traveled by the system during t', starting from x_0 and with $F(t)=+1$. Equation (9) is easily derived from the consideration that a crossing event occurs either during the first time interval [i.e., before the first switch in $F(t)$] or in later intervals (i.e., after the first switch). Thus, the first term on the right-hand side (rhs) of (9) assures that the crossing event has taken place at $t=\tau_1$ before the first switch in $F(t)$. The second term on the rhs of (9) comes from the following: If a crossing event has not occurred prior to the first switch then the switch has necessarily taken place just after a time interval t', which is less than the ballistic time τ_1 to reach the upper boundary [recall we start with $F(0)=+1$]. Moreover, when a switch from $+1$ to -1 takes place in t', a new renewal process is generated, but now this process is an ordinary renewal process and hence with the density $p_{\rm ord}^{-}(t,x_1)$. Recalling that we are dealing now with an ordinary renewal process and following the above line of reasoning, it is not difficult to see that $p_{\rm ord}^{-}(t,x_1)$ obeys the integral equation

$$p_{\rm ord}^{-}(t,x_1)=\delta(t-\tau_2)\int_{\tau_2}^{\infty}dt''\psi(t'') + \int_0^{\tau_2}dt''\psi(t'')p_{\rm ord}^{+}(t-t'',x_2) , \quad (11)$$

where τ_2 is the ballistic time to reach the lower boundary z_2 under $F(t)=-1$ and starting from x_1, i.e.,

$$\tau_2\equiv\int_{x_1}^{z_2}\frac{dx}{f(x)-1} , \quad (12)$$

and $x_2\equiv x(t''|x_1)$ is the distance traveled by the system during t'' under $F(t)=-1$ and starting from x_1. By combining (11) and (9), we obtain

$$p^{+}(t;x_0)=q^{+}(t;x_0) + \int_0^{\tau_1}dt'\phi(t') \times\int_0^{\tau_2}dt''\psi(t'')p_{\rm ord}^{+}(t-t'-t'',x_2) , \quad (13)$$

where

$$q^{+}(t;x_0)\equiv\delta(t-\tau_1)\int_{\tau_1}^{\infty}dt'\phi(t') + \int_0^{\tau_1}dt'\phi(t')\delta(t-t'-\tau_2)\int_{\tau_2}^{\infty}dt''\psi(t'') . \quad (14)$$

When $\phi(t)=\psi(t)$, then $p^{+}(t;x_0)=p_{\rm ord}^{+}(t;x_0)$ and (13) reads

$$p_{\rm ord}^{+}(t;x_0)=q_{\rm ord}^{+}(t;x_0) + \int_0^{\tau_1}dt'\psi(t') \times\int_0^{\tau_2}dt''\psi(t'')p_{\rm ord}^{+}(t-t'-t'',x_2) , \quad (15)$$

where $q_{\rm ord}^{+}(t;x_0)$ is given by (14) with $\phi(t)=\psi(t)$.

Equation (15) is a closed integral equation for $p_{\rm ord}^{+}(t;x_0)$. Once we obtain $p_{\rm ord}^{+}(t;x_0)$ from (15) its substitution into (13) allows us to find $p^{+}(t;x_0)$. The time Laplace transform of (13) is

$$\hat{p}^{+}(s;x_0)=\hat{q}^{+}(s;x_0) + \int_0^{\tau_1}dt'e^{-st'}\phi(t') \times\int_0^{\tau_2}dt''e^{-st''}\psi(t'')\hat{p}_{\rm ord}^{+}(s;x_2) , \quad (16)$$

where

$$\hat{q}^{+}(s;x_0)=e^{-s\tau_1}\int_{\tau_1}^{\infty}dt'\phi(t') + \int_0^{\tau_1}dt'\phi(t')e^{-s(t'+\tau_2)}\int_{\tau_2}^{\infty}dt''\psi(t'') . \quad (17)$$

We can now obtain an integral equation for the MFPT using the relation

$$T^{+}(x_0)=-\frac{\partial}{\partial s}\hat{p}^{+}(s;x_0)|_{s=0} . \quad (18)$$

We finally get

$$T^{+}(x_0)=\rho^{+}(x_0)+\int_0^{\tau_1}dt'\phi(t')\int_0^{\tau_2}dt''\psi(t'')T_{\rm ord}^{+}(x_2) \quad (19)$$

and

$$T_{\rm ord}^{+}(x_0)=\rho_{\rm ord}^{+}(x_0)+\int_0^{\tau_1}dt'\psi(t')\int_0^{\tau_2}dt''\psi(t'')T_{\rm ord}^{+}(x_2) , \quad (20)$$

where

$$\rho^+(x_0)=\tau_1\int_{\tau_1}^{\infty}dt'\phi(t') + \int_0^{\tau_1}dt'\phi(t')(t'+\tau_2)\int_{\tau_2}^{\infty}dt''\psi(t'') + \int_0^{\tau_1}dt'\phi(t')\int_0^{\tau_2}dt''(t'+t'')\psi(t'') \quad (21)$$

and $\rho^+_{\rm ord}(x_0)$ equals $\rho^+(x_0)$ with $\phi(t)=\psi(t)$.

Following a completely analogous reasoning we can obtain the integral equation satisfied by $p^-(t;x_0)$ and $T^-(x_0)$. The result is

$$T^-(x_0)=\rho^-(x_0)+\int_0^{\tilde\tau_1}dt'\phi(t')\int_0^{\tilde\tau_2}dt''\psi(t'')T^-_{\rm ord}(x_2) , \quad (22)$$

where

$$\tilde\tau_1\equiv\int_{x_0}^{z_2}\frac{dx}{f(x)-1} \text{ and } \tilde\tau_2\equiv\int_{\tilde x_1}^{z_1}\frac{dx}{f(x)+1} ,$$

where $\tilde x_1\equiv x(t'|x_0)$ is the distance traveled by the system during t' starting from x_0 and with $F(t)=-1$ and

$$\rho^-(x_0)=\tilde\tau_1\int_{\tilde\tau_1}^{\infty}dt'\phi(t') + \int_0^{\tilde\tau_1}dt'\phi(t')(t'+\tilde\tau_2)\int_{\tilde\tau_2}^{\infty}dt''\psi(t'') + \int_0^{\tilde\tau_1}dt'\phi(t')\int_0^{\tilde\tau_2}dt''(t'+t'')\psi(t'') . \quad (23)$$

Finally, the complete MFPT is given by

$$T_{z_2,z_1}(x_0)=\tfrac{1}{2}[T^+(x_0)+T^-(x_0)] . \quad (24)$$

The integral equations for $T^{\pm}(x_0)$ can be converted to differential equations by differentiation. The method for doing this is outlined in Appendix A. We now apply these results to the McFadden and gamma dichotomous noises.

IV. MFPT FOR McFADDEN DICHOTOMOUS NOISE

In this section we will obtain the MFPT for a McFadden dichotomous noise with $b=a+2$ and we will assume that we are dealing with an ordinary renewal process. Since much of our analyses will be concerned with the starting value $F(0)=+1$, we will frequently delete the superscript defining the MFPT and for brevity write $T(x_0)=T^+(x_0)$. Also, the starting value of the trajectory x_0 will always be taken to lie between two of the system's asymptotically fixed stable points x_s^- and x_s^+ which the trajectory can never cross. Then, substituting the McFadden interval density

$$\psi(t)=a(a+1)e^{-at}(1-e^{-t}) \quad (25)$$

into the integral equation for the MFPT and performing the necessary differentiations and limits leads to the ordinary differential equation (see Appendix A)

$$[\mathcal{L}_4\mathcal{L}_3\mathcal{L}_2\mathcal{L}_1-a^2(a+1)^2]T(x_0)=2a(a+1)(2a+1) , \quad (26)$$

where

$$\mathcal{L}_1=[f(x_0)+1]\frac{d}{dx_0}-a , \quad (27a)$$

$$\mathcal{L}_2=[f(x_0)+1]\frac{d}{dx_0}-(a+1) , \quad (27b)$$

$$\mathcal{L}_3=[f(x_0)-1]\frac{d}{dx_0}-a , \quad (27c)$$

$$\mathcal{L}_4=[f(x_0)-1]\frac{d}{dx_0}-(a+1) . \quad (27d)$$

The boundary conditions for the correct solution also follow from the integral equation. For critical values in between the asymptotically fixed stable points, we find

1. $T(z_1)=0$, (28a)
2. $\mathcal{L}_1T(z_1)=-1$, (28b)
3. $\mathcal{L}_2\mathcal{L}_1T(z_2)=2a+1$, (28c)
4. $\mathcal{L}_3\mathcal{L}_2\mathcal{L}_1T(z_2)=-a(3a+2)$. (28d)

We will also be interested in the situation in which one of the critical values is an asymptotically fixed stable point, say $z_2=x_s^-$. In this case the last two boundary conditions must be replaced by

3′. $\lim_{z_2\to x_s^-}[\mathcal{L}_2\mathcal{L}_1-a(a+1)]T(z_2)=2(2a+1)$, (29a)

4′. $\lim_{z_2\to x_s^-}[\mathcal{L}_3\mathcal{L}_2\mathcal{L}_1+a^2(a+1)]T(z_2)=-2a(2a+1)$. (29b)

We now apply these to the driftless case and the case of the linearly bound particle.

A. Driftless case $f(x)=0$

In this case the differential equation and boundary conditions for the MFPT become

$$T^{\rm IV}-k^2T''=2k^2\Omega , \quad (30)$$

where k and Ω are parameters depending on a,

$$k^2\equiv(a+1)^2+a^2 , \quad (31)$$

$$\Omega\equiv\frac{a(a+1)(2a+1)}{k^2} , \quad (32)$$

and

1. $T(z_1)=0$, (33a)
2. $T'(z_1)=-1$, (33b)
3. $T''(z_2)-(2a+1)T'(z_2)+a(a+1)T(z_2)=2a+1$, (33c)
4. $T'''(z_2)-(2a+1)T''(z_2)+a(a+1)T'(z_2)=a(a+1)$. (33d)

For the case $z_1=-z_2=z$, the solution, after a considerable amount of algebra, is

$$T(x_0)\equiv T^{+}_{-z,z}(x_0)$$
$$=A\,[\cosh kx_0-\cosh kz+k(z-x_0)\sinh kz]+B\,[\sinh kx_0-\sinh kz+k(z-x_0)\cosh kz]+z-x_0-\Omega(z-x_0)^2\ , \quad (34)$$

where

$$A=\frac{2\Omega}{k}\frac{1+(2a+1)z}{k\cosh kz+(2a+1)\sinh kz}\ , \quad (35a)$$

$$B=\frac{k}{2a+1}A\ . \quad (35b)$$

By symmetry, we have

$$T^{-}_{-z,z}(x_0)=T^{+}_{-z,z}(-x_0)\ . \quad (36)$$

Finally, for the average MFPT defined in (24), we get

$$T_{-z,z}(x_0)=\frac{(2a+1)^2}{k^2}z+\Omega(z^2-x_0^2)+A\left[\cosh kx_0-\cosh kz-\frac{k}{2a+1}\sinh kz\right]\ . \quad (37)$$

This is shown in Fig. 2 and compared with the same results for the Markov dichotomous noise [Eq. (5.14) of [2]]

$$T_{-z,z}(x_0)_{\exp}=z+a(z^2-x_0^2)\ .$$

B. Linear drift $f(x)=-x$

This case is far more interesting than the driftless case. Although the initial equations appear formidable, we will still be able to find a complete solution. The differential equation and boundary conditions for the MFPT become

$$(1-x_0^2)^2T^{\mathrm{IV}}-4(1-x_0^2)[(a+2)x_0+1]T'''+2[(3a^2+9a+7)x_0^2+2(2a+3)x_0-a^2-3a-1]T''$$
$$+4(a+1)^2[(a+1)x_0+1]T'=2a(a+1)(2a+1)\ , \quad (38)$$

and

1. $T(z_1)=0\ ,$ (39a)

2. $(1-z_1)T'(z_1)=-1\ ,$ (39b)

3. $(1-z_2)^2T''(z_2)$
$-2(a+1)(1-z_2)T'(z_2)+a(a+1)T(z_2)=2a+1\ ,$ (39c)

4. $(1-z_2)^2T'''(z_2)-2(a+2)(1-z_2)T''(z_2)$
$+(a+1)(a+2)T'(z_2)=\dfrac{a(a+1)}{1+z_2}$ (39d)

and

3′. $\lim_{z_2\to -1}[T''(z_2)-(a+1)T'(z_2)]=a+\frac{1}{2}\ ,$ (40a)

4′. $\lim_{z_2\to -1}(1+z_2)T'''(z_2)=0\ .$ (40b)

In these, boundary conditions 3 and 3′ have been used to simplify 4 and 4′, respectively, and the lower asymptotically fixed stable point is $x_s^-=-1$. Although (38) appears formidable, it can be transformed into a second-order equation by the substitution

$$T'(x_0)=(1+x_0)^{-a}(1-x_0)^{-(a+2)}u(x_0) \quad (41)$$

which carries (38) into

$$(1-x_0^2)u'''+2[(a-1)x_0+1]u''-2a^2u'$$
$$=2a(a+1)(2a+1)(1+x_0)^{a-1}(1-x_0)^{a+1} \quad (42)$$

or

$$(1-x_0^2)\theta''+2[(a-1)x_0+1]\theta'-2a^2\theta$$
$$=2a(a+1)(2a+1)(1+x_0)^{a-1}(1-x_0)^{a+1}\ , \quad (43)$$

in which $u'=\theta$. A particular solution of this last equation for all a is

$$\theta_p(x_0)=-(2a+1)(1+x_0)^{a-1}(1-x_0)^{a+1} \quad (44)$$

so that we need only consider further the homogeneous equation

$$(1-x_0^2)\theta''+2[(a-1)x_0+1]\theta'-2a^2\theta=0 \quad (45)$$

which, by the change of independent variable $\xi=(1+x_0)/2$ goes into

$$\xi(1-\xi)\frac{d^2\theta}{d\xi^2}+[2-a+2(a-1)\xi]\frac{d\theta}{d\xi}-2a^2\theta=0\ . \quad (46)$$

This is now in the standard form of the hypergeometric differential equation and two solutions can be written in terms of hypergeometric functions using classical theory. The integrals of these solutions can also be expressed in terms of hypergeometric functions and we will find, combining all of the above transformations, that the general

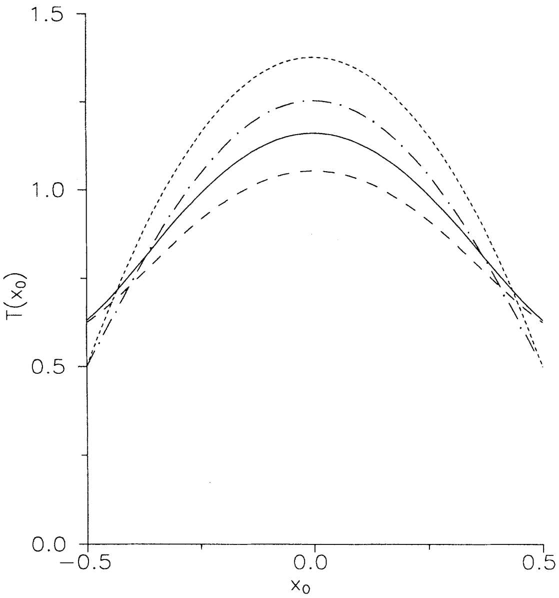

FIG. 2. Exact average MFPT's for the driftless case ($a=3.5$, $z_1=0.5$, and $z_2=-0.5$). Exponential case (dotted line), gamma case (dashed line), McFadden case (solid line), and equilibrium McFadden case (dashed-dotted line).

solution for $T(x_0)$ can be written as

$$T(x_0)=\int_{x_0}^{z_1} dx(1+x)^{-a}(1-x)^{-(a+2)} \times[Au_1(x)+Bu_2(x)+C-u_p(x)] , \quad (47)$$

in which

$$u_1(x)=F\left[-\alpha,-\alpha^*;1-a;\frac{1+x}{2}\right] , \quad (48)$$

$$u_2(x)=(1+x)^a F\left[a-\alpha,a-\alpha^*;a+1;\frac{1+x}{2}\right] , \quad (49)$$

$$\alpha=a+\tfrac{1}{2}+i\sqrt{a^2+a-\tfrac{1}{4}} , \quad (50)$$

and

$$u_p(x)=-(2a+1)\int_{-1}^{x}(1+y)^{a-1}(1-y)^{a+1}dy . \quad (51)$$

The first boundary condition has already been satisfied in writing (47). The constants A, B, and C can now be found by applying boundary conditions 2, 3, and 4. Although this is straightforward, the equations we get are messy and complicated and there seems little to be gained by writing them here.

There is some simplification in the case in which the second critical point lies at the lower asymptotically fixed stable point, in which case we need to apply boundary conditions 2, 3′, and 4′. In applying 3′, we will find that it is necessary to equate powers of $(1+z_2)$ to ensure that all terms cancel in the limit as $z_2 \to -1$. This will lead to two independent equations and we will find $A=0$ and $C=0$. Boundary condition 4′ will be automatically satisfied and B will be given by applying boundary condition 2 and is

$$B(z_1)=\frac{(1+z_1)^a(1-z_1)^{a+1}+u_p(z_1)}{u_2(z_1)} . \quad (52)$$

Substituting this back into (47), writing $z=z_1$, and using Eq. 2.8 (24) of [14] leads to the final form

$$\begin{aligned}T(x_0)&\equiv T^{+}_{-1,z}(x_0)\\&=(2a+1)\int_{x_0}^{z} dx(1+x)^{-a}(1-x)^{-(a+2)}\\&\quad\times\int_{-1}^{x} dy(1+y)^{a-1}(1-y)^{a+1}\\&\quad+B(z)[G(z)-G(x_0)] , \end{aligned}\quad (53)$$

where

$$G(x)=\frac{2^{-(a+2)}}{a+1}F\left[\alpha,\alpha^*;a;\frac{1+x}{2}\right] . \quad (54)$$

Before proceeding further, it is instructive to compare this with the same result for the case of Markov dichotomous noise which is [Eq. (A18) of [3]]

$$T^{+}_{-1,z}(x_0)_{\exp}=2a\int_{x_0}^{z} dx(1+x)^{-a}(1-x)^{-(a+1)} \times\int_{-1}^{x} dy(1+y)^{a-1}(1-y)^{a} . \quad (55)$$

Although there are some similarities, these are not the same. There had been some hope that they might be nearly identical since the steady-state probability density functions with the exponential and McFadden dichotomous noises are the same (see Sec. VII). The double integral in (53) can be evaluated when a is an integer. For

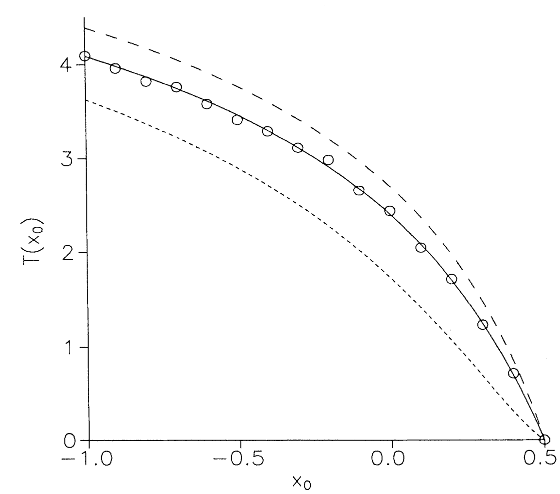

FIG. 3. MFPT for the McFadden dichotomous noise with linear drift $f(x)=-x$ ($a=1$, $z_1=0.5$, and $z_2=-1$). Ordinary renewal case (dashed line), equilibrium renewal case (solid line), and exponential case (dotted line). Circles represent simulation data for the equilibrium McFadden case.

$a=1$, we find

$$T^{+}_{-1,z}(x_0)=\ln\left[\frac{1-x_0}{1-z}\right]+\frac{2}{1-z}-\frac{2}{1-x_0}+\frac{2}{(1-z)^2}-\frac{2}{(1-x_0)^2}+\frac{2(z+1)(z-3)}{u_2(z)}[G(z)-G(x_0)] \quad (56)$$

and the corresponding result for the Markov dichotomous noise is

$$T^{+}_{-1,z}(x_0)_{\exp}=\ln\left[\frac{1-x_0}{1-z}\right]+\frac{2}{1-z}-\frac{2}{1-x_0} . \quad (57)$$

These are shown in Fig. 3.

V. MFPT FOR GAMMA DICHOTOMOUS NOISE ($N=1$)

There are many similarities in calculating the MFPT for the gamma dichotomous noise with the same calculations for the McFadden noise, but there are also some significant differences. For example, in the driftless case, the treatment is so close that we do not bother to give any of the analysis but simply state the result. Such is not the case for the linear drift and in this case we go into much greater detail.

Substituting the gamma interval density

$$\psi(t)=a^2te^{-at} \quad (58)$$

into the integral equation for $T^{+}(x_0)$ as in Appendix A leads to the differential equation and boundary conditions

$$(\mathcal{L}_3^2\mathcal{L}_1^2-a^4)T(x_0)=4a^3 , \quad (59)$$

1. $T(z_1)=0$, (60a)
2. $\mathcal{L}_1T(z_1)=-1$, (60b)
3. $\mathcal{L}_1^2T(z_2)=2a$, (60c)
4. $\mathcal{L}_3\mathcal{L}_1^2T(z_2)=-3a^2$, (60d)

and for one of the critical values at the lower asymptotically fixed stable point

3′. $\lim_{z_2\to x_s^-}(\mathcal{L}_1^2-a^2)T(z_2)=4a$, (61a)

4′. $\lim_{z_2\to x_s^-}(\mathcal{L}_3\mathcal{L}_1^2+a^3)T(z_2)=-4a^2$, (61b)

in which the operators $\mathcal{L}_1$ and $\mathcal{L}_3$ are defined by (27).

A. Driftless case $f(x)=0$

In this case the differential equation is identical to (30) but with the parameters $k=\sqrt{2a}$ and $\Omega=a$. The boundary conditions are extremely similar to (33) and the solution for the MFPT has the same form as (37) but with different constants. For the average MFPT (24) we find

$$T_{-z,z}(x_0)=2z+a(z^2-x_0^2)+(1+2az)\frac{\sqrt{2}\cosh kx_0-\sqrt{2}\cosh kz-\sinh kz}{k\cosh kz+\sqrt{2}k\sinh kz} , \quad (62)$$

which is shown plotted in Fig. 2.

B. Linear drift $f(x)=-x$

The differential equation for the MFPT becomes

$$(1-x_0^2)^2T^{\mathrm{IV}}-2(1-x_0^2)[(2a+3)x_0+2]T'''+[(6a^2+12a+7)x_0^2+8(a+1)x_0-2a^2-4a+1]T''+(2a+1)[(2a^2+2a+1)x_0+2a+1]T'=4a^3 , \quad (63)$$

with boundary conditions

1. $T(z_1)=0$, (64a)
2. $(1-z_1)T'(z_1)=-1$, (64b)
3. $(1-z_2)^2T''(z_2)-(2a+1)(1-z_2)T'(z_2)+a^2T(z_2)=2a$, (64c)
4. $(1-z_2)^2T'''(z_2)-(2a+3)(1-z_2)T''(z_2)+(a+1)^2T'(z_2)=\frac{a^2}{1+z_2}$ (64d)

and

3′. $\lim_{z_2\to -1}[2T''(z_2)-(2a+1)T'(z_2)]=2a$, (65a)

4′. $\lim_{z_2\to -1}(1+z_2)T'''(z_2)=0$. (65b)

We have been unsuccessful in finding a transformation to reduce the order of (63) as we did in the case of the McFadden dichotomous noise. Note, however, that (63) can be rewritten as

$$\frac{d}{dx_0}\{(1-x_0^2)^2T'''-2(1-x_0^2)[(2a+1)x_0+2]T''+[(6a^2+1)x_0^2+8ax_0-2a^2+3]T'\}$$
$$+(2a-1)[(2a^2-2a+1)x_0+2a-1]T'=4a^3 \ . \quad (66)$$

For the special case $a=\frac{1}{2}$, the last term on the left-hand side vanishes and the resulting equation can be integrated once yielding

$$(1-x_0^2)^2T'''-4(1+x_0)(1-x_0^2)T''+\tfrac{1}{2}(5x_0^2+8x_0+5)T' = \tfrac{1}{2}(1+x_0+C) \ , \quad (67)$$

in which C is a constant of integration. The boundary conditions become

1. $T(z_1)=0$, (68a)
2. $(1-z_1)T'(z_1)=-1$, (68b)
3. $(1-z_2)^2T''(z_2)-2(1-z_2)T'(z_2)+\frac{1}{4}T(z_2)=1$, (68c)
4. $T'(z_2)=\dfrac{2C+1+z_2}{(1-z_2)^2}$, (68d)

and

3′. $T''(-1)=\frac{1}{2}(C+1)$, (69a)

4′. $T'(-1)=\frac{1}{2}C$, (69b)

and we have made use of (67). Now, we make the transformation

$$T'(x_0)=(1+x_0)^{1/2}(1-x_0)^{-3/2}u(x_0) \quad (70)$$

and this carries (67) into

$$(1-x_0^2)u''-2x_0u'-\tfrac{1}{4}u = \tfrac{1}{2}(1+x_0)^{-3/2}(1-x_0)^{1/2}(1+x_0+C) \ . \quad (71)$$

The homogeneous equation is a form of the Legendre equation

$$(1-x_0^2)u''-2x_0u'+\nu(\nu+1)u=0, \quad \nu=-\tfrac{1}{2}+\frac{i}{2} \quad (72)$$

which is discussed in great detail in Chap. III of [14]. However, most of the discussion there is for a generalized form of (72) with an additional parameter μ (this μ has no relation to the mean time between switches used earlier). The discussion is for the two solutions $P_\nu^\mu(x_0)$ and $Q_\nu^\mu(x_0)$; however, the second solution is not real and this generality masks to some degree what we actually need. To satisfy (72), we merely need the two linearly independent solutions

$$u_1(x_0)=P_\nu(x_0) \ , \quad (73a)$$
$$u_2(x_0)=P_\nu(-x_0) \ , \quad (73b)$$

where $P_\nu(x)$ is the spherical Legendre function given by the hypergeometric function

$$P_\nu(x)=F\left[-\nu,\nu+1;1;\frac{1+x}{2}\right] . \quad (74)$$

A particular solution to (71) can be written using the Wronskian of $u_1(x)$ and $u_2(x)$ as is done in Sec. 16.516 of Gradshteyn and Ryzhik [15], and the complete solution to (67) written in the form

$$T(x_0)=\int_{x_0}^{z_1} dx\,(1+x)^{1/2}(1-x)^{-3/2} \times \tilde{A}u_1(x)+\tilde{B}u_2(x)-\tilde{u}_p(x)] \ , \quad (75)$$

in which

$$\tilde{u}_p(x)=\frac{1}{2W_0}\int_{x_l}^{x}[u_2(x)u_1(y)-u_1(x)u_2(y)](1+y)^{-3/2} \times(1-y)^{1/2}(1+y+C)dy \quad (76)$$

and x_l is some lower limit we are free to choose. W_0 is the value of the Wronskian $W(x)=W_0/(1-x^2)$ at $x=0$ and has the value $W_0=(2/\pi)\cosh(\pi/2)$. The integral defining this particular solution diverges when $x_l=-1$. To get around this undesirable feature, we can integrate by parts twice to effectively raise the exponent of the $(1+y)$ factor in the integrand and make use of the Legendre equation to simplify the resulting integrand. We find the two identities

$$\int_{x_l}^{x}(1+y)^{-3/2}(1-y)^{1/2}u_n(y)dy = v_n(x)-v_n(x_l)-\int_{x_l}^{x}(1+y)^{1/2}(1-y)^{-3/2}u_n(y)dy \ , \quad (77a)$$

$$\int_{x_l}^{x}(1+y)^{-1/2}(1-y)^{1/2}u_n(y)dy = w_n(x)-w_n(x_l)-\int_{x_l}^{x}(1+y)^{3/2}(1-y)^{-3/2}u_n(y)dy \ , \quad (77b)$$

where

$$v_n(x)=4(1-x^2)^{-1/2}[xu_n(x)-(1-x^2)u_n'(x)] \ , \quad (78a)$$
$$w_n(x)=4(1+x)^{1/2}(1-x)^{-1/2}[u_n(x)-(1-x^2)u_n'(x)] \ . \quad (78b)$$

When these are substituted into (76) and the Wronskian used to simplify the terms involving the v_n's and w_n's, we will find that the resulting terms can be either integrated or combined with the u_n's and will be led to the equivalent form for (75), but with different A and B constants

$$T(x_0)=2\ln\left[\frac{1-x_0}{1-z_1}\right]+\frac{2(C+2)}{1-x_0}-\frac{2(C+2)}{1-z_1} +\int_{x_0}^{z_1}dx\,(1+x)^{1/2}(1-x)^{-3/2} \times[Au_1(x)+Bu_2(x)+u_p(x)] \quad (79)$$

with

$$u_p(x)=\frac{1}{2W_0}\int_{-1}^{x}[u_2(x)u_1(y)-u_1(x)u_2(y)]\times(1+y)^{1/2}(1-y)^{-3/2}(1+y+C)dy \tag{80}$$

and we have taken $x_l=-1$. It is now straightforward to apply boundary conditions 2, 3, and 4 to get constants A, B, and C.

As in the case of the McFadden dichotomous noise, we will consider further only the case in which $z_2=-1$ and $z_1=z$ and apply boundary conditions 2, 3′, and 4′. The first three terms in (79) automatically satisfy boundary condition 4′. Applying condition 3′ leads to the equation

$$\lim_{x\to -1}(1+x)^{-1/2}\{A[u_1(x)+2(1+x)u_1'(x)]+B[u_2(x)+2(1+x)u_2'(x)]\}=0 \tag{81}$$

which, due to the logarithmic behavior of $u_1(x)$ in the vicinity of $x=-1$, can only be satisfied by $A=0$ and $B=0$. Applying boundary condition 2 then determines C and we find the final result

$$T(x_0)\equiv T^{+}_{-1,z}(x_0)=2\ln\left[\frac{1-x_0}{1-z}\right]+\frac{2(C+2)}{1-x_0}-\frac{2(C+2)}{1-z}+\int_{x_0}^{z}dx(1+x)^{1/2}(1-x)^{-3/2}u_p(x)\ , \tag{82}$$

where

$$C=\frac{R(z)(1-z^2)^{1/2}-3-z}{2-S(z)(1-z^2)^{1/2}} \tag{83}$$

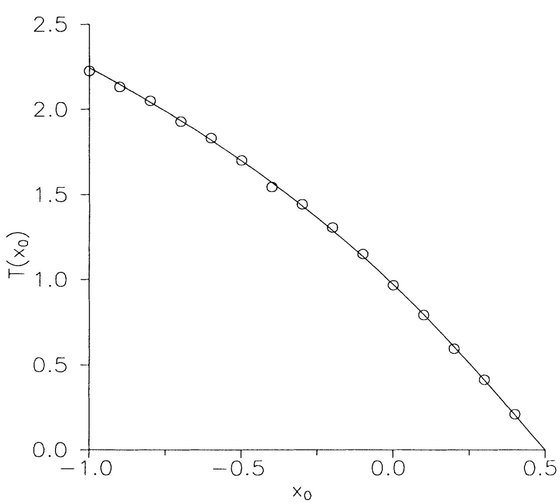

FIG. 4. MFPT for the gamma dichotomous noise with linear drift $f(x)=-x$ ($a=0.5$, $z_1=0.5$, and $z_2=-1$). Exact result (solid line), simulation (circles).

and

$$R(z)=\frac{1}{2W_0}\int_{-1}^{z}[u_2(z)u_1(y)-u_1(z)u_2(y)]\times(1+y)^{3/2}(1-y)^{-3/2}dy\ , \tag{84a}$$

$$S(z)=\frac{1}{2W_0}\int_{-1}^{z}[u_2(z)u_1(y)-u_1(z)u_2(y)]\times(1+y)^{1/2}(1-y)^{-3/2}dy\ . \tag{84b}$$

The MFPT given by (82) is plotted in Fig. 4 for the case $z=0.5$ (for this case, $C=-1.9142\ldots$) and is compared with the results of Monte Carlo simulations.

VI. MFPT FOR EQUILIBRIUM RENEWAL PROCESSES

We will now extend the results of Sec. IV to the McFadden dichotomous noise generated by equilibrium renewal processes. As noted in Sec. III, the equilibrium renewal process is a special case of the modified renewal process with the first interval governed by (6). For the McFadden probability density function, we find

$$\mu=\frac{2a+1}{a(a+1)}\ , \tag{85}$$

$$\phi(t)=\frac{a(a+1)}{2a+1}[(a+1)e^{-at}-ae^{-(a+1)t}]\ . \tag{86}$$

In order to find the MFPT for this case, it would be sufficient to use (19) with our results in Sec. IV for the ordinary renewal process. But the amount of algebra involved in finding $\rho^+(x_0)$ and in doing the double integral is excessive. Instead, we make use of the fact that $\phi(t)$ can be written as a linear combination of two probability density functions as

$$\phi(t)=\beta\psi(t)+(1-\beta)\psi_{\rm exp}(t)\ , \tag{87}$$

where $\beta=a/(2a+1)$, $\psi(t)$ is the McFadden density (25), and $\psi_{\rm exp}(t)=ae^{-at}$. Substituting (87) into (19), we find

$$T^{+}_{\rm eq}(x_0)=\beta T^{+}_{\rm ord}(x_0)+(1-\beta)T^{+}_{\rm mex}(x_0)\ , \tag{88}$$

where $T^{+}_{\rm eq}$ is the MFPT for the equilibrium renewal process, $T^{+}_{\rm ord}(x_0)$ the MFPT for the ordinary renewal process previously calculated, and $T^{+}_{\rm mex}(x_0)$ is the MFPT for a modified renewal process with $\phi(t)=\psi_{\rm exp}(t)$.

We could use again (19) to calculate $T^{+}_{\rm mex}(x_0)$, but there is an easier way. From (A1) of the Appendix we have

$$\mathcal{L}_1(T^{+}_{\rm mex}-\rho^{+}_{\rm mex})=-\frac{1}{a+1}\mathcal{L}_2\mathcal{L}_1(T^{+}_{\rm ord}-\rho^{+}_{\rm ord})\ , \tag{89}$$

where the operators $\mathcal{L}_1$ and $\mathcal{L}_2$ have been defined in (27) and $\rho^{+}_{\rm mex}[\rho^{+}_{\rm ord}]=\rho^{+}$ with $\phi=\psi_{\rm exp}(t)[\psi(t)]$. As a consequence of the commutativity of $\mathcal{L}_1$ and $\mathcal{L}_2$ we obtain

$$\mathcal{L}_1\left[T^{+}_{\rm mex}-\rho^{+}_{\rm mex}+\frac{1}{a+1}\mathcal{L}_2(T^{+}_{\rm ord}-\rho^{+}_{\rm ord})\right]=0\ , \tag{90}$$

which implies

$$T^{+}_{\rm mex}-\rho^{+}_{\rm mex}+\frac{1}{a+1}\mathcal{L}_2(T^{+}_{\rm ord}-\rho^{+}_{\rm ord})=Ce^{ag(x_0)}\ , \tag{91}$$

where C is a constant and

$$g(x_0)=\int^{x_0}\frac{dy}{f(y)+1} .$$

We see that $C=0$ from the boundary conditions

$$T^+_{\text{mex}}(z_1)=0 , \tag{92}$$

$$\rho^+_{\text{mex}}(z_1)=0 , \tag{93a}$$

$$\mathcal{L}_2[T^+_{\text{ord}}(z_1)-\rho^+_{\text{ord}}(z_1)]=0 . \tag{93b}$$

Finally

$$T^+_{\text{mex}}(x_0)=\rho^+_{\text{mex}}(x_0)-\frac{1}{a+1}\mathcal{L}_2[T^+_{\text{ord}}(x_0)-\rho^+_{\text{ord}}(x_0)] . \tag{94}$$

Introducing (94) into (88), we get

$$T^+_{\text{eq}}(x_0)=T^+_{\text{ord}}(x_0)-\frac{f(x_0)+1}{2a+1}\frac{d}{dx_0}T^+_{\text{ord}}(x_0)+(1-\beta)\left[\rho^+_{\text{mex}}(x_0)+\frac{1}{a+1}\mathcal{L}_2\rho^+_{\text{ord}}(x_0)\right] . \tag{95}$$

Whenever $\psi(t)$ is the convolution of any two exponentials, the last term can be shown to simplify greatly. In particular, for the McFadden probability density function, it can be verified directly that

$$\psi_{\text{exp}}(t)-\psi(t)=\frac{1}{a+1}\frac{d\psi(t)}{dt} . \tag{96}$$

This can be employed in the integrals defining $\rho^+_{\text{mex}}(x_0)$ and $\rho^+_{\text{ord}}(x_0)$ to show that

$$\rho^+_{\text{mex}}(x_0)+\frac{1}{a+1}\mathcal{L}_2\rho^+_{\text{ord}}(x_0)=-\frac{1}{a+1} \tag{97}$$

so that (95) simplifies to

$$T^+_{\text{eq}}(x_0)=T^+_{\text{ord}}(x_0)-\frac{1}{2a+1}-\frac{f(x_0)+1}{2a+1}\frac{d}{dx_0}T^+_{\text{ord}}(x_0) . \tag{98a}$$

This expression only involves a result we already know and its first derivative. Following the same reasoning with $F(0)=-1$, we obtain

$$T^-_{\text{eq}}(x_0)=T^-_{\text{ord}}(x_0)-\frac{1}{2a+1}-\frac{f(x_0)-1}{2a+1}\frac{d}{dx_0}T^-_{\text{ord}}(x_0) \tag{98b}$$

and then

$$T_{\text{eq}}(x_0)=\tfrac{1}{2}[T^+_{\text{eq}}(x_0)+T^-_{\text{eq}}(x_0)] . \tag{99}$$

We now find $T_{\text{eq}}(x_0)$ for the cases studied in Sec. IV.

A. Driftless case $f(x)=0$

When $f(x)=0$ and $z_1=z$, $z_2=-z$, we get from (34)–(36) and (98a)–(98b) the result

$$T_{\text{eq}}(x_0)=z+\Omega(z^2-x_0^2)+\frac{2a(a+1)}{(2a+1)^2}A(\cosh kx_0-\cosh kz) , \tag{100}$$

where k and Ω are defined in (31) and (32) and A is given by (35a). Equation (100) is shown plotted in Fig. 2.

B. Linear drift $f(x)=-x$

We will concentrate on the case $z_2=-1$ and $z_1=z$. Using (56) in (98a) for the case $a=1$, we find

$$T^+_{\text{eq}}(x_0)=T^+_{\text{ord}}(x_0)+\frac{2}{3(1-x_0)^2}\left[(3-x_0)-(3-z)\frac{F\left[\gamma,\gamma^*;2;\dfrac{1+x_0}{2}\right]}{F\left[\gamma,\gamma^*,2;\dfrac{1+z}{2}\right]}\right] , \tag{101}$$

where $\gamma=-\frac{1}{2}+i(\sqrt{7}/2)$. This is plotted in Fig. 3.

This same procedure can be followed in the case of the gamma dichotomous noise; however, it will not be considered here. We merely point out that the relation (87) holds for the gamma probability density function with $\beta=\frac{1}{2}$.

VII. STEADY-STATE PROBABILITY DENSITY FUNCTIONS

The steady-state probability density function $p(x)$ for the system state variable has been obtained in the case of the linear drift $f(x)=-x$ for the McFadden dichotomous noise by McFadden [7] and partial results for the gamma noise by Pawula and Rice [8]. We very briefly summarize some of these results to show the similarities with the MFPT.

A. gamma dichotomous noise ($N=1$)

By differentiating the governing integral equation to reduce it to a differential equation, the following third-order equation can be derived for $p(x)$ [cf. (52) of [8]]

$$(1-x^2)^2p'''-2(5-2a)x(1-x^2)p''+2[(3a^2-11a+12)x^2-a^2+3a-4]p'+2(3-2a)(a^2-2a+2)xp=0 . \tag{102}$$

This equation has a great similarity to (63) for the MFPT. As with (63), for $a=\frac{1}{2}$ this can be integrated once. Then imposing the conditions that the solution be symmetric, integrate to one, and give the correct second moment leads to the solution

$$p(x)=\frac{|\Gamma(1+\nu/2)|^4}{2\pi^2}(1-x^2)^{-1/2}[P_\nu(x)+P_\nu(-x)] . \quad (103)$$

The reader is referred to [8] for the details. A closely related quantity is the probability density function $p_0(x)$ of $x(t)$ at a minimum point, i.e., a point at which $F(t)$ switches from -1 to $+1$. The fourth-order differential equation satisfied by $p_0(x)$ is [cf. (47) of [8]]

$$(1-x^2)^2p_0^{\mathrm{IV}}+2(1-x^2)[(2a-5)x+2]p_0'''+[(6a^2-24a+25)x^2+8(a-2)x-2a^2+8a-5]p_0'' +(3-2a)[(2a^2-6a+5)x+2a-3]p_0'+[(a-1)^4-a^4]p_0=0 . \quad (104)$$

Although we have not been successful in solving this equation directly, we note that the differential operator is the adjoint of the differential operator in (63) for the MFPT. This, we feel, is an important connection between the *nonstationary* MFPT and the *stationary* probability density at a minimum point.

B. McFadden dichotomous noise ($b=a+2$)

By following this exact same approach for the McFadden dichotomous noise, we are led to the third-order differential equation

$$(1-x^2)^2p'''+4(a-2)x(1-x^2)p'' +2(3a^2-8a+7)x^2-a^2+2a-3]p' +4(1-a)(a^2-a+1)xp=0 , \quad (105)$$

which is similar to (38). The general solution is a term proportional to $(1-x^2)^{a-1}$ plus the sum of two hypergeometric functions. The coefficients of the hypergeometric functions can be shown to vanish by applying appropriate boundary conditions and we are led to

$$p(x)=\frac{(1-x^2)^{a-1}}{2^{2a-1}B(a,a)} . \quad (106)$$

This is the same as the corresponding result for the Markov dichotomous noise

$$p_{\exp}(x)=\frac{(1-x^2)^{a-1}}{2^{2a-1}B(a,a)} . \quad (107)$$

This equality of the systems' marginal steady-state probability density functions does not appear to have been realized by McFadden. The simplicity afforded by (106) for the McFadden dichotomous noise over (103) for the gamma dichotomous noise was our reason for expecting that the MFPT for the McFadden noise might be simpler than for the gamma noise and, indeed, this suspicion was confirmed by our results.

The fourth-order differential equation for the probability density function at a minimum point can be determined by following the same procedure as in the case of the gamma dichotomous noise [8]. The result is

$$(1-x^2)^2p_0^{\mathrm{IV}}+4(1-x^2)[(a-2)x+1]p_0'''+2[(3a^2-9a+7)x^2+2(2a-3)x-a^2+3a-1]p_0'' -4(a-1)^2[(a-1)x+1]p_0'-4a^3p_0=0 . \quad (108)$$

Again, we note that the differential operator is the adjoint of the differential operator in the fourth-order differential equation for the MFPT (38). The solution to (108), which can be obtained in a much simpler way than directly solving the differential equation, is [(26) and (4b) of [8]]

$$p_0(x)=\frac{(1+x)^{a-1}(1-x)^{a+1}}{2^{2a+1}B(a,a+2)} . \quad (109)$$

VIII. SUMMARY AND CONCLUSIONS

We have considered the problem of first-passage time statistics for general one-dimensional processes driven by non-Markov dichotomous noise. The governing integral equations for the first-passage time probability density function have been rederived in order to minimize the combinatorial difficulties that accompany the exact enumeration of trajectories. This has also allowed us to include within the same formalism and in a more compact way the cases of ordinary, modified, and equilibrium renewal processes.

We have applied the formalism to two non-Markov driving noises of physical interest: McFadden and gamma dichotomous noise. In both cases we have been able to convert the governing integral equations for the MFPT into fourth-order differential equations with appropriate boundary conditions. The differential equations have been completely solved in the driftless case and in the linear drift case. The exact solutions for the latter are written in terms of hypergeometric functions (McFadden case) or spherical Legendre functions (gamma case).

We have also shown similarities between the steady-state probability density problem and the MFPT problem. In particular, the differential operator for the MFPT was shown to be the adjoint of the differential operator for the probability density function of a minimum point. Finally all results have been confirmed through simulations.

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APPENDIX: CONVERSION OF INTEGRAL EQUATIONS INTO DIFFERENTIAL EQUATIONS

In order to obtain differential equations for the MFPT we first make changes of variables suggested by the dynamics of the system

$$t'=\int_{x_0}^{x_1}\frac{dx}{f(x)+1}\equiv t^+(x_1,x_0)\ ,$$

$$t''=\int_{x_1}^{x_2}\frac{dx}{f(x)-1}\equiv t^-(x_2,x_1)\ .$$

In terms of these new variables, we can write (19) as

$$T^+(x_0)=\rho^+(x_0)+\int_{x_0}^{z_1}dx_1\frac{\phi(t^+(x_1,x_0))}{f(x_1)+1}\int_{x_1}^{z_2}dx_2\frac{\psi(t^-(x_2,x_1))}{f(x_2)-1}T^+_{\rm ord}(x_2)\ . \tag{A1}$$

A similar replacement can be made for (20). In what follows we will only consider the ordinary renewal case which implies $\phi(t)=\psi(t)$ and $T_{\rm ord}(x)=T(x)$. Then (A1) is a closed integral equation for the MFPT.

For the McFadden dichotomous noise $\psi(t)$ is the combination of exponentials given by (25). In this case (A1) reads

$$T(x_0)=\rho(x_0)+a^2(a+1)^2\int_{x_0}^{z_1}dx_1\frac{e^{-at^+(x_1,x_0)}[1-e^{-t^+(x_1,x_0)}]}{f(x_1)+1}\int_{x_1}^{z_2}dx_2\frac{e^{-at^-(x_2,x_1)}[1-e^{-t^-(x_2,x_1)}]}{f(x_2)-1}T(x_2)\ , \tag{A2}$$

where [cf. (17) and (21)]

$$\rho(x_0)\equiv\rho^+(x_0)=\tau_1\int_{\tau_1}^{\infty}dt'\psi(t')+\int_0^{\tau_1}dt'(t'+\tau_2)\psi(t')\int_{\tau_2}^{\infty}dt''\psi(t'')+\int_0^{\tau_1}dt'\psi(t')\int_0^{\tau_2}dt''(t'+t'')\psi(t'')\ , \tag{A3}$$

with $\tau_1=t^+(z_1,x_0)$ and $\tau_2=t^-(z_2,x_1)$. Taking the x_0 derivative of (A2) and reorganizing the terms gives

$$\mathcal{L}_1[T(x_0)-\rho(x_0)]=-a^2(a+1)^2\int_{x_0}^{z_1}dx_1\frac{e^{-(a+1)t^+(x_1,x_0)}}{f(x_1)+1}\int_{x_1}^{z_2}dx_2\frac{e^{-at^-(x_2,x_1)}[1-e^{-t^-(x_2,x_1)}]}{f(x_2)-1}T(x_2)\ . \tag{A4}$$

Higher-order derivatives of this yield

$$\mathcal{L}_2\mathcal{L}_1[T(x_0)-\rho(x_0)]=a^2(a+1)^2\int_{x_0}^{z_2}dx_2\frac{e^{-at^-(x_2,x_0)}[1-e^{-t^-(x_2,x_0)}]}{f(x_2)-1}T(x_2)\ , \tag{A5}$$

$$\mathcal{L}_3\mathcal{L}_2\mathcal{L}_1[T(x_0)-\rho(x_0)]=-a^2(a+1)^2\int_{x_0}^{z_2}dx_2\frac{e^{-(a+1)t^-(x_2,x_0)}}{f(x_2)-1}T(x_2)\ , \tag{A6}$$

and

$$\mathcal{L}_4\mathcal{L}_3\mathcal{L}_2\mathcal{L}_1[T(x_0)-\rho(x_0)]=a^2(a+1)^2T(x_0)\ , \tag{A7}$$

where the differential operators $\mathcal{L}_i$ are defined in (27). One easily shows from (A3) that

$$\mathcal{L}_4\mathcal{L}_3\mathcal{L}_2\mathcal{L}_1\rho(x_0)=2a(a+1)(2a+1)\ . \tag{A8}$$

The substitution of (A8) into (A7) yields (26).

The first and second boundary conditions are obtained by setting $x_0=z_1$ in (A2)–(A4). The integral terms vanish, $\rho(z_1)=0$, and $\mathcal{L}_1\rho(z_1)=-1$. The third and fourth boundary conditions are obtained by setting $x_0=z_2$ in (A5) and (A6). The final result is (28).

When the critical value z_2 is the asymptotically fixed stable point x_s^-, i.e., $f(x_s^-)=1$, then the integrands on the right-hand side of (A5) and (A6) are singular and the integrals are not defined at $x_0=x_s^-$. In order to find the correct values, we integrate the rhs of (A5) twice by parts and in the limit $x_0\to z_2$ we obtain

$$\lim_{x_0\to z_2}\int_{x_0}^{z_2}dx_2\frac{e^{-at^-(x_2,x_0)}[1-e^{-t^-(x_2,x_0)}]}{f(x_2)-1}T(x_2)=\frac{1}{a(a+1)}\lim_{x_0\to z_2}[T(x_0)-T(z_2)e^{-(a+1)t^-(z_2,x_0)}]\ . \tag{A9}$$

Now if $z_2\neq x_s^-$ the right-hand side of (A9) vanishes and the third boundary condition is again obtained. But if $z_2=x_s^-$ then $t^-(x_s^-,x_0)=\infty$ and (A9) gives

$$\lim_{x_0\to x_s^-}\int_{x_0}^{x_s^-}dx_2\frac{e^{-at^-(x_2,x_0)}[1-e^{-t^-(x_2,x_0)}]}{f(x_2)-1}T(x_2)=\frac{1}{a(a+1)}T(x_s^-)\ . \tag{A10}$$

In an analogous way one easily sees that the integral on the rhs of (A6) can be written as

$$\lim_{x_0 \to x_s^-} \int_{x_0}^{x_s^-} dx_2 \frac{e^{-(a+1)t^-(x_2,x_0)}}{f(x_2)-1} T(x_2) = -\frac{1}{a+1} T(x_s^-) \ . \quad \text{(A11)}$$

On the other hand, when $z_2 = x_s^-$ the time interval τ_2 that appears in the expression for $\rho(x_0)$ [cf. (A3)] becomes infinite. This divergence can be removed by taking the limit and the result is

$$\rho(x_0) = \tau_1 \int_{\tau_1}^{\infty} dt' \psi(t') + \int_0^{\tau_1} dt' t' \psi(t') + \int_0^{\tau_1} dt' \psi(t') \int_0^{\infty} dt'' t'' \psi(t'') \ . \quad \text{(A12)}$$

From (A12) one can easily show that

$$\lim_{x_0 \to z_2} \mathcal{L}_2 \mathcal{L}_1 \rho(x_0) = 2(2a+1) \quad \text{(A13)}$$

and

$$\lim_{x_0 \to z_2} \mathcal{L}_3 \mathcal{L}_2 \mathcal{L}_1 \rho(x_0) = -2a(2a+1) \ . \quad \text{(A14)}$$

Collecting the results, we obtain the *singular* boundary conditions (29).

For the gamma dichotomous noise ($N=1$) the density $\psi(t)$ is given by (58) and (A1) becomes

$$T(x_0) = \rho(x_0) + a^4 \int_{x_0}^{z_1} dx_1 \frac{t^+(x_1,x_0) e^{-at^+(x_1,x_0)}}{f(x_1)+1} \int_{x_1}^{z_2} dx_2 \frac{t^-(x_2,x_1) e^{-at^-(x_2,x_1)}}{f(x_2)-1} T(x_2) \ . \quad \text{(A15)}$$

Following the procedure outlined above, it is straightforward to show that (A15) is equivalent to the differential equation (59) with boundary conditions (60) and (61).

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